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A partial Bombieri–Vinogradov theorem with explicit constants

A PARTIAL BOMBIERI–VINOGRADOV THEOREM WITH EXPLICIT CONSTANTS

by

Alisa Sedunova

Abstract. — In this paper we improve the result of [1] with getting $(\log x)^{\frac{7}{2}}$ instead of $(\log x)^{\frac{9}{2}}$. In particular we obtain a better version of Vaughan's inequality by applying the explicit variant of an inequality connected to the Möbius function from [5].

Résumé. — (*Aspects explicites d'un théorème de Bombieri–Vinogradov*) Dans cet article, nous améliorons un résultat de [1] en remplaçant le $(\log x)^{\frac{7}{2}}$ par un $(\log x)^{\frac{9}{2}}$. En particulier, nous obtenons une version améliorée de l'inégalité de Vaughan en appliquant une version explicite d'une inégalité dans [5] liée à la fonction de Möbius.

1. Introduction

For integer number a and $q \geq 1$, let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function. The Bombieri–Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all $q \leq x^{1/2}$.

***Theorem 1.1* (Bombieri–Vinogradov).** — *Let A be a given positive number and $Q \leq \frac{x^{1/2}}{(\log x)^B}$ where $B = B(A)$, then*

$$\sum_{q \leq Q} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y, q, a) - \frac{y}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

The implied constant in this theorem is not effective, since we have to take care of characters, associated with those q that have small prime factors. The main result of this paper is

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Theorem 1.2 **(Bombieri–Vinogradov theorem with explicit constants).** — *Let* $x \geq 4$, $1 \leq Q_1 \leq Q \leq x^{\frac{1}{2}}$. *Let also* $l(q)$ *denote the least prime divisor of* q. *Define* $F(x, Q, Q_1)$ *by*

$$F(x, Q, Q_1) = \frac{14x}{Q_1} + 4x^{\frac{1}{2}}Q + 15x^{\frac{2}{3}}Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1}.$$

Then

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\varphi(q)} \right| < c_1 F(x, Q, Q_1)(\log x)^{\frac{7}{2}},$$

where $c_1 = 42.2$.

Previously the best result obtained by these methods in the literature is due to Akbary, Hambrook (see [1, Theorem 1.3]), where they proved that under assumptions of Theorem 1.2 we have.

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\varphi(q)} \right| < c_1 F(x, Q, Q_1)(\log x)^{\frac{9}{2}},$$

where $F(x, Q, Q_1)$ is defined by

$$F(x, Q, Q_1) = \frac{4x}{Q_1} + 4x^{\frac{1}{2}}Q + 18x^{\frac{2}{3}}Q^{\frac{1}{2}} + 5x^{\frac{5}{6}} \log \frac{eQ}{Q_1}.$$

Here we reduce this power to $(\log x)^{\frac{7}{2}}$ by applying an explicit version for an upper bound for

$$b_k = \sum_{\substack{d \leq V \\ d|k}} \mu(d),$$

where $\mu(d)$ is Mobius function, V is a given number. This version can be found in [5, Chapter 6], namely we have

Lemma 1.3 **(Helfgott [5]).** — *There exists a constant* $L \leq 0.441$, *such that for* V *large enough we have*

$$\sum_{k \leq Y} |b_k|^2 = Y(L + O^*(C)) + O^*(V^2),$$

where $C = 0.000023$ *and* $O^*(x)$ *means that it is less in absolute value than* x.

This Lemma is a variant of the sum considered in [3], where it is shown that

$$\sum_{d_1, d_2 \leq Y} \frac{\mu(d_1)\mu(d_2)}{\gcd(d_1, d_2)}$$

tends to a positive constant as $Y \to \infty$. It is also suggested without proving that L can be about 0.440729.

Notice, that by sharpening the inequality in Lemma 1.3 we will not be able to reduce the power of $\log x$, since the upper bound is optimal there, so by these methods the power $\frac{7}{2}$ is the best possible. Going further seems to be a hard problem which involves among simpler things a very careful analysis of the logarithmic mean of Möbius function twisted by a Dirichlet character.

Corollary 1.4. — *Let $Q = \frac{x^{\frac{1}{2}}}{(\log x)^B}$, where $B > \frac{7}{2}$. Then Theorem 1.2 gives us the following bound*

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y;q,a) - \frac{\psi(y)}{\varphi(q)} \right| < c_1 \left(\frac{14x}{Q_1} (\log x)^{\frac{7}{2}} + 19x(\log x)^{\frac{7}{2}-B} \right).$$

Remark. — It would be very good for applications to get $(\log x)^2$ in Theorem 1.2, however it seems impossible to get by present methods.

Corollary 1.5. — *Define*

$$\pi(x) = \sum_{p \leq x} 1 \qquad and \qquad \pi(x;q,a) = \sum_{\substack{p \leq x \\ p \equiv a \bmod q}} 1.$$

Then Theorem 1.2 under the same assumptions can be also formulated for $\pi(x)$ and $\pi(x;q,a)$:

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \pi(y;q,a) - \frac{\pi(y)}{\varphi(q)} \right| < c_2 F(x,Q,Q_1)(\log x)^{\frac{7}{2}},$$

where $c_2 = 1 + \frac{2c_1}{\log 2}$.

Proof of Corollary 1.5 is exactly the same as in [1], we just have to change the power of log. The key tool for the proof of Theorem 1.2 is Vaughan's identity, which we have to get in an explicit version for our goal. Define

$$\psi(y,\chi) = \sum_{n \leq y} \Lambda(n)\chi(n),$$

the twisted summatory function for the von Mangoldt function Λ and a Dirichlet character χ modulo q. One of two main results of this paper is

Proposition 1.6 (Vaughan's inequality in an explicit form). — *For $x \geq 4$*

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |\psi(y,\chi)| < c_0 \left(7x + 2Q^2 x^{\frac{1}{2}} + 5Q^{\frac{3}{2}} x^{\frac{2}{3}} + 4Q x^{\frac{5}{6}} \right) (\log x)^{\frac{5}{2}},$$

where Q is any positive real number and $\sum_{\chi(q)}^{*}$ means a sum over all primitive characters $\chi(\bmod q)$.

The goal is to get an explicit version of $f(x,Q)$ by applying an improved version of Pólya–Vinogradov inequality (see [6]), that will reduce the coefficients of $f(x,Q)$ and then we can apply Lemma 1.3.

2. Proof of Proposition 1.6

Fix arbitrary real numbers $Q > 0$ and $x \geq 4$. In this section, we shall establish Proposition 1.6, which is the main ingredient in the proof of Theorem 1.2. Here we follow the ideas of [1] and

applying the results from [5]. The main tool in the proof is the large sieve inequality (see, for example [4, Lemma 2])

$$(2.1) \qquad \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \left| \sum_{m=m_0+1}^{m_0+M} a_m \chi(m) \right|^2 \leq (M+Q^2) \sum_{m=m_0+1}^{m_0+M} |a_m|^2,$$

from which it follows (see [2, Theorem 8.8.3] or [1, Lemma 6.1]) that

$$(2.2) \qquad \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y} \left| \sum_{m=m_0}^{M} \sum_{\substack{n=n_0 \\ mn \leq y}}^{N} a_m b_n \chi(mn) \right|$$

$$\leq c_3 (M'+Q^2)^{\frac{1}{2}} (N'+Q^2)^{\frac{1}{2}} \left(\sum_{m=m_0}^{M} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{n=n_0}^{N} |b_n|^2 \right)^{\frac{1}{2}} L(M,N),$$

where $c_3 = 2.65$, $L(M,N) = \log(2MN)$ and $M' = M - m_0 + 1$, $N' = N - n_0 + 1$ are the number of terms in the sums over m and n respectively. Here the a_m, b_n are arbitrary complex numbers.

2.1. Sieving and Vaughan's identity. — We reduce to the case $2 \leq Q \leq x^{1/2}$. If $Q < 1$, then the sum on the left-hand side of (1.6) is empty and we are done. Next, $1 \leq Q < 2$ then only the $q = 1$ term exists and we have

$$(2.3) \qquad \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |\psi(y,\chi)| = \max_{y \leq x} \left| \sum_{n \leq y} \Lambda(n) \right| = \psi(x) \leq A_0 x,$$

which is better than the theorem. As in [1] we use Vaughan's identity

$$\Lambda(n) = \lambda_1(n) + \lambda_2(n) + \lambda_3(n) + \lambda_4(n),$$

where

$$\lambda_1(n) = \begin{cases} \Lambda(n), & \text{if } n \leq U, \\ 0, & \text{if } n > U, \end{cases} \qquad \lambda_2(n) = \sum_{\substack{hd=n \\ d \leq V}} \mu(d) \log h,$$

$$\lambda_3(n) = -\sum_{\substack{mdr=n \\ m \leq U, d \leq V}} \Lambda(m)\mu(d), \qquad \lambda_4(n) = -\sum_{\substack{mk=n \\ m > U, k > V}} \Lambda(m) \sum_{\substack{d|k \\ d \leq V}} \mu(d).$$

Assume $y \leq x$, $q \leq Q$, and χ is a character mod q. We use the above decomposition to write

$$\psi(y,\chi) = s_1 + s_2 + s_3 + s_4,$$

where

$$s_i = \sum_{n \leq y} \lambda_i(n) \chi(n).$$

Let U, V be non-negative functions of x and Q to be set later and denote the contributions to our main sum by

$$S_i = \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |s_i|.$$

Easily we obtain

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |\psi(y, \chi)| \leq S_1 + S_2 + S_3 + S_4.$$

The heart of the proof of Theorem 1.3 in [1] are the following estimates:

Lemma 2.1 (Akbary, Hambrook [1], Section 7). — *We have*

$$S_1 \leq A_0 U Q^2, \quad S_2 < \left(x + Q^{\frac{5}{2}} V\right) (\log x V)^2, \quad S_3 < S_3^{'} + S_3^{''},$$

$$S_3^{'} < (x + Q^{\frac{5}{2}} U)(\log x U)^2,$$

$$S_3^{''} < \frac{c_3}{\log 2} \left(x + Q x^{\frac{1}{2}} U^{\frac{1}{2}} V^{\frac{1}{2}} + 2^{\frac{1}{2}} Q x U^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}\right) (\log 2UV)^2 (\log 4x),$$

$$S_4 < \frac{2^{\frac{3}{2}} A_0^{\frac{1}{2}} c_3}{\log 2} (x + Q x V^{-\frac{1}{2}} + 2^{\frac{1}{2}} Q x U^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}) \left(\log \frac{2x}{V}\right)^{\frac{3}{2}} (\log e^3 V)(\log 4x).$$

where $c_3 = 2.65$ and $A_0 = \max_{x>0} \frac{\psi(x)}{x} = \frac{\psi(113)}{113} < 1.04$.

Here we improve the estimate for S_4 in Lemma 2.1 above.

Lemma 2.2. — *Under previous notations we have*

$$S_4 \leq \frac{c_3}{\log 2} (2A_0)^{\frac{1}{2}} \left(x + 2^{\frac{1}{2}} Q^{\frac{1}{2}} x U^{-\frac{1}{2}} + Q x V^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}\right) (\log 4x) \left(\log \frac{2x}{V}\right)^{\frac{3}{2}}.$$

Proof. — We estimate S_4 contribution with the use of Lemma 1.3. Writing s_4 as a dyadic sum we have

$$s_4 = - \sum_{\substack{M=2^\alpha \\ \frac{1}{2} U < M \leq x/V}} \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} \sum_{\substack{V < k \leq x/M \\ mk \leq y}} \Lambda(m) \left(\sum_{\substack{d|k \\ d \leq V}} \mu(d)\right) \chi(mk).$$

Using the triangle inequality

$$S_4 \leq \sum_{\substack{M=2^\alpha \\ \frac{1}{2} U < M \leq x/V}} \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} \left| \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} \sum_{\substack{V < k \leq x/M \\ mk \leq y}} a_m b_k \chi(mk) \right|,$$

where $a_m = \Lambda(m)$, and, as it was defined in the introduction $b_k = \sum_{d|k, \, d \leq V} \mu(d)$. Now apply the large sieve inequality (2.2) to get

$$S_4 \leq c_3 \sum_{\substack{M=2^\alpha \\ \frac{1}{2} U < M \leq x/V}} (M' + Q^2)^{\frac{1}{2}} (K' + Q^2)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}} \sigma_2(M)^{\frac{1}{2}} L(M)$$

where

$$\sigma_1(M) = \sum_{V < k \leq x/M} |b_k|^2, \qquad \sigma_2(M) = \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} |a_m|^2,$$

and

$$L(M) = \log \left(\frac{2x}{M} \min \left(\frac{x}{V}, 2M\right)\right) \leq \log 4x,$$

where M' and K' denote the number of terms in the sums over m and k, respectively. From the definition of M' and N' we conclude

$$M' = \min\left(2M, \frac{x}{V}\right) - \max\left(M+1, U+1\right) \le M,$$

$$K' = \frac{x}{M} - (V+1) + 1 \le \frac{x}{M}.$$

By Chebyshev estimate we have an upper bound

$$\sigma_2(M) \le \sum_{m \le 2M} \Lambda(m)^2 \le \psi(2M) \log 2M \le 2A_0 M \log 2M.$$

Thus by Cauchy inequality

(2.4) $\qquad S_4 \le c_3(\log 4x) \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \le x/V}} (M+Q^2)^{\frac{1}{2}} \left(\frac{x}{M} + Q^2\right)^{\frac{1}{2}} (2A_0 M \log 2M)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}}.$

Further

$$M(M+Q^2)\left(\frac{x}{M} + Q^2\right) = Mx + Q^2 x + M^2 Q^2 + MQ^4$$

and

$$(\log 2M)^{\frac{1}{2}} \le \left(\log \frac{2x}{V}\right)^{\frac{1}{2}}.$$

Using Lemma 1.3 we get

$$(\sigma_1(M))^{\frac{1}{2}} \le \frac{x}{M}(L+C) - V(L+C) + 2V^2,$$

that implies

$$S_4 \le c_3(2A_0)^{\frac{1}{2}} \left(x + 2^{\frac{1}{2}}Q^{\frac{1}{2}}xU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}\right)(\log 4x)\left(\log \frac{2x}{V}\right)^{\frac{1}{2}} \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \le x/V}} 1.$$

Since

$$\sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \le x/V}} 1 \le \frac{\log \frac{2x}{V}}{\log 2},$$

then

$$S_4 \le \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}} \left(x + 2^{\frac{1}{2}}Q^{\frac{1}{2}}xU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}\right)(\log 4x)\left(\log \frac{2x}{V}\right)^{\frac{3}{2}}. \qquad \square$$

Combining Lemma 2.2 with results of Lemma 2.1 we get

(2.5) $\qquad\qquad S = \sum_{q \le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \max_{y \le x} |\psi(y, \chi)| \le c_4 \, \mathrm{R}_{x,Q,U,V} \, \mathrm{G}_{x,V,U},$

where

$$c_4 = \max \left\{ A_0, \frac{c_3}{\log 2}, \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}} \right\} = \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}},$$

$$\mathrm{R}_{x,Q,U,V} = 4x + Q^2 \left(2x^{\frac{1}{2}} + U \right) + Q^{\frac{5}{2}}(U+V) + \frac{2^{\frac{1}{2}}Q^{\frac{1}{2}}x}{U^{\frac{1}{2}}} + \frac{2^{\frac{1}{2}}Qx}{U^{\frac{1}{2}}} + Qx^{\frac{1}{2}}U^{\frac{1}{2}}V^{\frac{1}{2}} + \frac{Qx}{V^{\frac{1}{2}}},$$

$$\mathrm{G}_{x,V,U} = \max \left\{ (\log xV)^2, (\log xU)^2, (\log 2UV)^2 \log 4x, \left(\log \frac{2x}{V} \right)^{\frac{3}{2}} \log 4x \right\},$$

Now let's specify U and V. If $x^{\frac{1}{3}} \le Q \le x^{\frac{1}{2}}$, then $U = V = x^{\frac{2}{3}}Q^{-1}$. Then putting that into previous expression we get for the factor

$$\mathrm{R}_{x,Q,U,V} = \mathrm{R}_1(x,Q) = 4x + 2Q^2x^{\frac{1}{2}} + Qx^{\frac{2}{3}} \left(1 + 2^{\frac{1}{2}} \right) + Q^{\frac{3}{2}}x^{\frac{2}{3}} \left(2 + 2^{\frac{1}{2}} + 1 \right) + x^{\frac{7}{6}}$$

$$\le 4x + 2Q^2x^{\frac{1}{2}} + 2Qx^{\frac{5}{6}} + Q^{\frac{3}{2}}x^{\frac{2}{3}} \left(2 + 2^{\frac{1}{2}} + 1 \right).$$

where we used the fact that $x^{\frac{7}{6}} \le Qx^{\frac{5}{6}}$ and $Qx^{\frac{2}{3}} \le Qx^{\frac{5}{6}}$. Working in the same manner with G and keeping in mind the condition $x \ge 4$ we find that

$$\mathrm{G}_{x,V,U} = \mathrm{G}_1(x,V,U) \le \left(\frac{4}{3}\log x \right)^{\frac{3}{2}} 2\log x = \frac{2^4}{3^{\frac{3}{2}}}(\log x)^{\frac{5}{2}}.$$

If $Q \le x^{\frac{1}{3}}$, we let $U = V = x^{\frac{1}{3}}$ and get

$$\mathrm{R}_2(x,Q) = 4x + 2Q^2x^{\frac{1}{2}} + Q^2x^{\frac{1}{3}} + 2Q^{\frac{5}{2}}x^{\frac{1}{3}} + 2^{\frac{1}{2}}Q^{\frac{1}{2}}x^{\frac{5}{6}} + Qx^{\frac{5}{6}}(2^{\frac{1}{2}} + 2)$$

$$\le x \left(5 + 2^{\frac{1}{2}} \right) + 2Q^2x^{\frac{1}{2}} + 2Q^{\frac{3}{2}}x^{\frac{2}{3}} + Qx^{\frac{5}{6}} \left(2^{\frac{1}{2}} + 2 \right),$$

where we used $Q^2x^{\frac{1}{3}} \le x$, $Q^{\frac{5}{2}}x^{\frac{1}{3}} \le Q^{\frac{3}{2}}x^{\frac{2}{3}}$ and $Q^{\frac{1}{2}}x^{\frac{5}{6}} \le x$. Similarly we get for

$$\mathrm{G}_2(x,V,U) \le 2 \left(\frac{7}{6} \right)^{\frac{3}{2}} (\log x)^{\frac{5}{2}}.$$

Finally, we have in (2.5)

$$S \le c_4 \frac{2^4}{3^{\frac{3}{2}}} \left(7x + 2Q^2x^{\frac{1}{2}} + 5Q^{\frac{3}{2}}x^{\frac{2}{3}} + 4Qx^{\frac{5}{6}} \right) (\log x)^{\frac{5}{2}},$$

as demanded.

3. Proof of Theorem 1.2

Let $y \ge 2, (a,q) = 1$. By orthogonality of characters modulo q, we have

$$\psi(y; q, a) = \frac{1}{\varphi(q)} \sum_\chi \overline{\chi}(a)\psi(y, \chi).$$

Define $\psi'(y,\chi) = \psi(y,\chi)$ if $\chi \ne \chi_0$ and $\psi'(y,\chi) = \psi(y,\chi) - \psi(y)$ otherwise, χ_0 is the principal character mod q. Then

$$\psi(y, q, a) - \frac{\psi(y)}{\varphi(q)} = \frac{1}{\varphi(q)} \sum_\chi \overline{\chi}(a)\psi'(y, \chi).$$

For a character $\chi \pmod{q}$, we let χ^* be the primitive character modulo q^* inducing χ. Follow the way of [1] we obtain

$$\psi'(y, \chi^*) - \psi'(y, \chi) = \psi(y, \chi^*) - \psi(y, \chi) = \sum_{p^k \leq y} (\log p)(\chi^*(p^k) - \chi(p^k)).$$

If $p \mid q$ then $(p^k, q^*) = 1$, and hence $\chi^*(p^k) = \chi(p^k)$. If $p \mid q$ then $\chi(p^k) = 0$. Therefore

$$|\psi'(y, \chi^*) - \psi'(y, \chi)| \leq \sum_{\substack{p^k \leq y \\ p \mid q}} (\log p) \leq (\log y) \sum_{p \mid q} 1 \leq (\log qy)^2.$$

Denote the quantity we want to estimate as

$$\mathcal{M} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\varphi(q)} \right|.$$

Since

$$\left| \psi(y, q, a) - \frac{\psi(y)}{\varphi(q)} \right| \leq \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y, \chi)| \leq (\log qy)^2 + \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y, \chi^*)|,$$

then

$$\mathcal{M} \leq Q(\log Qx)^2 + \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\varphi(q)} \sum_{\chi} \max_{2 \leq y \leq x} |\psi'(y, \chi^*)|.$$

We have to take care just of the second term in the inequality above, since the first one is smaller than the desired bound. It remains to prove

$$\mathcal{N} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\varphi(q)} \sum_{\chi} \max_{2 \leq y \leq x} |\psi'(y, \chi^*)| \leq (c_1 - 1) F(x, Q, Q_1)(\log x)^4,$$

where $F(x, Q, Q_1)$ is the function from Theorem 1.2. A primitive character $\chi^* \bmod q^*$ induces characters of moduli dq^* and $\psi'(y, \chi^*) = 0$ for χ principal, we observe

$$\mathcal{N} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\varphi(q)} \sum_{\substack{q^* \mid q \\ q^* \neq 1}} \sum_{\chi(q^*)}^{*} \max_{2 \leq y \leq x} |\psi'(y, \chi)| \leq \sum_{\substack{q^* \leq Q \\ l(q^*) > Q_1}} \sum_{\chi(q^*)}^{*} \max_{2 \leq y \leq x} |\psi'(y, \chi)| \sum_{k \leq \frac{Q}{q^*}} \frac{1}{\varphi(kq^*)}.$$

As it was noted in [1] for $x > 0$

$$\sum_{k \leq x} \frac{1}{\varphi(k)} \leq E_0 \log(ex)$$

and as $q^* \leq Q \leq x^{1/2}$, $\varphi(k)\varphi(q^*) \leq \varphi(kq^*)$ and $x \geq 4$, we have

$$\sum_{k \leq \frac{Q}{q^*}} \frac{1}{\varphi(kq^*)} < \frac{5E_0}{4\varphi(q^*)} \log x.$$

For $q > 1$ and χ primitive character $\pmod{q}$, we know that χ is non-principal and $\psi(y, \chi) = \psi'(y, \chi)$. Since we assumed $Q_1 \geq 1$ then we can replace $\psi'(y, \chi)$ by $\psi(y, \chi)$ inside the internal

sum for $\mathcal{N}$. Combining it with an expression for $\mathcal{N}$ we get

$$\mathcal{N} \leq \frac{5E_0}{4}(\log x) \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\varphi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| = \mathcal{R}.$$

Thus it remains to show that

$$\mathcal{R} \leq \frac{4(c_1 - 1)}{5E_0} F(x, Q, Q_1)(\log x)^{\frac{5}{2}}.$$

Let

$$\mathcal{R}(q) = \frac{q}{\varphi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)|.$$

Partial summation gives us

$$\sum_{Q_1 < q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| = \frac{1}{Q} \sum_{q \leq Q} \mathcal{R}(q) - \frac{1}{Q_1} \sum_{q \leq Q_1} \mathcal{R}(q) + \int_{Q_1}^Q \left(\sum_{q \leq t} \mathcal{R}(q) \right) \frac{dt}{t}.$$

Now we apply Theorem 1.6

$$\sum_{q \leq Q} \mathcal{R}(q) < c_0 f(x, Q)(\log x)^{\frac{5}{2}},$$

where $f(x, Q) = 7x + 2Q^2 x^{\frac{1}{2}} + 5Q^{\frac{3}{2}} x^{\frac{2}{3}} + 4Q x^{\frac{5}{6}}$. Then

$$\sum_{Q_1 < q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| < c_0 \left(\Delta_f(Q, Q_1) + \int_{Q_1}^Q f(x, t) \frac{dt}{t} \right) (\log x)^{\frac{5}{2}},$$

where

$$\Delta_f(Q, Q_1) = \frac{f(x, Q)}{Q} - \frac{f(x, Q_1)}{Q_1} \leq \frac{7x}{Q_1} + 2x^{\frac{1}{2}} Q + 5x^{\frac{2}{3}} Q^{\frac{1}{2}}.$$

Calculating the integrals gives us

$$\int_{Q_1}^Q f(x, t) \frac{dt}{t} < \frac{7x}{Q_1} + 2x^{\frac{1}{2}} Q + 10x^{\frac{2}{3}} Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1}.$$

Finally

$$\mathcal{N} \leq \frac{4(c_1 - 1)}{5E_0} \left(\frac{14x}{Q_1} + 4x^{\frac{1}{2}} Q + 15x^{\frac{2}{3}} Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1} \right) (\log x)^{\frac{5}{2}}.$$

3.1. Proof of Corollary 1.5. — Define two functions

$$\pi_1(y) = \sum_{2 \leq n \leq y} \frac{\Lambda(n)}{\log n} \quad \text{and} \quad \pi_1(y; q, a) = \sum_{\substack{2 \leq n \leq y \\ n \equiv a \pmod q}} \frac{\Lambda(n)}{\log n}.$$

Since

$$\pi_1(y; q, a) - \pi(y; q, a) = \sum_{\substack{2 \leq k \leq \frac{\log y}{\log 2}}} \sum_{\substack{p^k \leq y \\ p^k \equiv a \pmod q}} \frac{1}{k} \leq \sum_{2 \leq k \leq \frac{\log y}{\log 2}} \frac{\pi(y^{\frac{1}{2}})}{2} < 2y^{\frac{1}{2}},$$

where we used the fact that for $x > 1$(see for example [1, Lemma 3.1])

$$\pi(x) < 1.25506 \frac{x}{\log x}.$$

Similarly, $\pi_1(y) - \pi(y) < 2y^{\frac{1}{2}}$. Thus by partial summation we obtain the bound

$$\left| \pi_1(y; q, a) - \frac{\pi_1(y)}{\varphi(q)} \right| = \left| \frac{\psi(y; q, a) - \psi(y)/\varphi(q)}{\log y} - \int_2^y \frac{\psi(t; q, a) - \psi(t)/\varphi(q)}{t \log^2 t} \mathrm{d}t \right|$$

$$\leq \frac{1}{\log 2} \left| \psi(y; q, a) - \frac{\psi(y)}{\varphi(q)} \right| + \max_{2 \leq t \leq y} \left| \psi(t; q, a) - \frac{\psi(t)}{\varphi(q)} \right| \left(\frac{1}{\log 2} - \frac{1}{\log y} \right).$$

We have

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a,q)=1}} \left| \pi(y; q, a) - \frac{\pi(y)}{\varphi(q)} \right|$$

$$\leq \frac{2}{\log 2} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{a,(a,q)=1} \left| \psi(y, q, a) - \frac{\psi(y)}{\varphi(q)} \right| + 2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\varphi(q)} \right)$$

$$< \frac{2c_1}{\log 2} F(x, Q, Q_1)(\log x)^{\frac{7}{2}} + 2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\varphi(q)} \right),$$

where we used Theorem 1.2 to estimate the first summand. For $x \geq 4$

$$2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\varphi(q)} \right) < \frac{2c_1}{\log 2} F(x, Q, Q_1)(\log x)^{\frac{7}{2}}.$$

and we are done.

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