ON L-ADIC GALOIS PERIODS, RELATIONS BETWEEN COEFFICIENTS OF GALOIS REPRESENTATIONS ON FUNDAMENTAL GROUPS OF A PROJECTIVE LINE MINUS A FINITE NUMBER OF POINTS

by

Zdzisław Wojtkowiak

Contents

Abstract. — The coefficients of the Drinfeld associator are known to satisfy two kind of shuffle relations. The first relations come from the formula for the multiplication of iterated integrals. The second ones come from the multiplication of multi zeta functions. Our aim is to study the analogous relations for the Ihara-Drinfeld element describing the action of the Galois group on the étale fundamental group of $\mathbb{P}_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$.

1. Introduction

The mixed Hodge structure of the fundamental group of $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ based at $\overrightarrow{01}$ is described by the element

$$a_{\overrightarrow{01}}^{\overrightarrow{10}}(X,Y) \in \mathbb{C}\{\{X,Y\}\}$$

called the Drinfeld associator. This element is a formal power series in non commuting variable X and Y given explicitly by the formula

$$a_{\vec{01}}^{\vec{10}}(X,Y) :=$$

$$1 + \sum_{w=X^{i_1}\cdot Y^{j_1}\cdot\ldots\cdot X^{i_n}\cdot Y^{j_n}} (\int_{\vec{01}}^{\vec{10}} (\frac{-dz}{z-1})^{j_n}, (\frac{-dz}{z})^{i_n}, \ldots, (\frac{-dz}{z-1})^{j_1}, (\frac{-dz}{z})^{i_1})w$$

We briefly sketch the definition of iterated integrals starting from tangential points $\vec{01}$ or $\vec{10}$. If $j_n > 0$ then we integrate from 0. If $j_n = 0$ and $i_n > 0$ then we integrate from 0 the iterated integral $\int_0^z \frac{(-logz)^{i_n}}{i_n!}(\frac{-dz}{z-1})^{j_{n-1}}, \ldots$. Similarly we define iterated integrals from $\vec{10}$ to z. Both power series whose coefficients are iterated integrals from $\vec{01}$ to z and from $\vec{10}$ to z are flat sections of the principal $\mathbb{C}\{\{X,Y\}\}^*$-fibre bundle over $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ equipped with an integrable connection given by a one form $\frac{dz}{z} \otimes X + \frac{dz}{z-1} \otimes Y$. Hence comparing these two flat sections which differ by a constant element we get iterated integrals from $\vec{01}$ to $\vec{10}$ and the power series $a_{\vec{01}}^{\vec{10}}(X,Y)$. (See [22] and [23] for more details as well as [4] and [16] for different approach).

The element $a_{\vec{01}}^{\vec{10}}(X,Y)$ satisfies $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$-cycle relations (see [4]). The coefficients of $a_{\vec{01}}^{\vec{10}}(X,Y)$ satisfy also three types of shuffle relations. The shuffle relations of type I or iterated integrals type are consequence of the Chen formula

$$(\int_a^b \omega_1, \ldots, \omega_p) \cdot (\int_a^b \omega_{p+1}, \ldots, \omega_{p+q}) = \sum_{\pi \in Sh(p,q)} \int_a^b \omega_{\pi(1)}, \ldots, \omega_{\pi(p+q)}$$

for iterated integrals (see [2]) which remains true if a or (and) b are tangential points (see [22] and [23]). The shuffle relations of type II or multi zeta type relations are generalization of the following identity

$$(\sum_{n=1}^{\infty} \frac{1}{n^2}) \cdot (\sum_{m=1}^{\infty} \frac{1}{m^3}) = \sum_{n>m=1}^{\infty} \frac{1}{m^3 \cdot n^2} + \sum_{m>n=1}^{\infty} \frac{1}{n^2 \cdot m^3} + \sum_{n=1}^{\infty} \frac{1}{n^5}.$$

The iterated integrals

$$\int_0^z \frac{-dz}{z-1}, \ldots, \frac{-dz}{z}, (\frac{-dz}{z-1})^k$$

are divergent when $z \to 1$ and $k > 0$.

The iterated integrals $\int_{\vec{01}}^{\vec{10}} \frac{-dz}{z-1}, \dots, \frac{-dz}{z}, (\frac{-dz}{z-1})^k$ regularized them in such a way that the Chen formula still holds (see [22], [23] and [16]) but the multi zeta type relations do not hold.
These divergent iterated integrals one can also regularized in such a way that the multi zeta type relations hold. The shuffle relations of type III compare these two regularizations.

The absolute Galois group $G_{\mathbb{Q}}$ acts on the étale fundamental group $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01})$, hence we get a Galois representation

$$\varphi : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01})),$$

which was studied by Ihara (see [10]), Deligne (see [3]), Grothendieck (see [8]) and other persons. The representation φ is completely described by the cocycle

$$G_{\mathbb{Q}} \ni \sigma \longrightarrow \mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p) \in \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01}),$$

where p is the canonical path from $\vec{01}$ to $\vec{10}$, the open interval $(0,1)$.
We embed the pro-l completion of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01})$ into the $\mathbb{Q}_l$-algebra of non-commutative formal power series $\mathbb{Q}_l\{\{X,Y\}\}$ in two non-commuting variables X and Y sending x onto e^X and y onto e^Y (see [18]). Hence we get a Galois representation

$$\varphi_l : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\mathbb{Q}_l\{\{X,Y\}\}).$$

Let $\Lambda_p(X,Y)(\sigma) \in \mathbb{Q}_l\{\{X,Y\}\}$ be the image of $\mathfrak{f}_p(\sigma)$ in $\mathbb{Q}_l\{\{X,Y\}\}$. The representation φ_l is completely described by the cocycle

$$G_{\mathbb{Q}} \ni \sigma \longrightarrow \Lambda_p(X,Y)(\sigma) \in \mathbb{Q}_l\{\{X,Y\}\}.$$

The element $\mathfrak{f}_p(\sigma)$ satisfies the $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$-cycle relations (see [11]). Hence after embedding of the pro-l completion of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01})$ into $\mathbb{Q}_l\{\{X,Y\}\}$ we get some kind of $\mathbb{Z}/2$ and $\mathbb{Z}/3$-cycle relations satisfied by the element $\Lambda_p(X,Y)(\sigma)$ (see section 3 of this note).
The cocycles $\mathfrak{f}_p$ and Λ_p are respectively pro-finite and l-adic analogs of the Drinfeld associator $a_{\vec{01}}^{\vec{10}}(X,Y)$. Hence it is a natural question to ask if the elements $\mathfrak{f}_p(\sigma)$ and $\Lambda_p(\sigma)$ satisfy shuffle relations of type I, II and III.

In this note we shall show that the coefficients of the formal power series $\Lambda_p(X,Y)(\sigma)$ satisfy trivially shuffle relations of type I because of the very definition of the embedding of the pro-l completion of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \vec{01})$ into the $\mathbb{Q}_l$-algebra of non-commutative formal power series $\mathbb{Q}_l\{\{X,Y\}\}$.

The formal power series $\Lambda_p(X,Y)(\sigma)$ satisfies $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$-cycle relations, as it does the Drinfeld associator $a_{\overrightarrow{01}}^{\overrightarrow{10}}(X,Y)$. However there are some differences. For example in the $\mathbb{Z}/3$-cycle relation $Z=-X-Y$ is replaced by $-(X \bigcirc Y) := -\log(e^X \cdot e^Y)$. Unfortunately we do not know how to formulate shuffle relations of type II and III in l-adic case. However when one passes to associated graded Lie algebras the $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$-cycle relations satisfied by the element $\Lambda_p(X,Y)(\sigma)$ become more familier and then we can formulate analogs of shuffle relations of type II and III for this element. Though we do not know how to prove them.

2. Shuffle relations of type I (of iterated integrals type)

Let K be a number field. Let $a_1,\dots,a_n$ belong to K and let

$$V := \mathbb{P}^1_K \setminus \{a_1,\dots,a_n,\infty\}.$$

Let v be a K-point of V or a tangential point defined over K. Let $x_1,\dots,x_n$ be geometric generators of $\pi_1(V_{\bar{K}};v)$ – loops around $a_1,\dots,a_n$ respectively (see [18]).

Let $\mathbb{X} := \{X_1,\dots,X_n\}$, let $\mathbb{Q}_l\{\mathbb{X}\}$ be a $\mathbb{Q}_l$-algebra of polynomials in non-commuting variables $X_1,\dots,X_n$ and let $\mathbb{Q}_l\{\{\mathbb{X}\}\}$ be a $\mathbb{Q}_l$-algebra of formal power series in non-commuting variables $X_1,\dots,X_n$. Let $Lie(\mathbb{X})$ be a Lie algebra of Lie polynomials in $\mathbb{Q}_l\{\mathbb{X}\}$ and let $L(\mathbb{X})$ be a Lie algebra of Lie formal power series in $\mathbb{Q}_l\{\{\mathbb{X}\}\}$.

If A and B belong to a Lie algebra L then we set $[A,B^{(0)}] := A$ and $[A,B^{(n+1)}] := [[A,B^{(n)}],B]$ for $n \geq 0$.

Let

$$k : \pi_1(V_{\bar{K}};v)_l \longrightarrow \mathbb{Q}_l\{\{\mathbb{X}\}\}$$

be a continous multiplicative embedding of the pro-l completion of $\pi_1(V_{\bar{K}};v)$ into $\mathbb{Q}_l\{\{\mathbb{X}\}\}$ given by

$$k(x_i) = e^{X_i}$$

for $i=1,\dots,n$.

Let z be also a K-point or a tangential point defined over K. Let γ be an l-adic path from v to z and let $\sigma \in G_K$. We recall the definitions from [18] by setting

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(V_{\bar{K}};v)_l$$

and

$$\Lambda_\gamma(\sigma) := k(\mathfrak{f}_\gamma(\sigma)) \in \mathbb{Q}_l\{\{\mathbb{X}\}\}.$$

Let $\mathcal{W}(\mathbb{X})$ be a set of all monomials in non-commuting variables $X_1,\dots,X_n$. We include 1 in $\mathcal{W}(\mathbb{X})$. Then $\mathcal{W}(\mathbb{X})$ is a base of a $\mathbb{Q}_l$-vector space $\mathbb{Q}_l\{\mathbb{X}\}$. Let

$$\mathcal{W}(\mathbb{X})^* := \{w^* \mid w \in \mathcal{W}(\mathbb{X})\}$$

be the dual base, i.e., $w^* \in \mathrm{Hom}(\mathbb{Q}_l\{\mathbb{X}\}, \mathbb{Q}_l)$ and $w^*(w') = \delta^w_{w'}$ for any $w' \in \mathcal{W}(\mathbb{X})$. We extend w^* to a linear form on $\mathbb{Q}_l\{\{\mathbb{X}\}\}$ setting $w^*(\Lambda)$ to be equal a coefficient of Λ at the monomial w. Then we can write

$$\Lambda = \sum_{w \in \mathcal{W}(\mathbb{X})} w^*(\Lambda) \cdot w$$

for any $\Lambda \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$.
If $w \in \mathcal{W}(\mathbb{X})$ we denote by $|w|$ the degree of the monomial w. We define the group of (p, q)-shuffle permutations of the set $\underline{p+q} := \{1, 2, \dots, p+q\}$ by

$$Sh(p, q) := \{\pi \in S_{p+q} \mid \forall a, b \in \underline{p+q}, a < b \leq p \ \text{ or } \ p < a < b \ \text{ implies}$$
$$\pi^{-1}(a) < \pi^{-1}(b)\}.$$

If $w = X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_p}$, $w_1 = X_{i_{p+1}} \cdot X_{i_{p+2}} \cdot \ldots \cdot X_{i_{p+q}}$ and $\pi \in Sh(p, q)$ then we set

$$\pi(w, w_1) := X_{i_{\pi(1)}} \cdot X_{i_{\pi(2)}} \cdot \ldots \cdot X_{i_{\pi(p+q)}}.$$

Definition 2.1. — We say that coefficients of a formal power series $\Lambda \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$ satisfy shuffle relations of type I if

$$w^*(\Lambda) \cdot w_1^*(\Lambda) = \sum_{\pi \in Sh(|w|, |w_1|)} \pi(w, w_1)^*(\Lambda)$$

for any $w, w_1 \in \mathcal{W}(\mathbb{X})$.

Let $\mathbb{Q}_l\{\{\mathbb{X}\}\}^\diamond$ be a $\mathbb{Q}_l$-vector subspace of $\mathrm{Hom}_{\mathbb{Q}_l}(\mathbb{Q}_l\{\{\mathbb{X}\}\}, \mathbb{Q}_l)$ generated by the set $\mathcal{W}(\mathbb{X})^*$. Then the formula

$$w^* \odot w_1^* := \sum_{\pi \in Sh(|w|, |w_1|)} \pi(w, w_1)^*$$

defines a commutative product on $\mathbb{Q}_l\{\{\mathbb{X}\}\}^\diamond$ and the obtained $\mathbb{Q}_l$-algebra we denote by $\mathbb{Q}_l\{\{\mathbb{X}\}\}^\diamond_\odot$.
If $\Lambda \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$ then we define a linear map

$$ev_\Lambda : \mathbb{Q}_l\{\{\mathbb{X}\}\}^\diamond \to \mathbb{Q}_l$$

by setting $ev_\Lambda(w^*) := w^*(\Lambda)$.
Let us define a continous homomorphism of $\mathbb{Q}_l$-algebras

$$\Delta : \mathbb{Q}_l\{\{\mathbb{X}\}\} \to \mathbb{Q}_l\{\{\mathbb{X}\}\} \hat{\otimes} \mathbb{Q}_l\{\{\mathbb{X}\}\}$$

by setting

$$\Delta(X_i) := X_i \otimes 1 + 1 \otimes X_i$$

on topological generators $X_1, X_2, \dots, X_n, \dots$ of $\mathbb{Q}_l\{\{\mathbb{X}\}\}$. We recall the classical result of Ree.

Theorem 2.2. — *(See [17]) Let* $\Lambda \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$ *be such that* $1^*(\Lambda) = 1$. *The following conditions are equivalent:*

i) *the coefficients of* Λ *satisfy shuffle relations of type I;*
ii) $log\Lambda$ *is a Lie formal power series in* $\mathbb{Q}_l\{\{\mathbb{X}\}\}$*;*
iii) $\Delta(\log\Lambda) = \log\Lambda \otimes 1 + 1 \otimes \log\Lambda$*;*
iv) $\Delta(\Lambda) = \Lambda \otimes \Lambda$*;*
v) *the map* $ev_\Lambda : \mathbb{Q}_l\{\{\mathbb{X}\}\}^{\diamond}_{\odot} \to \mathbb{Q}_l$ *is a homomorphism of* $\mathbb{Q}_l$*-algebras.*

Proposition 2.3. — *The coefficients of the formal power series* $\Lambda_\gamma(\sigma) \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$ *satisfy shuffle relations of type I.*

Proof. It follows from the Baker-Campbell-Hausdorff formula that the image of the embedding k is contained in the subgroup $\exp(L(\mathbb{X}))$ of the multiplicative group of $\mathbb{Q}_l\{\{\mathbb{X}\}\}$. Hence $\log\Lambda_\gamma(\sigma) \in L(\mathbb{X})$, i.e. $\log\Lambda_\gamma(\sigma)$ is a Lie formal power series. Now the proposition follows from Theorem 2.2. □

Iterated integrals satisfy the formula

2.3.1. $$\int_{\gamma^{-1}} \omega_1, \dots, \omega_n = (-1)^n \int_\gamma \omega_n, \dots, \omega_1$$

(see [2]). Our next aim is to show the analogue of this formula for the power series $\Lambda_\gamma(\sigma) \in \mathbb{Q}_l\{\{\mathbb{X}\}\}$ (see [19], pages 118 and 119, where this problem was raised).

The path γ^{-1} is a path from z to v, hence $\mathfrak{f}_{\gamma^{-1}}(\sigma) \in \pi_1(V_{\bar{K}}; z)$.

Lemma 2.4. — *(see [18] Lemma 1.0.6.)*
We have

$$(\gamma^{-1} \cdot \mathfrak{f}_{\gamma^{-1}}(\sigma) \cdot \gamma) \cdot \mathfrak{f}_\gamma(\sigma) = 1.$$

The elements $x'_i := \gamma \cdot x_i \cdot \gamma^{-1}$ for $i = 1, \dots, n$ are geometrical generators of $\pi_1(V_{\bar{K}}; z)$. Let

$$k' : \pi_1(V_{\bar{K}}; z)_l \longrightarrow \mathbb{Q}_l\{\{\mathbb{X}\}\}$$

be a continous multiplicative embedding given by $k'(x'_i) = e^{X_i}$ for $i = 1, \dots, n$. Let

$$\Lambda_{\gamma^{-1}}(\sigma) := k'(\mathfrak{f}_{\gamma^{-1}}(\sigma)).$$

Lemma 2.5. — *We have*

$$\Lambda_{\gamma^{-1}}(\sigma) \cdot \Lambda_\gamma(\sigma) = 1.$$

Proof. Observe that $k(x_i) = k'(x'_i)$ for $i = 1, \dots, n$. Hence $k(\gamma^{-1} \cdot \mathfrak{f}_{\gamma^{-1}}(\sigma) \cdot \gamma) = k'(\mathfrak{f}_{\gamma^{-1}}(\sigma))$. Therefore it follows from Lemma 2.4 that $\Lambda_{\gamma^{-1}}(\sigma) \cdot \Lambda_\gamma(\sigma) = 1$. □

Let $t : \mathbb{Q}_l\{\{\mathbb{X}\}\} \to \mathbb{Q}_l\{\{\mathbb{X}\}\}$ be a continous linear mapping given by

$$t(X_{i_1} \cdot \ldots \cdot X_{i_m}) = X_{i_m} \cdot \ldots \cdot X_{i_1}$$

on elements of $\mathcal{W}(\mathbb{X})$. Then t is an anti-automorphism of $\mathbb{Q}_l$-algebras, i.e. $t(a \cdot b) = t(b) \cdot t(a)$.

Proposition 2.6. — *Let $w \in \mathcal{W}(\mathbb{X})$. Then we have*

$$w^*(\Lambda_{\gamma^{-1}}(\sigma)) = (-1)^{|w|}(t(w))^*(\Lambda_\gamma(\sigma)).$$

Proof. It follows from Lemma 2.5 that $\Lambda_{\gamma^{-1}}(\sigma) = (\Lambda_\gamma(\sigma))^{-1}$. The coefficients of the power series $\Lambda_\gamma(\sigma)$ satisfy shuffle relations of type I by Proposition 2.3. Hence it follows from [17] Theorem 2.6 that

$$w^*(\Lambda_{\gamma^{-1}}(\sigma)) = w^*(\Lambda_\gamma(\sigma)^{-1}) = (-1)^{|w|}(t(w))^*(\Lambda_\gamma(\sigma)).$$

□

Remark 2.7. — If $\Lambda_\gamma(z, v)$ is a flat section of the canonical pro-nilpotent connection on $V(\mathbb{C})$ along a path γ from v to z then the formula 2.3.1 can be written in the form

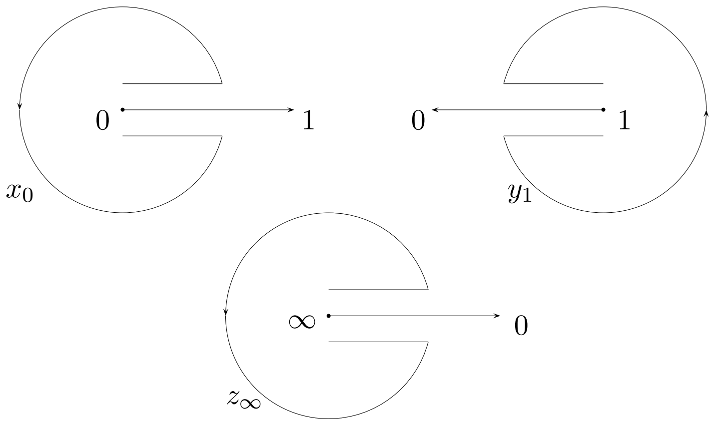
$$w^*(\Lambda_{\gamma^{-1}}(v, z)) = (-1)^{|w|}(t(w))^*(\Lambda_\gamma(z, v)).$$

3. l-adic iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ evaluated at 1

The coefficient of the Drinfeld associator $a_{\overrightarrow{01}}^{\overrightarrow{10}}(X, Y)$ at $X^{n-1}Y$ is equal $\int_0^1 \frac{-dz}{z-1}, \frac{-dz}{z}, \dots, \frac{-dz}{z} = (-1)^{n-1} \int_0^1 -\log(1-z)\frac{dz}{z}, \dots, \frac{dz}{z} = (-1)^{n-1} Li_n(1) = (-1)^{n-1}\zeta(n)$. It was shown by Euler that $\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2 \cdot (2k)!} b_{2k} = -\frac{(2\pi i)^{2k}}{2 \cdot (2k)!} b_{2k}$, where $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, ... are Bernoulli numbers. Using $\mathbb{Z}/2$ and $\mathbb{Z}/3$ symmetries of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the Baker-Campbell-Hausdorff formula one can reprove the Euler result (see [3]).

We shall prove here the analogous result in the l-adic setting.

Let $V := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Let $g : V \to V$ and $h : V \to V$ be given by $g(\mathfrak{z}) = 1 - \mathfrak{z}$ and $h(\mathfrak{z}) = \frac{\mathfrak{z}}{\mathfrak{z}-1}$. Let p be the canonical path from $\overrightarrow{01}$ to $\overrightarrow{10}$ – the interval $(0, 1)$. Let x_0, y_1 and z_∞ be loops around 0, 1 and ∞ based at $\overrightarrow{01}$, $\overrightarrow{10}$ and $\overrightarrow{\infty 0}$ respectively (see Picture 1).

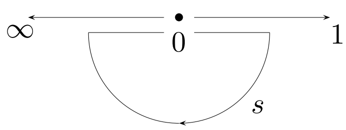


Picture 1

Let us set

$$x := x_0 \quad y := p^{-1} \cdot y_1 \cdot p \ \text{ and } \ z := s^{-1} \cdot h(p)^{-1} \cdot z_\infty \cdot h(p) \cdot s,$$

where s is a path from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$ as on Picture 2.

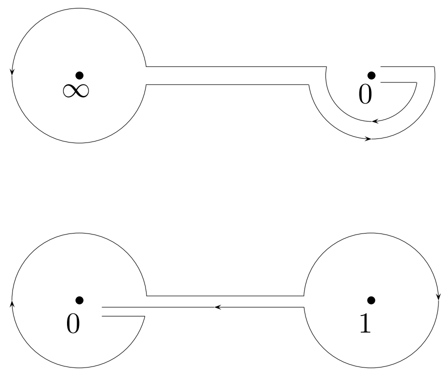


Picture 2

Observe that $x \cdot y \cdot z = 1$ in $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$, hence

3.1. $$s^{-1} \cdot h(p)^{-1} \cdot z_\infty \cdot h(p) \cdot s = (p^{-1} \cdot y_1 \cdot p)^{-1} \cdot x_0^{-1}$$

(see Picture 3).



Picture 3

We recall that for any $\sigma \in G_{\mathbb{Q}}$,

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p).$$

The element $\mathfrak{f}_p$ has been studied by Ihara (see [11]) and also by Nakamura and Schneps (see [14]). Observe that $g(p) = p^{-1}$. Hence we get $g_*(\mathfrak{f}_p) = \mathfrak{f}_{g(p)} = \mathfrak{f}_{p^{-1}} = p \cdot \mathfrak{f}_p^{-1} \cdot p^{-1}$, i.e. $p^{-1} \cdot (g_*(\mathfrak{f}_p)) \cdot p = \mathfrak{f}_p^{-1}$. The last equality implies

3.2. $$\mathfrak{f}_p(y,x) = \mathfrak{f}_p(x,y)^{-1}.$$

(This is of course the famous $\mathbb{Z}/2$-cycle relation of Drinfeld (see for example [11]). Observe that

$$s^{-1} \cdot h(x) \cdot s = x \text{ and } s^{-1} \cdot h(y) \cdot s = z.$$

This implies

3.3. $$s^{-1} \cdot \mathfrak{f}_{h(p)} \cdot s = s^{-1} \cdot h_*(\mathfrak{f}_p) \cdot s = \mathfrak{f}_p(x,z).$$

Let

$$\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_l^*$$

be the cyclotomic character. It follows from [18] Lemma 1.0.6 and the equality 3.3 that

3.4. $$\mathfrak{f}_{(p^{-1} \cdot y_1^{-1} \cdot p \cdot x_0^{-1})}(\sigma) = x \cdot y \cdot (\mathfrak{f}_p(x,y)(\sigma))^{-1} \cdot y^{-\chi(\sigma)} \cdot (\mathfrak{f}_p(x,y)(\sigma)) \cdot x^{-\chi(\sigma)}$$

and

3.5.
$$\mathfrak{f}_{(s^{-1} \cdot h(p)^{-1} \cdot z_\infty \cdot h(p) \cdot s)}(\sigma) = z^{-1} \cdot x^{\frac{(\chi(\sigma)-1)}{2}} \cdot (\mathfrak{f}_p(x,z)(\sigma))^{-1} \cdot z^{\chi(\sigma)} \cdot \mathfrak{f}_p(x,z)(\sigma) \cdot x^{-\frac{(\chi(\sigma)-1)}{2}}.$$

Let $k : \pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})_l \to \mathbb{Q}_l\{\{X, Y\}\}$ be a continous multiplicative embedding given by $k(x) = e^X$ and $k(y) = e^Y$. Then

$$\Lambda_p(X, Y)(\sigma) := k(\mathfrak{f}_p(x, y)(\sigma)).$$

It follows from 3.2 that

3.6. $$\Lambda_p(Y, X) = \Lambda_p(X, Y)^{-1}.$$

Let I_2 (resp. J_2) be a closed Lie ideal of $L(X, Y)$ generated by Lie brackets with 2 or more Y's (resp. X's). We recall that

$$\log\Lambda_p(X, Y) \equiv \sum_{n=2}^{\infty} l_n(1)[Y, X^{(n-1)}] \mod I_2$$

by the very definition of l-adic polylogarithms (see [19]). It follows from 3.6 that

3.7.

$$\log\Lambda_p(X, Y) \equiv l_2(1)[Y, X] + \sum_{n=2}^{\infty} l_n(1)[Y, X^{(n-1)}] + \sum_{n=3}^{\infty} -l_n(1)[X, Y^{(n-1)}] \mod I_2 \cap J_2.$$

Let us set

$$X \bigcirc Y := \log(e^X \cdot e^Y).$$

The right hand sides of 3.4 and 3.5 are equal by 3.1. Hence applying k and then taking logarithm we get the equality

3.8. $$(-\log\Lambda_p(X, Y)) \bigcirc (-\chi \cdot Y) \bigcirc (\log\Lambda_p(X, Y)) \bigcirc (-\chi \cdot X) =$$

$$(\frac{\chi - 1}{2} X) \bigcirc (-\log\Lambda_p(X, -(X \bigcirc Y)) \bigcirc (-\chi(X \bigcirc Y)) \bigcirc (\log\Lambda_p(X, -(X \bigcirc Y)) \bigcirc (-\frac{\chi - 1}{2} X).$$

It follows from the formulas

$$(-X) \bigcirc Y \bigcirc X = Y + \sum_{n=1}^{\infty} \frac{1}{n!}[Y, X^{(n)}]$$

(see [13]) and

$$X \bigcirc Y \equiv X + Y + \frac{1}{2}[X, Y] + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}[Y, X^{(2n)}] + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}[X, Y^{(2n)}] \mod I_2 \cap J_2$$

(see [1] or [6]) and from the congruences 3.7 and

$$\log\Lambda_p(X, -(X \bigcirc Y)) \equiv \sum_{n=2}^{\infty} (-1)^n l_n(1)[X, Y^{(n-1)}] \mod J_2$$

that the left hand side of the equality 3.8 is congruent to

3.9.

$$-\chi\cdot X-\chi\cdot Y-\sum_{n=2}^{\infty}\chi\cdot l_n(1)[X,Y^{(n)}]-\frac{1}{2}\chi^2[X,Y]-\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}\chi^{2k+1}[X,Y^{(2k)}] \mod J_2$$

and the right hand side of the equality 3.8 is congruent to

3.10.

$$-\chi\cdot X-\chi\cdot Y-\frac{1}{2}\chi^2[X,Y]-\chi\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}[X,Y^{(2k)}]+\sum_{n=2}^{\infty}(-1)^n\chi\cdot l_n(1)[X,Y^{(n)}] \mod J_2$$

Comparing 3.9 and 3.10 we get the following result.

Proposition 3.1. —

$$l_{2k}(1) = -\frac{b_{2k}}{2\cdot(2k)!}(\chi^{2k}-1).$$

(This result is stated without proof in [11] and in references given by Ihara in [11] only $l_n(1)$ for n odd is calculated.)

The l-adic polylogarithm $l_{2k}(1)$ is a function from $G_{\mathbb{Q}}$ to $\mathbb{Q}_l(2k)$, hence its cohomology class in $H^1(G_{\mathbb{Q}};\mathbb{Q}_l(2k))$ is zero. However the related function studied in [15] and in [21], which takes values in $\mathbb{Z}_l(2k)$ need not be zero in $H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$. We shall show below that it determines a torsion class in $H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$ and we shall calculate its order (see also [3]).

We start by recalling the arithmetic formula for l-adic polylogarithms from [15]. Let z be a $\mathbb{Q}$-point of V or a tangential point defined over $\mathbb{Q}$. Let q be a path on $V_{\bar{\mathbb{Q}}}$ from $\overrightarrow{01}$ to z. Let $\varphi_n : \pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})_l \to \mathbb{Z}/l^n$ be a homomorphism given by $\varphi_n(x)=1$ and $\varphi_n(y)=0$. Let us set

$$H_n := \ker(\varphi_n : \pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})_l \to \mathbb{Z}/l^n).$$

Then we have

3.11. $$x^{-l(z)_q(\sigma)}\cdot \mathfrak{f}_q(\sigma) \equiv \prod_{i=0}^{l^n-1}(x^i\cdot y\cdot x^{-i})^{\kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)} \mod (H_n,H_n)$$

where (H_n,H_n) is the commutator subgroup of H_n and where the coefficients $\kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)\in\mathbb{Z}_l$ are defined by the formula

$$\xi_{l^k}^{\kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)} = \frac{\sigma((1-\xi_{l^n}^{i\chi(\sigma)^{-1}}\cdot z^{\frac{1}{l^n}})^{\frac{1}{l^k}})}{(1-\xi_{l^n}^{i+l(z)_q(\sigma)}\cdot z^{\frac{1}{l^n}})^{\frac{1}{l^k}}}.$$

After the embedding k of $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ into $\mathbb{Q}_l\{\{X,Y\}\}$ and then taking logarithm we get

3.12. $$(-l(z)_q(\sigma)X) \bigcirc \log\Lambda_q(X,Y)(\sigma) \equiv$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Big(\sum_{i=0}^{l^n-1} i^m \kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)\Big)[Y,X^{(m)}] \mod \log(k((H_n,H_n))).$$

Let us denote by $c_{m+1}(z)(\sigma)$ the coefficient of the left hand side of 3.12 at the term $[Y,X^{(m)}]$. Then we have

3.13.
$$c_{m+1}(z)(\sigma) \equiv \frac{(-1)^m}{m!}\Big(\sum_{i=0}^{l^n-1} i^m \kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)\Big)[Y,X^{(m)}] \mod l^{n-v_l((m-1)!)}.$$

(The formula on the right hand side is related to the Gabber construction of the Heisenberg cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ (see [5]).
It is shown in [15] that $\boldsymbol{\kappa}(z)(\sigma) := \{\kappa(1-\xi_{l^n}^i z^{\frac{1}{l^n}})(\sigma)\}_{i\in\mathbb{Z}/l^n,\, n\in\mathbb{N}}$ is a measure on $\mathbb{Z}_l$. We set

$$\boldsymbol{\ell}_{m+1}(z)(\sigma) := \int_{\mathbb{Z}_l} x^m d\boldsymbol{\kappa}(z)(\sigma).$$

Therefore $\boldsymbol{\ell}_{m+1}(z)$ is a function from $G_{\bar{\mathbb{Q}}}$ to $\mathbb{Z}_l$. It follows from 3.13 that

3.14. $$c_{m+1}(z)(\sigma) = \frac{(-1)^m}{m!}\boldsymbol{\ell}_{m+1}(z)(\sigma).$$

Let q be the path p from $\overrightarrow{01}$ to $\overrightarrow{10}$. Then $l(1)(\sigma)=0$. Hence we get

3.15. $$c_{m+1}(1)(\sigma) = l_{m+1}(1)(\sigma).$$

Theorem 3.2. — i) *We have* $\boldsymbol{\ell}_{2k}(1) = \frac{b_{2k}}{2\cdot(2k)}\cdot(\chi^{2k}-1)$ *in* $Z^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$;

ii) *Let us write* $\frac{b_{2k}}{2\cdot(2k)} = \frac{a}{b}$, *where* $a,b\in\mathbb{Z}$ *are relatively prime. The class of the cocycle* $\boldsymbol{\ell}_{2k}(1)$ *is a torsion element of* $H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$ *of order* $l^{\mathrm{v}_l(b)}$.

iii) *The class of the cocycle* $\boldsymbol{\ell}_{2k}(1)$ *is a torsion element of maximal order in* $H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$.

Proof. We have already observed that $\boldsymbol{\ell}_{2k}(1)$ is a function from $G_{\bar{\mathbb{Q}}}$ to $\mathbb{Z}_l$. It follows immediately from Proposition 3.1 and the equalities 3.14 and 3.15 that $\boldsymbol{\ell}_{2k}(1) = \frac{b_{2k}}{2\cdot 2k}(\chi^{2k}-1)$. Hence $\boldsymbol{\ell}_{2k}(1)$ is in $Z^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$. The point ii) is a consequence immediate of the point i).
Let us suppose that l is an odd prime. The cyclotomic character $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_l^* = (1+l\mathbb{Z}_l)\times\mu_{l-1}$ is surjective. Therefore there are torsion elements in

$H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$ if and only if $2k \equiv 0 \bmod (l-1)$. Let us suppose that $2k = (l-1)\cdot l^{m-1}\cdot q$ with $(q,l)=1$. Then it follows immediately that $\chi^{2k}-1$ is divisible by l^m but not by l^{m+1}. On the other side Von Staudt Congruece (see [12], chapter 2, section 2, Corollary 2) implies that $\mathbf{v}_l(\frac{b_{2k}}{2\cdot 2k}) = -m$. Hence $\boldsymbol{\ell}_{2k}(1)$ is a torsion element in $H^1(G_{\mathbb{Q}};\mathbb{Z}_l(2k))$ of maximal order.
The proof in case $l=2$ is similar and we omit it. □

As a corollary we get a well known result about the Bernoulli numbers.

Corollary 3.3. — *Let $2k \not\equiv 0$ mod $(l-1)$. Then $\mathbf{v}_l(\frac{b_{2k}}{2\cdot 2k}) \geq 0$.*

Proof. If $2k \not\equiv 0 \bmod l-1$ then $\chi^{2k}-1$ is not divisible by l. Hence the point i) of the theorem implies that $\mathbf{v}_l(\frac{b_{2k}}{2\cdot 2k}) \geq 0$. □

Let $s : \mathbb{Q}_l\{\{X,Y\}\} \to \mathbb{Q}_l\{\{X,Y\}\}$ be an isomorphism of $\mathbb{Q}_l$-algebras given by $s(X)=Y$ and $s(Y)=X$. We recall from section 1 that $t:\mathbb{Q}_l\{\{X,Y\}\} \to \mathbb{Q}_l\{\{X,Y\}\}$ is an anti-isomorphism. Let us set

$$\tau := s\circ t.$$

Now we shall show the analog of the duality theorem for multi zeta values (see [9] page 53).

Proposition 3.4. — *Let $w \in \mathcal{W}(X,Y)$. We have*

$$w^*(\Lambda_p(X,Y)(\sigma)) = (-1)^{|w|}(\tau(w))^*(\Lambda_p(X,Y)(\sigma)).$$

Proof. It follows from Proposition 2.6 and Lemma 2.5 that

$$w^*(\Lambda_p(X,Y)(\sigma)) = (-1)^{|w|}(t(w))^*((\Lambda_p(X,Y)(\sigma))^{-1}).$$

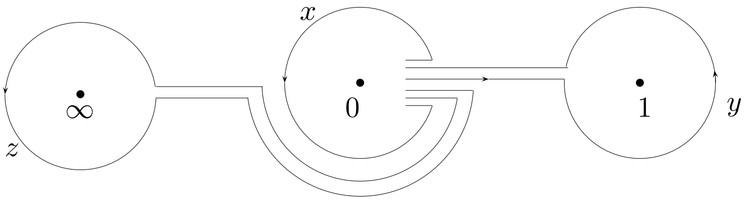
It follows from $\mathbb{Z}/2$-cycle relation that

$$(-1)^{|w|}(t(w))^*((\Lambda_p(X,Y)(\sigma))^{-1}) = (-1)^{|w|}(t(w))^*(\Lambda_p(Y,X)(\sigma)).$$

Observe that for any $u\in\mathcal{W}(X,Y)$ we have $u^*(\Lambda_p(Y,X)(\sigma)) = s(u)^*(\Lambda_p(X,Y)(\sigma)$.
Hence we get $w^*(\Lambda_p(X,Y)(\sigma)) = (-1)^{|w|}(\tau(w))^*(\Lambda_p(X,Y)(\sigma))$. □

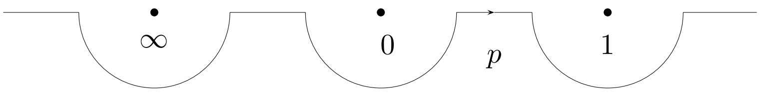
4. $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$-cycle relations

Let us chose generators of $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ as on the picture



Picture 4

Then we have $x \cdot y \cdot z = 1$.



Picture 5

Calculating the element $\mathfrak{f}$ along the path on the Picture 5 we get the $\mathbb{Z}/3$-cycle relation of Drinfeld in $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$

$$x^{\frac{\chi-1}{2}} \cdot \mathfrak{f}_p(z,x) \cdot z^{\frac{\chi-1}{2}} \cdot \mathfrak{f}_p(y,z) \cdot y^{\frac{\chi-1}{2}} \cdot \mathfrak{f}_p(x,y) = 1.$$

(see for example [11]). After the embedding k of $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ into $\mathbb{Q}_l\{\{X,Y\}\}$ the $\mathbb{Z}/2$ and $\mathbb{Z}/3$-cycle relations have the following form

$*_2$
$$\Lambda_p(X,Y) \cdot \Lambda_p(Y,X) = 1,$$

and

$*_3$
$$e^{\frac{\chi-1}{2}X} \cdot \Lambda_p(-(X \bigcirc Y), X) \cdot e^{\frac{\chi-1}{2}(-(X \bigcirc Y))} \cdot \Lambda_p(Y, -(X \bigcirc Y)) \cdot e^{\frac{\chi-1}{2}Y} \cdot \Lambda_p(X,Y) = 1.$$

In this moment we cannot expect that shuffle relations of type II and III have the same form as in De Rham–Betti setting (see [16]). However when we pass to associated graded Lie algebras the situation becomes more familiar. We recall that with the action of a Galois group on π_1 or on a torsor of paths there is associated a filtration of the Galois group (see [18] section 3).

Let $I := \ker(\varepsilon : \mathbb{Q}_l\{\{X,Y\}\} \longrightarrow \mathbb{Q}_l)$ be an augmentation ideal.
In the case of the action of $G_{\bar{\mathbb{Q}}}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ or on the torsor of paths on $V_{\bar{\mathbb{Q}}}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ this filtration is defined as follows

$$G_0(\mathbb{Q}) := G_{\bar{\mathbb{Q}}}, \quad G_1(\mathbb{Q}) := G_{\bar{\mathbb{Q}}(\mu_{l^\infty})},$$

$$G_n(\mathbb{Q}) := \{\sigma \in G_{\bar{\mathbb{Q}}(\mu_{l^\infty})} \mid \Lambda_p(X,Y)(\sigma) \equiv 1 \mod I^n\}$$

for $n > 1$.
If we pass to a graded Lie algebra

$$\mathrm{Lie}G_{\bar{\mathbb{Q}}} := \bigoplus_{i=1}^{\infty} \big(G_i(\mathbb{Q})/G_{i+1}(\mathbb{Q})\big) \otimes \mathbb{Q}$$

associated with the action of $G_{\bar{\mathbb{Q}}}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ the equations $*_2$ and $*_3$ have more familier form as we can see in the next proposition.

Proposition 4.1. — *Let $\sigma \in G_n(\mathbb{Q})$. Let us set $Z := -X - Y$. Then we have*

$$**_2 \qquad \Lambda_p(X,Y)(\sigma) + \Lambda_p(Y,X)(\sigma) = 0 \mod I^{n+1},$$

$$**_3 \qquad \Lambda_p(Z,Y)(\sigma) + \Lambda_p(Y,Z)(\sigma) + \Lambda_p(X,Y) = 0 \mod I^{n+1}.$$

The $\mathbb{Z}_5$-cycle relation can also be written in this form but we do not state it here in order not to make this paper too heavy. In the next section we formulate shuffle relations of type II and III for coefficients of $\Lambda_p(X,Y)$.

5. Shuffle relations of type II and III (multi zeta type and regularized multi zeta type)

We start this section by recalling some notations from [16].
Let $\mathcal{Y} := \{y_1, y_2, \ldots, y_n, \ldots\}$ be a set of non-commuting variables. We set deg $y_i := i$. Then the degree of a monomial $y_{i_1} \cdot y_{i_2} \cdot \ldots \cdot y_{i_k}$ is $\sum_{\alpha=1}^{k} i_\alpha$. We denote by $\mathbb{Q} << \mathcal{Y} >>$ the $\mathbb{Q}$-algebra of formal power series in non-commuting variables $y_1, y_2, \ldots, y_n, \ldots$ We define a coproduct of $\mathbb{Q}$-algebras

$$\Delta_* : \mathbb{Q} << \mathcal{Y} >> \longrightarrow \mathbb{Q} << \mathcal{Y} >> \hat{\otimes} \mathbb{Q} << \mathcal{Y} >>$$

by setting

$$\Delta_*(y_n) := \sum_{i=0}^{n} y_{n-i} \otimes y_i$$

on generators (with the convention that $y_0 = 1$).

Let $\mathcal{W}(X,Y)_{\hat{X}}$ be a subset of $\mathcal{W}(X,Y)$ containing 1 and all monomials whose last term is Y. Let $\mathbb{Q}\{\{X,Y\}\}_{\hat{X}}$ be a subalgebra of $\mathbb{Q}\{\{X,Y\}\}$ generated topologically by the set $\mathcal{W}(X,Y)_{\hat{X}}$.
We identify the $\mathbb{Q}$-algebra $\mathbb{Q} << \mathcal{Y} >>$ with the subalgebra $\mathbb{Q}\{\{X,Y\}\}_{\hat{X}}$ of $\mathbb{Q}\{\{X,Y\}\}$ sending

$$y_i \to X^{i-1}Y$$

for $i = 1, 2, \ldots$. Hence we get a coproduct

$$\Delta_* : \mathbb{Q}\{\{X,Y\}\}_{\hat{X}} \to \mathbb{Q}\{\{X,Y\}\}_{\hat{X}} \hat{\otimes} \mathbb{Q}\{\{X,Y\}\}_{\hat{X}},$$

which induces a commutative product

$$\bigstar : (\mathbb{Q}\{\{X,Y\}\}_{\hat{X}})^{\diamond} \otimes (\mathbb{Q}\{\{X,Y\}\}_{\hat{X}})^{\diamond} \longrightarrow (\mathbb{Q}\{\{X,Y\}\}_{\hat{X}})^{\diamond}.$$

The $\mathbb{Q}$-vector space $(\mathbb{Q}\{\{X,Y\}\}_{\hat{X}})^{\diamond}$ equipped with the product $\bigstar$ we denote by $(\mathbb{Q}\{\{X,Y\}\}_{\hat{X}})^{\diamond}_{\bigstar}$. If we tensored by $\mathbb{Q}_l$ or by $\mathbb{C}$ we obtain algebras $(\mathbb{Q}_l\{\{X,Y\}\}_{\hat{X}})^{\diamond}$ and $(\mathbb{C}\{\{X,Y\}\}_{\hat{X}})^{\diamond}_{\bigstar}$.

Conjecture 5.1. — *(associated graded Lie algebra version of shuffle relations of type II and III.) Let $\sigma \in G_n(\mathbb{Q})$. Then for any $w, w_1 \in \mathcal{W}(X,Y)_{\hat{X}}$ such that $|w| + |w_1| = n$ we have*

$$\sum_{u \in \mathcal{W}(X,Y)_{\hat{X}}} (-1)^{deg_Y(u)} (w^* \bigstar w_1^*)(u) \cdot u^*(\Lambda_p(X,Y)(\sigma)) = 0 \mod I^{n+1}.$$

We point out that the shuffle relations deduced from products of multi zeta functions are also studied in greater generality in [7].

References

[1] N. Bourbaki, Eléments de mathématiques, Groupes et algèbres de Lie, Diffusion C.C.L.S. Paris 1972.

[2] Kuo-Tsai Chen, Algebra of iterated path integrals and fundamental groups, Transactions of the American Mathematical Society, Vol. 156, May 1971, pp. 359-379.

[3] P. Deligne, Le groupe fondamental de la droite projective moins trois points, *in* Galois groups over Q (ed. Y. Ihara, K. Ribet and J.-P. Serre), *Mathematical Sciences Research Institute Publications* **16** (1989), pp. 79-297.

[4] V.G. Drinfeld, On quasi-triangulated quasi-Hopf algebras and on a group that is closely connected with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2 (4), (1991), pp. 829-860.

[5] O. Gabber, notes of Deligne, not published.

[6] K. Goldberg, The formal power series for $\log e^x e^y$, Duke Math. J., 23 (1956), pp. 13-21.

[7] A. B. Goncharov, The dihedral Lie algebra and Galois symmetries of $\pi_1^{(l)}\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_N)$, Duke Math. J. Vol. 110, No. 3, 2001, pp. 397-487.

[8] A. Grothendieck, Esquisse d'un programme, *in* Geometric Galois Actions, (ed. L. Schneps and P. Lochak), L.M.S. Lecture Note Series 243, Cambridge University Press, pp. 5-48.

[9] M. E. Hoffman, Algebraic Aspects of Multiple Zeta Values, *in* Zeta Functions, Topology and Quantum Physics, Springer, 2005, pp. 51-73.

[10] Y. Ihara, Profinite braid groups, Galois representations and complex multiplications, *Annals of Math.* **123** (1986), pp. 43-106.

[11] Y. Ihara, Braids, Galois Groups and Some Arithmetic Functions, Proc. of the Int. Congress of Math. Kyoto 1990, Springer-Verlag pp. 99-120.

[12] S. Lang, Cyclotomic Fields I and II, 1990, Springer-Verlag New York Inc.

[13] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, *Pure and Applied Mathematics* **XIII**, Interscience Publishers (1966).

[14] H. Nakamura, L. Schneps, On a subgroup of the Grothendieck-teichmuller group acting on the tower of profinite Teichmuller modular groups, Invent. Math. 141, (2000), pp. 503-560.

[15] H. Nakamura, Z. Wojtkowiak, On the explicit formula for ℓ-adic polylogarithms, *in* Arithmetic Fundamental Groups and Noncommutative Algebra, *Proc. of Symposia in Pure Math.* **70**, AMS 2002, pp. 285-294.

[16] G. Racinet, Doubles mélange des polylogarithmes multiples aux racines de l'unité, Publ. Math. Inst. Hautes Etudes Sci. No. 95 (2002), pp. 185-231.

[17] R. Ree, Lie elements and an algebra associated with shuffles, Ann. of Math. (2) 68 (1958), pp. 210-220.

[18] Z. Wojtkowiak, On ℓ-adic iterated integrals, I Analog of Zagier Conjecture, Nagoya Math. Journal, Vol. 176 (2004), pp. 113-158.

[19] Z. Wojtkowiak, On ℓ-adic iterated integrals, II Functional equations and ℓ-adic polylogarithms, Nagoya Math. Journal, Vol. 177 (2005), pp. 117-153.

[20] Z. Wojtkowiak, On ℓ-adic iterated integrals, III Galois actions on fundamental groups, Nagoya Math. Journal, Vol. 178 (2005), pp. 1-36.

[21] Z. Wojtkowiak, A Note on Functional Equations of ℓ-adic Polylogarithms, Journal of the Inst. of Math. Jussieu (2004) (**3**), pp. 461-471.

[22] Z. Wojtkowiak, Monodromy of Iterated Integrals and Non abelian Unipotent Periods, *in* Geometric Galois Actions London Math. Soc. Lecture Note Series 243, Cambridge University Press 1997, 219-289.

[23] Z. Wojtkowiak, Mixed Hodge Structures and Iterated Integrals I *in* Motives, Polylogarithms and Hodge Theory. Part I: Motives and Polylogarithms. F. Bogomolov and L. Katzarkov, eds. International Press Lecture Series Vol. 3, part I, 2002, pp. 121-208.

30 mai 2008

Zdzisław Wojtkowiak, Université de Nice-Sophia Antipolis, Département de Mathématiques, Laboratoire Jean Alexandre Dieudonné, U.R.A. au C.N.R.S., N° 168, Parc Valrose – B.P. N° 71, 06108 Nice Cedex 2, France • *E-mail :* wojtkow@math.unice.fr