REGULAR FIELDS : NORMIC CRITERIA

IN P-EXTENSIONS

R.I. BERGER

G. GRAS

REGULAR FIELDS: NORMIC CRITERIA IN P-EXTENSIONS
by
R.I. BERGER *and* G. GRAS

ABSTRACT: We explain the connection between the different approaches taken in [Be] and [Gr] in examining the tame kernel $K_2(O_k)$, where O_k is the ring of integers of a number field k. In particular, we generalize the normic criterion in [Be, 2.4] by using Jaulent's work which gives links between K_2 and S-genera.

1. INTRODUCTION

From [Ta] we have a p^n-rank formula for $K_2(O_k)$, if k contains the p^n-th roots of unity. There are many publications that investigate the p^n-rank, in particular for $p = 2$ see for example [Ko]. It follows from [Ta] that the 2-rank of $K_2(O_k)$ is at least r_1, the number of real infinite places of k. Therefore the 2-primary subgroup of $K_2(O_k)$ is smallest if it is elementary abelian of that rank: $2\text{-}prim K_2(O_k) \cong (\mathbb{Z}/2)^{r_1}$. The totally real number fields with this property are examined in [Be], where they are denoted as having property (*). The methods used in [Be] are based on the work of Conner and Hurrelbrink, using the exact hexagon defined in [C-H$_2$] to examine the interplay between units, class numbers and ramification. The hexagon involves galois cohomology groups; parts of it were already introduced in [C-H$_1$], where one result is: "*Let k be an arbitrary number field. The 2-Sylow subgroup of $K_2(O_k)$ is elementary abelian of rank r_1 iff 2 does not split in $k/\mathbb{Q}$, the S-class number of k is odd, and k contains S-units with independent signs, where S is the set of infinite and dyadic places of k.*"

A characterization of these number fields has been obtained in [Gr], even for arbitrary primes, by examining p-ramification. In [Gr] the tame kernel is defined in a new sense (the narrow one), which is more convenient for the examination of the 2-part. Using the notation suggested in [Ke], we denote it by $K_2^+(O_k)$. It is related to the classical tame kernel by: $K_2(O_k)/K_2^+(O_k) \cong (\mathbb{Z}/2)^{r_1}$. The connection between p-ramification and K_2 is given by [Gr, Thm 1]: "*Let k be a number field and p a prime. If the maximal real subfield of $\mathbb{Q}(\mu_p)$ is contained in k, then $dim_p K_2^+(O_k) = dim_p T + \delta$; where δ is the defect of Leopoldt conjecture at p in k, and T is the finite p-group which, in the maximal abelian p-extension unramified outside p, fixes the composite of all $\mathbb{Z}_p$-extensions of k.*" The results in [Gr] assume the validity of the Leopoldt conjecture, but this restriction is lifted in [G-J]. There, the notion of p-regular fields is introduced and the Leopoldt conjecture is verified for certain galois p-extensions of regular fields.

Definition: A number field k is *p-regular* if the p-primary subgroup of $K_2^+(O_k)$ is trivial.

In [Gr, Cor to Thm 1] a criterion for the triviality of the p-part of $K_2^+(O_k)$ in terms of ideal class groups is derived: "*Let k be a number field that contains the p-th roots of unity. Then k is p-regular iff p does not split in $k/\mathbb{Q}$ and the p-primary subgroup of the narrow S-class group of k is trivial, where S is the set of places of k lying over p.*" More equivalent criteria are given in [G-J, 2.1].

For $p = 2$ one obtains the above criterion from [C-H$_1$], but to see this we need to clarify a few definitions. The ideal class groups used in [Gr] are the ones that are classically referred to as narrow ideal class groups. That is, they are the quotient of the group of fractional ideals of k by the subgroup of principal ideals with totally positive generators. Similarly, the group of units $U_k = \{x \in k^\times \mid v_P(x) = 0 \text{ for all finite places } P\}$ in [Gr] includes the corresponding restriction at infinite places. The resulting group of totally positive units will be referred to as U_k^+. Let S be a set of places of k. The S-ideal class group $Cl_{k,S}$ is the quotient of the ideal class group Cl_k by $Cl_k(S)$, the subgroup generated by classes of places in S. The group of S-units is $U_{k,S} = \{x \in k^\times \mid v_P(x) = 0 \text{ for all places } P \notin S\}$. Again, excluding infinite places from S results in the narrow S-ideal class group $Cl_{k,S}^+$, and the totally positive S-units $U_{k,S}^+$. The following exact sequence relates $Cl_{k,S}^+$ to $Cl_{k,S}$ (note that $k^\times / \left((k^\times)^+ \cdot U_{k,S}\right) = 1$ means that k contains S-units with independent signs):

$$1 \to k^\times / \left((k^\times)^+ \cdot U_{k,S}\right) \to Cl_k^+ / Cl_k^+(S) \to Cl_k / Cl_k(S) \to 1$$

2. NORMIC CRITERION

Since property (*) in [Be] is only a special case of 2-regularity, some of the explicit results in [Be] can be obtained by interpreting the results from [Gr] in the case $p = 2$. Some, however, do not follow in a straightforward manner. For example, the normic condition in the lift theorem [Be, 2.4] can be derived using S-genera theory. It follows as a special case of the following theorem. We will assume μ_p to be contained in the base field, otherwise the notion of p-regular would have to be changed into p-rational; see [G-J], [J-N] and [M-N].

Theorem: *Let $p \geq 2$ be a prime, and let F be a number field that contains μ_p. Let E/F be a Galois p-extension, and let S be the set of places lying over p, in F or E respectively. Then, E is p-regular if and only if: F is p-regular, p does not split in E, and $\left(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\right) = \prod_{P \in T} e_P^{ab}(E/F)$; where the product is taken over T the set of all tame ramified primes P in E/F, and $e_P^{ab}(E/F)$ indicates the ramification index of P in the maximal abelian sub-extension in E_P/F_P.*

Proof: By the above characterization of p-regularity, we need to show that under the assumtion that p does not split in E we have: p-prim $Cl_{E,S}^+ = 1 \Leftrightarrow p$-prim $Cl_{F,S}^+ = 1$ and $\left(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\right) = \prod_{P \in T} e_P^{ab}(E/F)$. If we let $_p Cl$ denote the p-primary subgroup of Cl, we have from [Ja, Thm. III.2.12]:

$$|_p Cl_{E,S}^{+G}| = |_p Cl_{F,S}^+| \cdot \frac{\prod_{P \in S} d_P^{ab}}{[E^{ab} : F]} \cdot \frac{\prod_{P \in T} e_P^{ab}(E/F)}{\left(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\right)} \cdot |\kappa_{E/F}| \ ,$$

where $d_P^{ab}(E/F)$ is the local degree and $e_P^{ab}(E/F)$ the ramification index in the maximal abelian sub-extension of E_P/F_P, and $\kappa_{E/F}$ is the knot number. If p does not split in E,

$\kappa = 1$ by [G-J, 2.5], and the above simplifies to:

$$|_p Cl_{E,S}^{+G}| = |_p Cl_{F,S}^{+}| \cdot 1 \cdot \frac{\prod_{P \in T} e_P^{ab}(E/F)}{\left(U_{F,S}^{+} : U_{F,S}^{+} \cap N_{E/F}(E^*) \right)} \cdot 1$$

$\square$

For computational purposes, note that the p-rank of $U_{F,S}^{+}/(U_{F,S}^{+})^p$ is $r_2(F) + 1$, hence $\left(U_{F,S}^{+} : U_{F,S}^{+} \cap N_{E/F}(E^*) \right) \leq r_2(F) + 1$. Furthermore, the concept of primitive ramification may yield a more practically computable criterion; for the definitions we refer to [G-J, 1.1] and [G-J, 1.3]. From this point of view we obtain by [G-J, 2.3]:

Corollary: *Let F be a number field that contains μ_p, and let E/F be a Galois p-extension. If F is p-regular then: the set T of tame ramified primes in E/F is p-primitive if and only if p does not split in E/F and $\left(U_{F,S}^{+} : U_{F,S}^{+} \cap N_{E/F}(E^*) \right) = \prod_{P \in T} e_P^{ab}(E/F)$.*
$\square$

REFERENCES

[Be] R.I. Berger, Quadratic extensions of number fields with elementary abelian $K_2(O_F)$ of smallest rank, J. Number Theory **36** (1990) 284-292. Addendum in J. Number Theory **37** (1991), 122.

[C-H$_1$] P.E. Conner and J. Hurrelbrink, A comparison theorem for the 2-rank of $K_2(O)$, AMS Contemporary Math. **55** part II (1986) 411-420.

[C-H$_2$] P.E. Conner and J. Hurrelbrink, Class number parity, Pure Math. Series 8, World Scientific Publishing Co., Singapore, 1988.

[Gr] G. Gras, Remarks on K_2 of Number Fields, J. Number Theory **23** (1986) 322-335.

[G-J] G. Gras and J.-F. Jaulent, Sur les corps de nombres réguliers, Math. Z. **202** (1989) 343-365.

[Ja] J.-F. Jaulent, L'arithmétique des l-extensions (Thèse), Publ. Math. Fac. Sci. Besançon, Théor. Nombres 1984/1985, 1985/1986 (1986).

[J-N] J.-F. Jaulent and T. Nguyen Quang Do, Corps p-rationnels, corps p-reguliers et ramification restreinte, Séminaire de Théor. des Nombres, Talence 1987-88.

[Ke] F. Keune, On the Structure of the K_2 of the Ring of Integers in a Number Field, K-Theory **2** (1989), 625-645.

[Ko] M. Kolster, The structure of the 2-Sylow-subgroup of $K_2(O)$, I, Comm. Math. Helv. **61** (1986), 376-388.

[M-N] A. Movahhedi and T. Nguyen Quang Do, Sur l'arithmétique des corps de nombres p-rationnels, Séminaire de Théor. des Nombres, Paris 1987-88.

[Ta] J. Tate, Relations between K_2 and Galois cohomology, Invent. math. **36** (1976), 257-274.

R.I. Berger
Dept. of Mathematical Sciences
Memphis State University
Memphis, TN 38152
USA

G. Gras
Faculté des Sciences
Laboratoire de Mathématiques
U.A. au C.N.R.S. 741
F-25030 Besançon Cédex