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The linear Vlasov-Poisson-Ampère equation from the viewpoint of abstract scattering theory

Bruno Després *

Abstract

We review some recent results on the scattering structure of the linearized Vlasov-Poisson equation in $d = 1$ space dimension. It started with [9] where the linearized Vlasov-Poisson equation is rewritten as a linear Vlasov-Ampère set of equations which makes the L^2 structure more visible. A consequence is that the linear Landau damping becomes an application of the scattering theory for Hamiltonian systems. Then we review the extension, firstly of the linearization around non homogeneous profiles which is treated with the theory of trace-class operators, secondly of the case with a forcing magnetic field which has the ability to eliminate the possibility of a linear Landau damping effect. Finally, we evoke some possibility for extension to space dimension $\mathbf{x} \in \mathbb{R}^d$ with $d > 1$.

1 Introduction

The presentation starts with a non linear Vlasov-Poisson model problem for negatively charged particules in a constant magnetic field $\mathbf{B}_0$ and in a bath of ions with non homogeneous density ρ_{ions}

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - (\mathbf{E}(t,\mathbf{x}) + \mathbf{v} \times \mathbf{B}_0) \cdot \nabla_{\mathbf{v}} f = 0, \quad f(t,\mathbf{x},\mathbf{v}) \geq 0, \\ -\Delta \varphi = \rho_{\text{ions}}(\mathbf{x}) - \int f dv, \\ \mathbf{E}(t,\mathbf{x}) = -\nabla_{\mathbf{x}} \varphi(t,\mathbf{x}). \end{cases} \tag{1}$$

The long time dynamics of the linearized version of this system and of related linear systems presents many similarities with abstract scattering theory, some references in this direction are [1, 7, 12]. The recent observations from [10, 11] explain how the linear Vlasov-Poisson issued from (1) can be rewritten[1] in the framework of abstract scattering theory [17, 19]. As a consequence the powerful artillery of spectral theory and scattering theory is remarkably adapted to analyze the long time dynamics of linear systems. This is a way to discuss

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[1] Non linear considerations [24, 23, 20] are outside the scope of this work.

with another approach the seminal Landau's contribution on what is called the linear Landau damping effect [18], see also [22, 5, 21, 6, 2, 3, 7, 1, 7, 12, 4].

A simplified system issued from (1) is the 1d-2v kinetic equation, one dimension in space in the torus $\mathbb{T}$ and two dimensions in velocity in the plane,

$$\partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right) f = 0, \quad t > 0, \ x \in \mathbb{T}, \ \mathbf{v} \in \mathbb{R}^2, \quad (2)$$

coupled with the Poisson equation

$$\begin{cases} \partial_x E = \rho_{\rm ions}(x) - \int_{\mathbb{R}^2} f dv, & t > 0, \ x \in \mathbb{T}, \\ \int_{\mathbb{T}} E dx = 0, & t > 0. \end{cases} \quad (3)$$

The last integral relation is deduced from the representation of the electric as minus the gradient of the electric potential (1). For simplicity, the mass of electrons and ions is normalized $m_i = m_e = 1$, the charge is also normalized as $e = +1$ for ions and e for electrons. Here the cyclotron frequency is $\omega_c = e|\mathbf{B}_0|/m_e = |\mathbf{B}_0|$. It can be checked that it is equivalent to consider the Ampère equation complemented with the Poisson equation at initial time $t = 0$

$$\begin{cases} \partial_t E = 1^* \int_{\mathbb{R}^2} v_1 f dv, & t > 0, \ x \in \mathbb{T}, \\ \partial_x E = \rho_{\rm ions}(x) - \int_{\mathbb{R}^2} f dv, & t = 0, \ x \in \mathbb{T}, \\ \int_{\mathbb{T}} E dx = 0, & t = 0, \end{cases} \quad (4)$$

where $(1^* g)(x) = g(x) - \int_I g(y) dy$. An important idea in the approach developed hereafter is that the "non stationary" Ampère equation is better than the "stationary" Poisson equation because it provides or comprehensive understanding of the energetic structure of the system.

The total initial charge of electrons is $\int_{\mathbb{T}\times\mathbb{R}^2} f_{\rm ini} dx dv = \int_{\mathbb{T}} \rho_{\rm ions}(x) dx$. The solutions of the system preserve the physical energy

$$\frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{R}^2} f v^2 dv dx + \frac{1}{2} \int_{\mathbb{T}} E^2 dx$$

where $f \geq 0$. They also preserve the charge of electrons, equal to the charge of ions

$$\int_{\mathbb{T}} \int_{\mathbb{R}^2} f(t,x,v) dx dv = \int_{\mathbb{T}} \int_{\mathbb{R}^2} f_{\rm ini}(x,v) dx dv = \int_{\mathbb{T}} \rho_{\rm ions}(x) dx.$$

The initial data (f, E) can be considered as a small perturbation of a stationary state (f_0, E_0) such that $v_1 \partial_x f_0 - E_0(x) \partial_{v_1} f_0 + \omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right) f_0 = 0$ with $E_0 = -\varphi_0'$. The Boltzmanian hypothesis is natural to represent physically sound stationary states. It writes $f_0(x, v_1, v_2) = \exp\left(-|\mathbf{v}|^2/2 + \varphi_0(x) \right)$ where φ_0 is the reference electric potential: this Boltzmanian hypothesis is a strong hypothesis that probably can be modified [2, 3, 14] or relaxed as in [20]. We consider a linearization under the form $f(t,x,v_1,v_2) = f_0(x,v_1,v_2) + \varepsilon \sqrt{f_0(x,v_1,v_2)} u(t,x,v_1,v_2) + O(\varepsilon^2)$ and $E(t,x) = E_0(x) + \varepsilon F(t,x) + O(\varepsilon^2)$. Then we inject in the equations and drop the quadratic terms. It yields the equations at order 0 $f_0(x, v_1, v_2) = n_0(x) e^{-|\mathbf{v}|^2/2}$, $n_0(x) = e^{\varphi_0(x)}$, $E_0(x) = -\varphi_0'(x)$

and $-\varphi_0''(x)+2\pi e^{\varphi_0(x)} = \rho_{\text{ions}}(x)$ where the last equation is a non linear Poisson equation which is solvable, and the equations at order 1

$$\begin{cases} \partial_t u + v_1 \partial_x u - E_0 \partial_{v_1} u + F v_1 \sqrt{f_0} + \omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right) u = 0, \\ \partial_t F = 1^* \int_{\mathbb{R}} v u \sqrt{f_0} dv. \end{cases} \tag{5}$$

equipped with periodic boundary conditions on the Torus. A fundamental physical property is the preservation of energy

$$\frac{d}{dt} \left(\int_{\mathbb{T}} \int_{\mathbb{R}} u^2 dv dx + \int_{\mathbb{T}} F^2 dx \right) = 0. \tag{6}$$

The Gauss law

$$\partial_x F = - \int_{\mathbb{R}^2} u M dv \tag{7}$$

can be understood as a constraint satisfied by the initial data and propagated by the equation. It can be characterized in the weak sense

$$\int_{\mathbb{T}\times\mathbb{R}} u(x,v) M(x,v) \varphi(x) dx dv - \int_{\mathbb{T}} F(x) \varphi'(x) dx = 0 \quad \forall \varphi \in H^1(\mathbb{T}). \tag{8}$$

Define the space $L^2_0(\mathbb{T}) := \left\{ F \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} F(x) dx = 0 \right\}$. One has the identity

$$\int_{\mathbb{T}} \int_{\mathbb{R}^2} u (v_1 \sqrt{f_0} F) dv dx = \int_{\mathbb{T}} 1^* \left(\int_{\mathbb{R}^2} u v_1 \sqrt{f_0} dv \right) F dx$$

which shows that the adjoint operator of the multiplication operator

$$\begin{array}{rccc} v_1 \sqrt{f_0} : & L^2_0(\mathbb{T}) & \to & L^2(\mathbb{T}\times\mathbb{R}^2) \\ & F & \mapsto & v_1 \sqrt{f_0} F \end{array}$$

is the multiplication operator

$$\begin{array}{rccc} (v\sqrt{f_0})^* : & L^2(\mathbb{T}\times\mathbb{R}^2 & \to & L^2_0(\mathbb{T}) \\ & u & \mapsto & 1^* \int_{\mathbb{R}} v_1 u \sqrt{f_0} dv. \end{array}$$

Define for convenience the space of complex-valued functions $X = L^2(\mathbb{T}\times\mathbb{R}^2) \times L^2_0(\mathbb{T})$. The subspace $GL = \{(u,F) \in X,$ the identity (8) holds$\} \subset X$ characterizes pairs which satisfy the Gauss law in the weak sense. From now on, one systematically introduces the pure imaginary number $i^2 = -1$ to obtain compatibility with more standard notations in scattering theory [17, 19, 25].

One recasts the linear Vlasov-Ampère equations as

$$\partial_t U(t) + iHU(t) = 0 \tag{9}$$

where the unknown is $U = \begin{pmatrix} u \\ F \end{pmatrix} \in GL$ and the symmetric total operator is $H = H_0 + K$ where the unperturbed symmetric operator H_0 is given by

$$iH_0 = \left(\begin{array}{c|c} v\partial_x - E_0 \partial_v & 0 \\ \hline 0 & 0 \end{array} \right) \tag{10}$$

and the perturbation is given by

$$iK = \left(\begin{array}{c|c} 0 & vM \\ \hline -1^* \int_v vM & 0 \end{array} \right). \tag{11}$$

The main observation [9] is that the structure of (9) with the decomposition $H = H_0 + K$ is the one of abstract scattering theory [17, 19, 25].

Then the main question is to explore the dynamics of the full Hamiltonian e^{-iHt} with respect to the dynamics of the reduced Hamiltonian e^{-iH_0t}. Among the tools of abstract scattering theory, one can distinguish the Lipmann-Schwinger equation which calculates the generalized eigenvectors of H in function of those of H_0, the construction of wave-operators $W^{\pm}(H, H_0)$ which establishes an isometric equivalence between the absolutely continuous part of the spectrum of H and the one of H_0 and the trace-class method which justifies the existence of the wave operators provided the perturbation K satisfies the trace-class criterion. Going back to the physical problem attached to (9), square integrable initial data in the absolutely continuous part of the spectrum H converge weakly to zero in quadratic norm and this is essentially a generic proof of the linear Landau damping (the electric field tends to zero in time).

Using the Ampère equation instead of the Poisson equation is a key. However technical difficulties remain. Indeed one can check that all pairs in GL are null eigenvectors for H, but are not for H_0 in the general case. So the dynamics of e^{-iH_0t} has no knowledge of the Poisson constraint. On physical grounds this is of course a naive approach. On mathematical grounds it is shown in [11] that this decomposition is not adapted. Nevertheless the structure (9) with the decomposition $H = H_0 + K$ is the core of the approach explained in these notes.

2 Non homogeneous profiles

The material presented in this section comes from [11, 10]. A convenient change of unknown is possible for (9). The idea is to consider a new kinetic function

$$w(x, v) = u(x, v) + \gamma(x)M(x, v)F(x) \tag{12}$$

and the 1-periodic function γ solution to the Riccati equation

$$\partial_x \gamma + \alpha^2 \gamma^2 \exp \varphi_0 = 1, \qquad \alpha = (2\pi)^{1/4}. \tag{13}$$

One has by construction the preservation of the quadratic norm

$$\int_{\mathbb{T}} \int_{\mathbb{R}} |w(x, v, t)|^2 dxdv = \int_{\mathbb{T}} \int_{\mathbb{R}} |w(x, v, t)|^2 dxdv + \int_{\mathbb{T}} |F(x, v, t)|^2 dx. \tag{14}$$

For a the pair (u, F) which satisfies (9)-(7), then purely algebraic manipulations show that w is solution to the autonomous equation

$$w'(t) = i\mathcal{H}w(t), \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{K} \tag{15}$$

where $i\mathcal{H}_0 = v\partial_x - E_0\partial_v$ is (once again) the transport operator and $i\mathcal{K} = i\mathcal{K}_1 + i\mathcal{K}_2$ is integral operator defined by

$$i\mathcal{K}_1 w = \gamma\left(vM\int_{\mathbb{R}} wMdv - M\int_{\mathbb{R}} wvMdv\right)$$

and

$$i\mathcal{K}_2 w = +\gamma M\int_{\mathbb{R}\times\mathbb{T}} wvMdv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right)Mv.$$

This is an equivalence, that is there is a bijection between the space GL and the space

$$\mathcal{X} := \left\{w \in L^2(\mathbb{T}\times\mathbb{R}),\ (w,\gamma M) = 0\right\} \subset L^2(\mathbb{T}\times\mathbb{R}),$$

with the property $\mathcal{H}\gamma M = 0$. It appears the new formulation (15) is well adapted to proof the trace-class criterion. The operators $\mathcal{H}_0$ and $\mathcal{K}$ are self-adjoint and the integral operator $\mathcal{K}$ regularizes in v but not in x. Let $(\psi_n)_{n\in\mathbb{N}}$ be the orthonormal complete family of Hermite functions

$$\psi_0(v) = \exp(-v^2/4)/\alpha, \quad \psi_1(v) = v\exp(-v^2/4)/\alpha, \quad \psi_2(v) = \dots$$

Lemma 2.1. $\mathrm{Ker}(\mathcal{K}) = \underset{n\geq 2}{\mathrm{Span}}\left\{a_n(x)\psi_n(v)\right\}$ *where* $a_n \in L^2(\mathbb{T})$ *for* $n \geq 2$.

The trace-class criterion (in the resolvent sense) expresses that some infinite sum of eigenvalues must be convergent [17, 25, 19]. To check the trace-class criterion, one can consider

$$(\mathcal{H}-z)^{-1} - (\mathcal{H}_0 - z)^{-1} = -(\mathcal{T}+\mathcal{S})\mathcal{C}$$

where $z \in \mathbb{C}$, $z \notin \mathbb{R}$, and

$$\begin{cases} \mathcal{T} = (\mathcal{H}_0 - z)^{-1}\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}, \\ \mathcal{S} = (\mathcal{H}_0 - z)^{-1}\mathcal{K}_2(\mathcal{H}_0 - z)^{-1}, \\ \mathcal{C} = I - \mathcal{K}(\mathcal{H} - z)^{-1} \text{ is bounded operator.} \end{cases}$$

The main result obtained in [11] is that the trace class property holds for this decomposition. The main steps of the proof concentrate on showing that $\mathcal{T}$ is trace-class. We take for simplicity $z = i\beta \in i\mathbb{R}^*$. One has $\mathcal{T}^*\mathcal{T} = (\mathcal{H}_0 - \overline{z})^{-1}\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}$.

Lemma 2.2. *One has* $\mathrm{Ker}(\mathcal{T}^*\mathcal{T}) = \mathrm{Span}_{n\geq 2}\left\{(\mathcal{H}_0 - z)a_n\psi_n, a_n \in H^1(\mathbb{T})\right\}$.

Lemma 2.3. *Let* $\lambda \in \mathbb{R}^*$. *The equation* $(\mathcal{T}^*\mathcal{T})w = \lambda w$ *for* $w \neq 0$ *is equivalent to two decoupled integral equations*

$$\begin{pmatrix} \gamma e^{\varphi_0}\mathcal{T}_2\gamma e^{\varphi_0}\mathcal{T}_1 a \\ \gamma e^{\varphi_0}\mathcal{T}_1\gamma e^{\varphi_0}\mathcal{T}_2 b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}, \quad (a,b) \neq (0,0), \tag{16}$$

where $\mathcal{T}_1, \mathcal{T}_2 : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *are integral operators*

$$\begin{cases} \mathcal{T}_1 a(x) = \alpha^2 \int_{\mathbb{R}} \psi_0(v) \left[(\mathcal{H}_0^2 + |z|^2)^{-1} (a\psi_0) \right](y,v) dv, \\ \mathcal{T}_2 b(x) = \alpha^2 \int_{\mathbb{R}} \psi_1(v) \left[(\mathcal{H}_0^2 + |z|^2)^{-1} (b\psi_1) \right](y,v) dv. \end{cases} \tag{17}$$

Moreover $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are self adjoint, bounded, positive and injective.*

Proof. Take $w = (\mathcal{H}_0 - \overline{z})^{-1} \left(a(x)\psi_0(v) + b(x)\psi_1(v) \right) \in \mathrm{Ker}(\mathcal{T}^* \mathcal{T})^\perp$, plug inside the equation and expand. □

The regularity of operators $\mathcal{T}_1$ and $\mathcal{T}_2$ is as follows.

Lemma 2.4. *Assume* $E_0 \in W^{s+1,\infty}(\mathbb{T})$ *for* $s \geq 0$. *Then there exists* $C_s > 0$ *such that* $\|\mathcal{T}_2 b\|_{H^{s+2}(\mathbb{T})} \leq C_s \|b\|_{H^s(\mathbb{T})}$.

Hint of the proof. With over-simplification, take $E_0 = 0$ and $z = i$. Then

$$\begin{aligned} \mathcal{T}_2 b(x) &= C \int_{\mathbb{R}} v e^{-v^2/4} \left[\left(-v^2 \partial_{xx} + 1 \right)^{-1} v e^{-v^2/4} b \right](y,v) dv \\ &\approx \widetilde{\mathcal{T}}_2 b(x) = C \int_{\mathbb{R}} v^2 \left[\left(-v^2 \partial_{xx} + 1 \right)^{-1} e^{-v^2/2} b \right](y,v) dv \end{aligned}$$

Clearly $\partial_{xx} \widetilde{\mathcal{T}}_2 b(x) = C \int_{\mathbb{R}} \left[(\partial_{xx} v^2) \left(-v^2 \partial_{xx} + 1 \right)^{-1} \right] e^{-v^2/2} b(y,v) dv$ so one has that $\|\widetilde{\mathcal{T}}_2 b\|_{H^{s+2}(\mathbb{T})} \leq C_s \|b\|_{H^s(\mathbb{T})}$. One checks it is the same for operator $\mathcal{T}_2$. □

The critical part is the operator $\mathcal{T}_1$.

Lemma 2.5. *Under the same assumptions, there exists a constant* $C_s > 0$ *such that* $\|\mathcal{T}_1 a\|_{H^{s+1/4}(\mathbb{T})} \leq C_s \|a\|_{H^s(\mathbb{T})}$.

Hint of the proof. With the same over-simplification $E_0 = 0$ and $z = i$, one gets

$$\begin{aligned} \mathcal{T}_1 a(x) &= C \int_{\mathbb{R}} e^{-v^2/4} \left[\left(-v^2 \partial_{xx} + 1 \right)^{-1} e^{-v^2/4} b \right](y,v) dv \\ &= C \int_{\mathbb{R}} e^{-v^2/4} \left(v\partial_x + 1 \right)^{-1} \underbrace{\left[\left(-v\partial_x + 1 \right)^{-1} e^{-v^2/4} b \right]}_{=g} (y,v) dv \end{aligned}$$

Extra-regularity comes from compactness by integration. Take

$$g = \left(-v\partial_x + 1 \right)^{-1} e^{-v^2/4} b \in L^2(\mathbb{T} \times \mathbb{R}), \quad \text{and} \quad f = \left(v\partial_x + 1 \right)^{-1} g \in L^2(\mathbb{T} \times \mathbb{R}).$$

So

$$v\partial_x f = g - f \in L^2(\mathbb{T} \times \mathbb{R}).$$

One can invoke compactness by integration [16, 15]. Using the notations from [13], one gets with $m = 0$ that $\int_{\mathbb{R}} e^{-v^2/4} f(x,v) dx dv \in H^{1/2}(\mathbb{T})$. The general case for a non zero $E_0 \neq 0$ is

$$g = \left(-v\partial_x + E_0(x)\partial_v + 1 \right)^{-1} b \in L^2(\mathbb{T} \times \mathbb{R}),$$

and $f = (v\partial_x - E_0(x)\partial_v + 1)^{-1} g \in L^2(\mathbb{T}\times\mathbb{R})$. So

$$v\partial_x f = g - f + \partial_v(E_0(x)f) \in L^2(\mathbb{T}; H^{-1}(\mathbb{R})).$$

With $m = 1$, one gets from [13] that $\int_{\mathbb{R}} e^{-v^2/4} f(x,v)dxdv \in H^{1/4}(\mathbb{T})$. □

Then [11], the extra regularity is transformed in estimates for the eigenvalues. One gain of regularity yields a power n^{-1} for the decreasing rate of the eigenvalues λ_n of $\mathcal{T}^*\mathcal{T}$. Since the gain of regularity is 2 (with Lemma 2.4) plus $\varepsilon > 0$ (with Lemma 2.5), one gets

$$\sum_{n\geq 1} \sqrt{\lambda_n} \leq C \frac{1}{\sqrt{n^2+\varepsilon}} < \infty$$

which shows the trace-class property. Ultimately, it yields the existence of waves operators which are key objects in abstract scattering theory.

Theorem 2.6 ([11])**.** *Assume the electric potential is smooth $\varphi_0 \in W^{4+\varepsilon,\infty}(\mathbb{T})$. Then the wave operators $\mathcal{W}_\pm(\mathcal{H},\mathcal{H}_0)$ exist and are complete. In particular one has the orthogonal decompositions between spaces associated to absolute continuous, singular continuous and discrete parts of the spectrum*

$$L^2(\mathbb{T}\times\mathbb{R}) = \mathcal{X}_0^{\rm ac} \oplus \mathcal{X}_0^{\rm sc} \oplus \mathcal{X}_0^{\rm pp} = \mathcal{X}^{\rm ac} \oplus \mathcal{X}^{\rm sc} \oplus \mathcal{X}^{\rm pp} \tag{18}$$

and there exists two complete wave operators $\mathcal{W}_\pm$ isometric on $\mathcal{X}_0^{\rm ac}$. Mutatis mutandis (that is up to the bijection between GL and $\mathcal{X}$), a similar result holds for the initial problem.

This result describes the strong constraints on the absolutely continuous part of the spectrum of the operators. Essentially, they are the same. More information on the singular continuous part and on the pure point spectrum is needed to describe completely the decomposition (18). As usual for problems which directly come from classical physics, one can expect that the singular continuous part is empty, that is $\mathcal{X}_0^{\rm sc} = \mathcal{X}^{\rm sc} = \emptyset$. Standard explicit representation prove it is indeed the case for some reasonable electric potential φ_0. The pure point spectrum which corresponds to classical eigenvectors of the operators can be studied by different means.

There are two cases where one knows exactly the absolutely continuous part of the spectrum of the different operators. The first one is the homogeneous case E_0, and the spectral decomposition of H_0 and H is explicitly calculated in [11]. The second case is in the recent work [10] where the explicit and complete calculation of the spectral decomposition is performed for a non homogeneous one-bump (in space) background electric field: a phase portrait method gives the spectral decomposition of H_0 and the absolutely continuous part of the spectrum; then a method based on a Lipmann-Schwinger equation gives the the spectral decomposition of H.

3 Magnetized equations

This part is based on [8] with F. Charles, A. Rege and R. Weder where we re-examine the magnetized equations (5) for $\varphi_0 = 0$ and $\omega_c \neq 0$ at the light of abstract scattering theory. One can note that there is a renewal on the mathematical analysis of such magnetized systems, we refer to [4] where the classical method Fourier-Laplace of Landau is justified. The Fourier-Laplace method was used by Bernstein in his seminal work [5] for magnetized equations.

The examination of the magnetized equations

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 \sqrt{f_0} + \omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right) u = 0, \\ \partial_t F = 1^* \int_{\mathbb{R}} v u \sqrt{f_0} dv. \end{cases} \tag{19}$$

at the light of abstract spectral theory and scattering theory yields the question of deciding what will be H_0 and what will be K, in other words what will be the reference Hamiltonian and what will be the perturbation. If one follows the classical physical intuition [5], one sees ω_s as a small parameter and expect that the operator $\omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right)$ induces some controlled modifications of the situation ω_c. It appears it is not the case, and this situation has been described in some works as the Landau-Bernstein paradox [26]: indeed the classical linear Landau damping for ω_c (the electric field tends to zero) is replaced by a cyclotron behavior with period $\frac{2\pi}{\omega_c}$ and the electric field does not tend to zero in time. For scattering theory it is the structure of operators which matters the most. It appears that H_0 corresponds to the full transport part $v_1 \partial_x u + \omega_c \left(v_2 \partial_{v_1} - v_1 \partial_{v_2} \right) u$, while K is the coupling of the particles with the electric field which corresponds to $F v_1 \sqrt{f_0}$ and $1^* \int_{\mathbb{R}} v u \sqrt{f_0} dv$.

It is easy to calculate explicitly the spectrum of H_0 which is made only of eigenvectors for eigenvalues $\lambda \in \omega_c \mathbb{Z}$. That is the countable eigenvalues are all multiple of the cyclotron frequency. Since the dimension of the eigenspaces is infinite, it corresponds to an essential spectrum equal $H_0^{\rm ess} = \omega_c \mathbb{Z}$. Then two approaches are possible. In the first one, one construct the eigenvectors of H from the ones of H_0 by a perturbation approach based on a secular equation. In the second one, one uses operator theoretic results to construct the spectrum of H.

Here we detail the operator theoretic approach, announcing results in [8]. Since the coefficients of the operators are constant in space, a Fourier decomposition in the torus simplifies the proof. Then it is sufficient to realize that the perturbation by K is trace-class (Fourier mode per Fourier mode). It is compact (Fourier mode per Fourier mode). By the Weyl Theorem the essential spectrum is preserved so $H^{\rm ess} = \omega_c \mathbb{Z}$, which shows it is atomic. However the continuous part of the spectrum is not atomic, so it is empty. In particular the absolutely continuous part of the spectrum is empty and the singular continuous part of the spectrum is empty. It yields that (19) does not have absolutely continuous spectrum, so it is the impossibility of a linear Landau damping phenomenon for this system. This phenomenon was sometimes the Berstein-Landau paradox in plasma physics [26].

4 Extension in nD

Previously, the dimensionality of the space variable was $d = 1$ and this was important for some algebraic manipulations. A natural question is then to extend to higher space dimensions. Hereafter we explain that such an extension should be doable in the non magnetized case ω_c. Considering the method in Section 2 for non homogeneous profiles, the two key steps are: a) the construction of a reduced equation like (15) adapted to the trace-class analysis; and b) the study of the various spectrum of the various operators ($\mathcal{T}$, $\mathcal{T}_1$, ...) by means of compactness by integration following [13]. The explanation below concern only the beginning of the first step a).

One starts from (1) in the non magnetized case $\mathbf{b}_0 = 0$ and rewrite it with a non stationary Ampère law instead of the stationary Poisson law. One writes

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{v}} f = 0, & t > 0, \\ \partial_t \mathbf{E} = \nabla \Delta^{-1} \nabla \cdot \int f \mathbf{v} dv, & t > 0, \end{cases} \tag{20}$$

and the Poisson equations are satisfied at initial time

$$\begin{cases} -\Delta \varphi = \rho_{\text{ions}}(\mathbf{x}) - \int f dv, & t = 0 \\ \mathbf{E}(t,\mathbf{x}) = -\nabla_{\mathbf{x}} \varphi(t,\mathbf{x}), & t = 0. \end{cases} \tag{21}$$

The non local operator $\nabla \Delta^{-1} \nabla \cdot$ is the generalization of 1^* defined in Section 2. The initial data verifies $\int \int f(0,\mathbf{x},\mathbf{v}) dx dv = \int \rho_{\text{ions}}(\mathbf{x}) dx$.

In this part, we focus only on the algebraic structure of the equations and disregard any analytical considerations. The evolutionary equation for the electric field can also be deduced formerly from the evolutionary equation

$$\partial_t \varphi = \Delta^{-1} \partial_t \int f dv = -\Delta^{-1} \nabla \cdot \int f \mathbf{v} dv \tag{22}$$

for the electric potential. Let us linearize around a non homogeneous Boltzmaniann equilibrium

$$\begin{cases} f_0(\mathbf{x},\mathbf{v}) = e^{-|\mathbf{v}|^2/2 + \varphi_0(\mathbf{x})}, \\ f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v}) + \varepsilon \sqrt{f_0(\mathbf{x},\mathbf{v})} u(t,\mathbf{x},\mathbf{v}) + O(\varepsilon^2), \\ E(t,\mathbf{x}) = E_0(\mathbf{x}) + \varepsilon F(t,\mathbf{x}) + O(\varepsilon^2). \end{cases} \tag{23}$$

The equations at order 0 yield $E_0(\mathbf{x}) = -\nabla \varphi_0(\mathbf{x})$ with the non linear Poisson

$$-\Delta \varphi_0(\mathbf{x}) + (2\pi)^{d/2} e^{\varphi_0(\mathbf{x})} = \rho_{\text{ions}}(\mathbf{x}).$$

The equations at order 1 write

$$\begin{cases} \partial_t u + \mathbf{v} \cdot \nabla_{\mathbf{x}} u - \mathbf{E}_0 \cdot \nabla_{\mathbf{v}} u + \mathbf{F} \cdot \mathbf{v} \sqrt{f_0} = 0, & t > 0, \\ \partial_t \mathbf{F} = \nabla \Delta^{-1} \nabla \cdot \int u \mathbf{v} \sqrt{f_0} dv, & t > 0. \end{cases} \tag{24}$$

The initial data is written as $\mathbf{F}(0) = -\nabla\varphi(0)$ which is a gradient at initial time, so $\mathbf{F} = -\nabla\varphi(t)$ is a gradient for $t \geq 0$. Since (21) is a linear relation, its structure is preserved by the linearization procedure so one can write

$$-\Delta\varphi = \int \sqrt{f_0} u dv \quad \text{and} \quad \partial_t \varphi = -\Delta^{-1}\nabla \cdot \int \sqrt{f_0} u \mathbf{v} dv.$$

It is tempting to identify H_0 as the transport part, and the perturbation K as the two remaining parts. However one is faced with the same kind of difficulties [11] as in space dimension $d = 1$, so it is better to try to define a new variable w to simplify the application of the scattering methodology. Unfortunately the transformation (12-13) seems to be restricted to space dimension $d = 1$.

After some tries, another possibility seems to be the following. It corresponds to define a complex valued new kinetic function $w = u + i\sqrt{f_0}L\varphi$ where L is a space operator to construct in such a way that the isometry relation (14) is generalized in higher dimension. In what follows we consider

$$L = (-\Delta)^{-1/2}\left[(-\Delta)^{-1/2}\alpha_0(-\Delta)^{-1/2}\right]^{-1/2}(-\Delta)^{1/2}.$$

where the weight is $\alpha_0(\mathbf{x}) = \int_{\mathbb{R}^d} f_0(\mathbf{x}, \mathbf{v})dv = (2\pi)^{d/2} n_0(\mathbf{x}) > 0$.

Lemma 4.1. *One has formally* $\|w\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} + \|F\|^2_{L^2(\mathbb{R}^d)}$.

Proof. One has

$$\|w\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} + 2\mathrm{Re}\left(u\sqrt{f_0}, iL\varphi\right) + \|\sqrt{f_0}L\varphi\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Note $M = \left[(-\Delta)^{-1/2}\alpha_0(-\Delta)^{-1/2}\right]^{-1/2}$ which is formally symmetric positive and $\psi = (-\Delta)^{1/2}\varphi$. With this notation, one has $L\varphi = (-\Delta)^{-1/2}M\psi$. So one can write

$$\left(u\sqrt{f_0}, iL\varphi\right)_{xv} = -i\left((-\Delta)\varphi, (-\Delta)^{-1/2}M\psi\right)_x = -i\left(\psi, M\psi\right)_x \in i\mathbb{R},$$

so its real part vanishes. The other term is

$$\begin{aligned}\|\sqrt{f_0}L\varphi\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} &= (f_0 L\varphi, L\varphi)_{xv} \\ &= \left((-\Delta)^{-1/2}\alpha_0(-\Delta)^{-1/2}M\psi, M\psi\right)_x \\ &= (\psi,\psi)_x = (-\Delta\varphi,\varphi)_x = \|F\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

□

Since the square of norm of w is the same as the quantity which is preserved by the dynamical system, it is possible that an autonomous equation exists for w, and that this equation has the form of an Hamiltonian system. So the algebraic step a) should be doable, even if the construction can be seen as artificial. It remains to complement with step b) about analytic and functional considerations. These issues are left for further researches.

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