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# Arithmetic and Dynamical Degrees on Abelian Varieties

par Joseph H. SILVERMAN

Résumé. Soit $\phi : X \dashrightarrow X$ une application rationnelle dominante d'une variété lisse et soit $x \in X$, tous deux définis sur $\bar{\mathbb{Q}}$. Le degré dynamique $\delta(\phi)$ mesure la complexité géométrique des itérations de $\phi$, tandis que le degré arithmétique $\alpha(\phi, x)$ mesure la complexité arithmétique de la $\phi$-orbite de $x$. Il est connu que $\alpha(\phi, x) \le \delta(\phi)$, et il est conjecturé que si la $\phi$-orbite de $x$ est Zariski dense dans $X$, alors $\alpha(\phi, x) = \delta(\phi)$. Dans cette note, nous prouvons cette conjecture dans le cas où $X$ est une variété abélienne, étendant des travaux antérieurs où la conjecture a été prouvée pour les isogénies.

Abstract. Let $\phi : X \dashrightarrow X$ be a dominant rational map of a smooth variety and let $x \in X$, all defined over $\bar{\mathbb{Q}}$. The dynamical degree $\delta(\phi)$ measures the geometric complexity of the iterates of $\phi$, and the arithmetic degree $\alpha(\phi, x)$ measures the arithmetic complexity of the forward $\phi$-orbit of $x$. It is known that $\alpha(\phi, x) \le \delta(\phi)$, and it is conjectured that if the $\phi$-orbit of $x$ is Zariski dense in $X$, then $\alpha(\phi, x) = \delta(\phi)$, i.e. arithmetic complexity equals geometric complexity. In this note we prove this conjecture in the case that $X$ is an abelian variety, extending earlier work in which the conjecture was proven for isogenies.

## 1. Introduction

Let $K$ be an algebraically closed field, let $X$ be a smooth projective variety of dimension $d$, let $\varphi : X \dashrightarrow X$ be a dominant rational map, and let $H$ be an ample divisor on $X$, all defined over $K$. We write $\varphi^n$ for the $n^{\text{th}}$ iterate of $\varphi$.

**Definition.** The *dynamical degree of* $\varphi$ is the quantity

$$\delta(\varphi) = \lim_{n\to\infty} \Big( ((\varphi^n)^* H) \cdot H^{d-1} \Big)^{1/n},$$

where $\cdot$ and exponentiation on divisors denote intersection.

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We remark that for morphisms $\varphi : \mathbb{P}^N \to \mathbb{P}^N$, or more generally if $\varphi$ is a morphism and $\mathrm{NS}(X)$ has rank 1, then the dynamical degree agrees with the usual notion of degree in the sense that $\varphi^* H \equiv \delta(\varphi) H$. It is known that the limit defining $\delta(\varphi)$ exists and is a birational invariant; see [5, Proposition 1.2(iii)] and [10, Corollary 16]. Bellon and Viallet [1] conjectured that $\delta(\varphi)$ is always an algebraic integer.

We now assume $K$ is a field of characteristic 0 on which one has a good theory of height functions, for example $K = \bar{\mathbb{Q}}$; see for example [7, Part B] or [11, Chapters 1–4]. We write $h_{X,H} : X(K) \to [1, \infty)$ for a Weil height function associated to our ample divisor $H$.

**Definition.** Let $x \in X$ be a point whose forward $\varphi$-orbit

$$\mathcal{O}_\varphi(x) = \{\varphi^n(x) : n \geq 0\}$$

is well-defined. The *arithmetic degree of $x$ (relative to $\varphi$)* is the quantity

$$\alpha(\varphi, x) = \lim_{n \to \infty} h_{X,H}\big(f^n(x)\big)^{1/n}.$$

Kawaguchi and the author [10] proved that $\alpha(\varphi, x) \leq \delta(\varphi)$, i.e. the arithmetic complexity of an orbit never exceeds the geometric complexity of the underlying dynamical system,[1] and they made the following conjectures.

**Conjecture 1.1** (Kawaguchi–Silverman [10, 13])**.**

(a) *The limit defining $\alpha(\varphi, x)$ exists.*
(b) *$\alpha(\varphi, x)$ is an algebraic integer.*
(c) *$\{\alpha(\varphi, x) : x \in X$ such that $\mathcal{O}_\varphi(x)$ exists$\}$ is a finite set.*
(d) *If the orbit $\mathcal{O}_\varphi(x)$ is Zariski dense in $X$, then $\alpha(\varphi, x) = \delta(\varphi)$.*

Conjecture 1.1(a,b,c) has been proven when $\varphi$ is a morphism [9, Theorem 3], and the full conjecture is known in a handful of situations, including:

(i) $\varphi$ is a morphism and $\mathrm{NS}(X)$ has rank 1.
(ii) $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ extends a regular affine automorphism $\mathbb{A}^N \to \mathbb{A}^N$.
(iii) $X$ is a smooth projective surface and $\varphi$ is an automorphism.
(iv) $X = \mathbb{G}_m^N$ is a torus, $x \in \mathbb{G}_m^N(\bar{K})$, and $\mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a monomial map.
(v) $X$ is an abelian variety and $\varphi$ is an isogeny.

See [8] for (i,ii,iii), see [13, Theorem 4] for (iv), and see [9, Theorem 4] for (v). The primary goal of this note is to extend the result in [9] to arbitrary dominant self-maps of abelian varieties.

**Theorem 1.2.** *Let $A/K$ be an abelian variety, let $\varphi : A \dashrightarrow A$ be a dominant rational map, and let $P \in A$ be a point whose orbit $\mathcal{O}_\varphi(P)$ is Zariski dense in $A$. Then*

$$\alpha(\varphi, P) = \delta(\varphi).$$

[1] Since the limit defining $\alpha(\varphi, x)$ is not known to converge, the inequality in [10] is proven with the limit defining $\alpha(\varphi, x)$ replaced by $\limsup$.

**Remark.** Every map as in Theorem 1.2 is a composition of a translation and an isogeny (see Remark 2), so in particular is a morphism. We can thus write $\varphi : A \to A$ as

$$\varphi(P) = f(P) + Q$$

with $f : A \to A$ an isogeny and $Q \in A$. As already noted, if $Q = 0$, then Theorem 1.2 was proven in [9], and it may seem that the introduction of translation by a non-zero $Q$ introduces only a minor complication to the problem. However, the potential interaction between the points $P$ and $Q$ may lead to significant changes in both the orbit of $P$ and the value of the arithmetic degree $\alpha(\varphi, P)$. To illustrate the extent to which taking $Q \neq 0$ is important, consider the following related question. For which $\varphi$ are there any points $P \in A$ whose $\varphi$-orbit $\mathcal{O}_\varphi(P)$ is Zariski dense in $A$? If $Q = 0$, this question is easy to answer, e.g., by using Poincaré reducibility [12, Section 19, Theorem 1]. But if $Q \neq 0$ and the field $K$ is countable, for example $K = \bar{\mathbb{Q}}$, then the problem becomes considerably more difficult. Indeed, the solution, which only recently appeared in [4], uses Faltings' theorem (Mordell–Lang conjecture) on the intersection of subvarieties of $A$ with finitely generated subgroups of $A$. So at present it requires deep tools to even determine whether there exist any points $P \in A(K)$ to which Theorem 1.2 applies.

**Remark.** In view of the results in [13] for tori and the results in [9] and the present paper for abelian varieties, it is natural to ask whether one might prove a general result for (translated) endomorphisms of semi-abelian varieties, i.e. when $X$ is an extension of an abelian variety by a torus. This may be possible, but since the proofs in [13] and [9] use quite different techniques, a general proof will probably not be a straightforward combination of the earlier proofs. In particular, the proof for tori in [13] requires at a crucial step Baker's theorem on linear forms in logarithms, while the proof for abelian varieties in [9] uses canonical heights and a characterization of nef divisors on abelian varieties coming from the embedding of $\mathrm{NS}(A) \otimes \mathbb{Q}$ in $\mathrm{End}(A) \otimes \mathbb{Q}$.

We briefly outline the contents of this note. We begin in Section 2 by setting notation. Section 3 contains a number of preliminary results describing how dynamical and arithmetic degrees vary in certain situations. We then apply these tools and results from earlier work to give the proof of Theorem 1.2 in Section 4. Finally, in Section 5 we prove an auxiliary lemma on pullbacks and pushforwards of divisors that is needed for one of the proofs in Section 3.

The basic strategy in proving Theorem 1.2 is to first note that the proof is reasonably straightforward in the case that a multiple of the translation point $Q$ is in the image of the isogeny $f - 1$. This case is proved at the beginning of Section 4. We then use the tools from Section 3 to reduce to this case. Roughly the idea is to find an $f$-compatible isogeny $A \sim A_1 \times A_2$ with $Q \leftrightarrow (Q_1, Q_2)$ so that a multiple of $Q_1$ is in the image of $(f-1)\big|_{A_1}$ and such that some power of $f-1$ kills $A_2$.

## 2. Notation

We set the following notation, which will be used for the remainder of this note.

$K$ — an algebraically closed field of characteristic 0 with a collection of absolute values such that there is a well-defined theory of Weil height functions, as explained for example in [11, Chapters 1–4]. Primary examples of interest would be algebraic closures of $\mathbb{Q}$ and of $\mathbb{C}(T)$.

$A/K$ — an abelian variety of dimension $d$ defined over $K$.

$Q$ — a point in $A(K)$.

$\tau_Q$ — the translation-by-$Q$ map,
$$\tau_Q : A \longrightarrow A, \quad \tau_Q(P) = P + Q.$$

$f$ — an isogeny $f : A \to A$ defined over $K$.

$\varphi$ — the finite map $\varphi : A \to A$ given by $\varphi = \tau_Q \circ f$, i.e.
$$\varphi(P) = f(P) + Q.$$

$h_{A,H}$ — a height function $h_{A,H} : A(K) \to \mathbb{R}$ associated to an ample divisor $H \in \operatorname{Div}(A)$; see for example [7, §B.3] or [11, Chapter 4].

$\varphi^n$ — the $n^{\text{th}}$ iterate of $\varphi$, i.e. $\varphi^n(P) = \varphi \circ \varphi \circ \cdots \circ \varphi(P)$.

$\mathcal{O}_\varphi(P)$ — the forward $\varphi$-orbit of $P$, i.e. the set $\{\varphi^n(P) : n \geq 0\}$.

**Remark.** It is a standard fact that every rational map $A \dashrightarrow A$ is a morphism, and that every finite morphism $A \to A$ is the composition of an isogeny and a translation [12, Section 4, Corollary 1]. Hence the set of dominant rational maps $A \dashrightarrow A$ is the same as the set of maps of the form $\varphi = \tau_Q \circ f$ as in our notation. Further, as noted earlier, since $\varphi : A \to A$ is a morphism, it is known [9, Theorem 3] that the limit defining $\alpha_\varphi(P)$ exists (and is an algebraic integer).

## 3. Preliminary material

In this section we collect some basic results that are needed to prove Theorem 1.2. We begin with a standard (undoubtedly well-known) decomposition theorem.

**Lemma 3.1.** *Let $A$ be an abelian variety, let $f : A \to A$ be an isogeny, and let $F(X) \in \mathbb{Z}[X]$ be a polynomial such that $F(f) = 0$ in* $\operatorname{End}(A)$. *Suppose that $F$ factors as*
$$F(X) = F_1(X) F_2(X) \quad \textit{with } F_1, F_2 \in \mathbb{Z}[X] \textit{ and } \gcd(F_1, F_2) = 1,$$

*where the gcd is computed in* $\mathbb{Q}[X]$. *Let*

$$A_1 = F_1(f)A \quad \textit{and} \quad A_2 = F_2(f)A,$$

*so* $A_1$ *and* $A_2$ *are abelian subvarieties of* $A$. *Then we have*:

(a) $A = A_1 + A_2$.

(b) $A_1 \cap A_2$ *is finite.*

*More precisely, if we let* $\rho = \operatorname{Res}(F_1, F_2)$, *then* $A_1 \cap A_2 \subset A[\rho]$.

*Proof.* The gcd assumption on $F_1$ and $F_2$ implies that their resultant is non-zero, so we can find polynomials $G_1, G_2 \in \mathbb{Z}[X]$ so that

$$G_1(X)F_1(X) + G_2(X)F_2(X) = \rho = \operatorname{Res}(F_1, F_2) \neq 0.$$

We observe that $fA_1 \subset A_1$ and $fA_2 \subset A_2$ and compute

$$\begin{aligned} A = \rho A &= \big(G_1(f)F_1(f) + G_2(f)F_2(f)\big)A \\ &= G_1(f)A_1 + G_2(f)A_2 \\ &\subset A_1 + A_2 \subset A. \end{aligned}$$

Hence $A = A_1 + A_2$. This proves (a). For (b), suppose that $P \in A_1 \cap A_2$, so

$$P = F_1(f)P_1 = F_2(f)P_2 \quad \text{for some } P_1 \in A_1 \text{ and } P_2 \in A_2.$$

Then

$$\begin{aligned} \rho P &= \big(G_1(f)F_1(f) + G_2(f)F_2(f)\big)P \\ &= G_1(f)F_1(f)F_2(f)P_2 + G_2(f)F_2(f)F_1(f)P_1 \\ &= G_1(f)F(f)P_2 + G_2(f)F(f)P_1 \quad \text{since } F = F_1F_2, \\ &= 0 \quad \text{since } F(f) = 0. \end{aligned}$$

Hence $A_1 \cap A_2 \subset A[\rho]$. □

The next two lemmas relate dynamical and arithmetic degrees. We state them somewhat more generally than needed in this note, since the proofs are little more difficult and they may be useful for future applications. The first lemma says that dynamical and arithmetic degrees are invariant under finite maps, and the second describes dynamical and arithmetic degrees on products.

**Lemma 3.2.** *Let* $X$ *and* $Y$ *be non-singular projective varieties, and let*

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & Y \\ \big\downarrow{\scriptstyle f_X} & & \big\downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\lambda} & Y \end{array}$$

*be a commutative diagram, where* $f_X$ *and* $f_Y$ *are dominant rational maps and* $\lambda$ *is a finite map, with everything defined over* $K$

(a) *Let $x \in X$ whose orbit $\mathcal{O}_{f_X}(x)$ is well-defined. Then $\mathcal{O}_{f_X}(x)$ is Zariski dense in $X$ if and only if $\mathcal{O}_{f_Y}(\lambda(x))$ is Zariski dense in $Y$.*

(b) *The dynamical degrees of $f_X$ and $f_Y$ are equal,*

$$\delta(f_X) = \delta(f_Y)$$

(c) *Let $P \in X$ be a point such that the forward $f_X$-orbit of $P$ and the arithmetic degree of $P$ relative to $f_X$ are well-defined. Then the arithmetic degrees of $P$ and $\lambda(P)$ satisfy*

$$\alpha(f_X, P) = \alpha(f_Y, \lambda(P)).$$

**Remark.** Lemma 3.2(b) is a special (relatively easy) case of results of Dinh–Nguyen [2] and Dinh–Nguyen–Truong [3]. For completeness, we give an algebraic proof, in the spirit of the present paper, which works in arbitrary characteristic.

*Proof of Lemma* 3.2.

(a) — We first remark that the $f_Y$ orbit of $\lambda(x)$ is also well-defined. To see this, let $n \geq 1$ and let $U$ be any Zariski open set on which $f_X^n$ is well-defined. Then $\lambda \circ f_X^n$ is also well-defined on $U$, since $\lambda$ is a morphism. Also, since $\lambda$ is a finite map, the image $\lambda(U)$ is a Zariski open set, and we note that $f_Y^n$ on the set $\lambda(U)$ agrees with $\lambda \circ f_X^n$ on $U$. Thus $f_Y^n$ is defined on $\lambda(U)$. In particular, since $f_X^n$ is assumed defined at $x$, we see that $f_Y^n$ is defined at $\lambda(x)$.

Suppose that $Z = \overline{\mathcal{O}_{f_X}(x)} \neq X$. Then $\lambda(Z)$ is a proper Zariski closed subset of $Y$, since finite maps send closed sets to closed sets. Further,

$$\mathcal{O}_{f_Y}(\lambda(x)) = \lambda(\mathcal{O}_{f_X}(x)) \subset \lambda(Z).$$

Hence $\mathcal{O}_{f_Y}(\lambda(x))$ is not Zariski dense. Conversely, suppose that $W = \overline{\mathcal{O}_{f_Y}(\lambda(x))} \neq Y$. Finite maps (and indeed, morphisms) are continuous for the Zariski topology, so $\lambda^{-1}(W)$ is a closed subset of $X$, and the fact that $\lambda$ is a finite map, hence surjective, implies that $\lambda^{-1}(W) \neq X$. Then

$$\mathcal{O}_{f_X}(x) \subset \lambda^{-1}\Big(\mathcal{O}_{f_Y}(\lambda(x))\Big) \subset \lambda^{-1}(W) \subsetneq X,$$

so $\mathcal{O}_{f_X}(x)$ is not Zariski dense in $X$.

(b) — Let $d = \dim(X) = \dim(Y)$, and let $H_Y$ be an ample divisor on $Y$. The assumption that $\lambda$ is a finite morphism implies that $H_X := \lambda^* H_Y$ is an ample divisor on $X$. This follows from [6, Exercise 5.7(d)], or we can use the Nakai–Moishezon Criterion [6, Theorem A.5.1] and note that for every irreducible subvariety $W \subset X$ of dimension $r$ we have

$$\lambda_*(H_X \cdot W^r) = \lambda_*(\lambda^* H_Y \cdot W^r) = H_Y \cdot (\lambda_* W)^r > 0,$$

since $\lambda_* W$ is a positive multiple of an $r$-dimensional irreducible subvariety of $Y$. This means that we can use $H_X$ to compute $\delta(f_X)$. In the following computation we use that fact that since $\lambda$ is a finite morphism, we have

$$(f_X^N)^* \circ \lambda^* = (\lambda \circ f_X^N)^* = (f_Y^N \circ \lambda)^* = \lambda^* \circ (f_Y^N)^*. \tag{3.1}$$

We give the justification for this formula at the end of this paper, see Lemma 5.1, but we note that for the proof of Theorem 1.2, all of the relevant maps are morphisms, so (3.1) is trivially true. We compute

$$\begin{aligned}
\delta(f_X) &= \lim_{n\to\infty}\Big((f_X^n)^* H_X \cdot H_X^{d-1}\Big)^{1/n} \\
&= \lim_{n\to\infty}\Big((f_X^n)^* \circ \lambda^* H_Y \cdot (\lambda^* H_Y)^{d-1}\Big)^{1/n} \\
&= \lim_{n\to\infty}\Big(\lambda^* \circ (f_Y^n)^* H_Y \cdot (\lambda^* H_Y)^{d-1}\Big)^{1/n} \quad \text{from (3.1)}, \\
&= \lim_{n\to\infty}\Big(\deg(\lambda)\big((f_Y^n)^* H_Y \cdot^* H_Y^{d-1}\big)\Big)^{1/n} \\
&= \lim_{n\to\infty}\Big((f_Y^n)^* H_Y \cdot H_Y^{d-1}\Big)^{1/n} \\
&= \delta(f_Y).
\end{aligned}$$

This completes the proof of (b).

(c) — We do an analogous height computation, where the $O(1)$ quantities depend on $X$, $Y$, $\lambda$, $f_X$, $f_Y$, and the choice of height functions for $H_X$ and $H_Y$, but do not depend of $n$.

$$\begin{aligned}
\alpha(f_X, P) &= \lim_{n\to\infty} h_{X,H_X}\big(f_X^n(P)\big)^{1/n} \\
&= \lim_{n\to\infty} h_{X,\lambda^* H_Y}\big(f_X^n(P)\big)^{1/n} \\
&= \lim_{n\to\infty}\Big(h_{X,H_Y}\big(\lambda \circ f_X^n(P)\big) + O(1)\Big)^{1/n} \\
&= \lim_{n\to\infty}\Big(h_{X,H_Y}\big(f_Y^n \circ \lambda(P)\big) + O(1)\Big)^{1/n} \\
&= \alpha\big(f_Y, \lambda(P)\big).
\end{aligned}$$

This completes the proof of (c). □

**Lemma 3.3.** *Let $Y$ and $Z$ be non-singular projective varieties, let*

$$f_Y : Y \to Y \quad \textit{and} \quad f_Z : Z \to Z$$

*be dominant rational maps, and let $f_{Y,Z} := f_Y \times f_Z$ be the induced map on the product $Y \times Z$, with everything defined over $K$.*

(a) *Let $y \in Y$ and $z \in Z$ be points whose forward orbits via $f_Y$, respectively $f_Z$, are well-defined, and suppose that $\mathcal{O}_{f_{Y,Z}}(y,z)$ is Zariski dense in $Y \times Z$. Then $\mathcal{O}_{f_Y}(y)$ is Zariski dense in $Y$ and $\mathcal{O}_{f_Z}(z)$ is Zariski dense in $Z$.*

(b) *The dynamical degrees of $f_Y$, $f_Z$, and $f_{Y,Z}$ are related by*

$$\delta(f_{Y,Z}) = \max\{\delta(f_Y), \delta(f_Z)\}.$$

(c) *Let $(P_Y, P_Z) \in (Y \times Z)(K)$ be a point such that the arithmetic degrees $\alpha(f_Y, P_Y)$ and $\alpha(f_Z, P_Z)$ are well-defined. Then*

$$\alpha(f_{Y,Z}, (P_Y, P_Z)) = \max\{\alpha(f_Y, P_Y), \alpha(f_Z, P_Z)\}.$$

*Proof.*

(a) — This elementary fact has nothing to do with orbits. Let $S \subset Y$ and $T \subset Z$ be sets of points. By symmetry, it suffices to prove that if $S \times T$ is Zariski dense in $Y \times Z$, then $S$ is Zariski dense in $Y$. We prove the contrapositive, so assume that $S$ is not Zariski dense in $Y$. This means that there is a proper Zariski closed subset $W \subset Y$ with $S \subset W$. Then $S \times T \subset W \times Z \subsetneq Y \times Z$, which shows that $S \times T$ is not Zariski dense in $Y \times Z$.

(b) — Let

$$\pi_Y : Y \times Z \to Y \quad \text{and} \quad \pi_Z : Y \times Z \to Z$$

denote the projection maps. Let $H_Y$ and $H_Z$ be, respectively, ample divisors on $Y$ and $Z$. Then

$$H_{Y,Z} := (H_Y \times Z) + (Y \times H_Z) = \pi_Y^* H_Y + \pi_Z^* H_Z$$

is an ample divisor on $Y \times Z$. We compute

$$\begin{aligned}(f_{Y,Z}^n)^* H_{Y,Z} &= (f_Y^n \times f_Z^n)^* (\pi_Y^* H_Y + \pi_Z^* H_Z) \\ &= \pi_Y^* \circ (f_Y^n)^* H_Y + \pi_Z^* \circ (f_Z^n)^* H_Z.\end{aligned}$$

We let

$$d_Y = \dim(Y), \quad d_Z = \dim(Z), \quad \text{so} \quad \dim(Y \times Z) = d_Y + d_Z.$$

We compute

$$\begin{aligned}&(f_{Y,Z}^n)^* H_{Y,Z} \cdot H_{Y,Z}^{d_Y+d_Z-1} \\ &\quad = \Bigl(\pi_Y^* \circ (f_Y^n)^* H_Y + \pi_Z^* \circ (f_Z^n)^* H_Z\Bigr) \cdot \Bigl(\pi_Y^* H_Y + \pi_Z^* H_Z\Bigr)^{d_Y+d_Z-1} \\ &\quad = \binom{d_Y+d_Z-1}{d_Z} ((f_Y^n)^* H_Y \cdot H_Y^{d_Y-1})(H_Z^{d_Z}) \\ &\qquad\qquad + \binom{d_Y+d_Z-1}{d_Y} ((f_Z^n)^* H_Z \cdot H_Z^{d_Z-1})(H_Y^{d_Y}).\end{aligned} \tag{3.2}$$

For any dominant rational self-map $f : X \to X$ of a non-singular projective variety of dimension $d$ and any ample divisor $H$ on $X$, the dynamical degree of $f$ is, by definition, the number $\delta(f)$ satisfying

$$\delta(f)^n = (f^n)^* H \cdot H^{d-1} \cdot 2^{o(n)} \quad \text{as } n \to \infty.$$

Using this formula three times in (3.2) yields

$$\delta(f_{Y,Z})^n \cdot 2^{o(n)} = \delta(f_Y)^n \cdot 2^{o(n)} \cdot H_Z^{d_Z} + \delta(f_Z)^n \cdot 2^{o(n)} \cdot H_Y^{d_Y}.$$

The quantities $H_Y^{d_Y}$ and $H_Z^{d_Z}$ are positive, since $H_Y$ and $H_Z$ are ample. Now taking the $n^{\text{th}}$ root of both sides and letting $n \to \infty$ gives the desired result, which completes the proof of (b).

(c) — We do a similar computation. Thus

$$\begin{aligned} &h_{Y\times Z,H_{Y,Z}}\big(f_{Y,Z}^n(P_Y,P_Z)\big) \\ &\qquad = h_{Y\times Z,\pi_Y^* H_Y}\big(f_{Y,Z}^n(P_Y,P_Z)\big) + h_{Y\times Z,\pi_Z^* H_Z}\big(f_{Y,Z}^n(P_Y,P_Z)\big) + O(1) \\ &\qquad = h_{Y,H_Y}\big(\pi_Y \circ f_{Y,Z}^n(P_Y,P_Z)\big) + h_{Z,H_Z}\big(\pi_Z \circ f_{Y,Z}^n(P_Y,P_Z)\big) + O(1) \\ &\qquad = h_{Y,H_Y}\big(f_Y^n(P_Y)\big) + h_{Z,H_Z}\big(f_Z^n(P_Z)\big) + O(1). \end{aligned}$$

For any dominant rational self-map $f : X \to X$ of a non-singular projective variety defined over $K$, any ample divisor $H$ on $X$, and any $P \in X(K)$ whose $f$-orbit is well-defined, the arithmetic degree is the limit (if it exists)

$$\alpha(f,P) := \lim_{n\to\infty} h_{X,H}^+\big(f^n(P)\big)^{1/n}.$$

(Here $h^+ = \max\{h,1\}$.) Hence

$$\begin{aligned} \alpha\big(f_{Y,Z},(P_Y,P_Z)\big) &= \lim_{n\to\infty} h_{Y\times Z,H_{Y,Z}}^+\big(f_{Y,Z}^n(P_Y,P_Z)\big)^{1/n} \\ &= \lim_{n\to\infty}\Big(h_{Y,H_Y}^+\big(f_Y^n(P_Y)\big) + h_{Z,H_Z}^+\big(f_Z^n(P_Z)\big) + O(1)\Big)^{1/n} \\ &= \max\{\alpha(f_Y,P_Y),\alpha(f_Z,P_Z)\}, \end{aligned}$$

which completes the proof of (c). □

## 4. Proof of Theorem 1.2

If $Q = 0$, i.e. if the map $\varphi$ is an isogeny, then Theorem 1.2 was proven in [9]. This fact, which we restate here, is used in a crucial way in the proof of Theorem 1.2 when $Q \neq 0$.

**Theorem 4.1.** *Let $A/K$ be an abelian variety, let $f : A \to A$ be an isogeny, and let $P \in A$ be a point whose orbit $\mathcal{O}_\varphi(P)$ is Zariski dense in $A$. Then*

$$\alpha(\varphi,P) = \delta(\varphi).$$

*Proof.* See [9, Theorem 4]. □

*Proof of Theorem* 1.2. The translation map $\tau_Q$ induces the identity map[2]

$$\tau_Q^* = \mathrm{id} : \mathrm{NS}(A) \to \mathrm{NS}(A),$$

[2] Let $\mu : A \times A \to A$ be $\mu(x,y) = x + y$, and let $D \in \mathrm{Div}(A)$. Then for any $P \in A$, the divisor $\mu^* D$ has the property that $\mu^* D|_{A\times\{P\}} = \tau_P^* D \times \{P\}$. Hence as $P$ varies, the divisors $\tau_P^* D$ are algebraically equivalent, so in particular $\tau_Q^* D \equiv \tau_0^* D \equiv D$, which shows that $\tau_Q^*$ is the identity map on $\mathrm{NS}(A)$.

from which we deduce that

$$\varphi^* = f^* \quad \text{and} \quad \delta_\varphi = \delta_f. \tag{4.1}$$

We begin by proving Theorem 1.2 under the assumption that a non-zero multiple of the point $Q$ is in the image of the map $f-1$, say

$$mQ = (f-1)(Q') \quad \text{for some } m \ne 0 \text{ and } Q' \in A.$$

Then we have

$$\begin{aligned} m\varphi^n(P) &= m\bigl(f^n(P) + (f^{n-1}+f^{n-2}+\cdots+f+1)(Q)\bigr) \\ &= f^n(mP) + (f^{n-1}+f^{n-2}+\cdots+f+1)(mQ) \\ &= f^n(mP) + (f^{n-1}+f^{n-2}+\cdots+f+1)\circ(f-1)(Q') \\ &= f^n(mP) + f^n(Q') - Q' \\ &= f^n(mP+Q') - Q'. \end{aligned} \tag{4.2}$$

In particular, the $\varphi$-orbit of $P$ and the $f$-orbit of $mP+Q'$ differ by translation by $-Q'$, so the assumption that $\mathcal{O}_\varphi(P)$ is Zariski dense and the fact that translation is an automorphism imply that $\mathcal{O}_f(mP+Q')$ is also Zariski dense. We will also use the standard formula

$$h_{A,H}\circ m = m^2 h_{A,H} + O\bigl(|m|\bigr), \tag{4.3}$$

where the big $O$ constant depends on the choice of height function $h_{A,H}$; see for example [11, Chapter 5, Theorem 3.1].

We now compute (with additional explanation for steps (4.4) and (4.5) following the computation)

$$\begin{aligned} \alpha_\varphi(P) &= \lim_{n\to\infty} h_{A,H}\bigl(\varphi^n(P)\bigr)^{1/n} && \text{by definition,} \\ &= \lim_{n\to\infty} h_{A,H}\bigl(m\varphi^n(P)\bigr)^{1/n} && \text{from (4.3),} \\ &= \lim_{n\to\infty} h_{A,H}\bigl(f^n(mP+Q')-Q'\bigr)^{1/n} && \text{from (4.2),} \\ &= \lim_{n\to\infty} h_{A,\tau^*_{-Q'}H}\bigl(f^n(mP+Q')\bigr)^{1/n} && \text{functoriality,} \\ &= \alpha_f(mP+Q') && \text{by definition,} \quad (4.4) \\ &= \delta_f && \text{from Theorem 4.1,} \quad (4.5) \\ &= \delta_\varphi && \text{from (4.1).} \end{aligned}$$

We note that (4.4) follows from [10, Proposition 12], which says that the arithmetic degree may be computed using the height relative to any ample divisor. (The map $\tau_{-Q'}$ is an isomorphism, so $\tau^*_{-Q'}H$ is ample.) For (4.5), we have applied Theorem 4.1 ([9, Theorem 4]) to the isogeny $f$ and the point $mP+Q'$, since we've already noted that $\mathcal{O}_f(mP+Q')$ is Zariski dense. This completes the proof of Theorem 1.2 if $mQ \in (f-1)(A)$ for some integer $m \ne 0$.

We now commence the proof in the general case. The Tate module $T_\ell(A)$ of $A$ has rank $2d$, and an isogeny is zero if and only if it induces the trivial map on the Tate module, from which we see that $f$ satisfies a monic integral polynomial equation of degree $2d$, say

$$F(f) = 0 \quad \text{with } F(X) \in \mathbb{Z}[X] \text{ monic.}$$

We factor $F(X)$ as

$$F(X) = F_1(X)F_2(X)$$

with

$$F_1(X) = (X-1)^r, \quad F_2(X) \in \mathbb{Z}[X], \quad \text{and} \quad F_2(1) \neq 0.$$

We first deal with the case that $r = 0$. This means that $F(1) \neq 0$. Writing $F(X) = (X-1)G(X) + F(1)$, we have

$$0 = F(f)Q = (f-1)G(f)Q + F(1)Q,$$

so

$$F(1)Q = -(f-1)G(f)Q \in (f-1)A.$$

Thus a non-zero multiple of $Q$ is in $(f-1)A$, which is the case that we handled earlier.

We now assume that $r \geq 1$, and we define abelian subvarieties of $A$ by

$$A_1 = F_1(f)A \quad \text{and} \quad A_2 = F_2(f)A$$

and consider the map

$$\lambda : A_1 \times A_2 \longrightarrow A, \quad \lambda(P_1, P_2) = P_1 + P_2.$$

Lemma 3.1 tells us that $\lambda$ is an isogeny. More precisely, Lemma 3.1(a) says that $\lambda$ is surjective, while Lemma 3.1(b) tells us that

$$\ker(\lambda) = \bigl\{(P, -P) : P \in A_1 \cap A_2\bigr\} \cong A_1 \cap A_2$$

is finite.

We recall the the map $\varphi : A \to A$ has the form $\varphi(P) = f(P) + Q$ for some fixed $Q \in A$. The map $\lambda$ is onto, so we can find a pair

$$(Q_1, Q_2) \in A_1 \times A_2 \quad \text{satisfying} \quad \lambda(Q_1, Q_2) = Q, \text{ i.e. } Q_1 + Q_2 = Q.$$

We observe that $fA_1 \subset A_1$ and $fA_2 \subset A_2$, since $f$ commutes with $F_1(f)$ and $F_2(f)$. Writing $f_1$ and $f_2$ for the restrictions of $f$ to $A_1$ and $A_2$, respectively, we define maps

$$\begin{aligned} \varphi_1 &: A_1 \longrightarrow A_1, & \varphi_1(P_1) &= f_1(P_1) + Q_1, \\ \varphi_2 &: A_2 \longrightarrow A_2, & \varphi_2(P_2) &= f_2(P_2) + Q_2. \end{aligned}$$

Then

$$\begin{aligned}\lambda\circ(\varphi_1\times\varphi_2)(P_1,P_2) &= \lambda\bigl(f_1(P_1)+Q_1, f_2(P_2)+Q_2\bigr)\\ &= f(P_1)+Q_1+f(P_2)+Q_2\\ &= f(P_1+P_2)+Q\\ &= \varphi\circ\lambda(P_1,P_2),\end{aligned}$$

which shows that we have a commutative diagram

$$\begin{array}{ccc} A_1\times A_2 & \xrightarrow{\ \lambda\ } & A\\ \Big\downarrow{\scriptstyle \varphi_1\times\varphi_2} & & \Big\downarrow{\scriptstyle\varphi}\\ A_1\times A_2 & \xrightarrow{\ \lambda\ } & A\end{array}$$

The map $\lambda$ is an isogeny, so in particular it is a finite morphism, so Lemma 3.2 with $X = A_1\times A_2$ and $Y = A$ says that

$$\delta(\varphi_1\times\varphi_2)=\delta(\varphi)\quad\text{and}\quad \alpha\bigl(\varphi_1\times\varphi_2,(P_1,P_2)\bigr)=\alpha(\varphi,P_1+P_2). \tag{4.6}$$

Next we apply Lemma 3.3 with $X = A_1$ and $Y = A_2$ to conclude that

$$\delta(\varphi_1\times\varphi_2)=\max\{\delta(\varphi_1),\delta(\varphi_2)\}, \tag{4.7}$$

$$\alpha\bigl(\varphi_1\times\varphi_2,(P_1,P_2)\bigr)=\max\{\alpha(\varphi_1,P_1),\alpha(\varphi_2,P_2)\}. \tag{4.8}$$

We now fix a point $P\in A$ whose orbit $\mathcal{O}_\varphi(P)$ is Zariski dense in $A$. Since $\lambda$ is onto, we can write

$$P=\lambda(P_1,P_2)=P_1+P_2\quad\text{for some } P_1\in A_1 \text{ and } P_2\in A_2. \tag{4.9}$$

Then Lemma 3.2(a) tells us that the $(\varphi_1\times\varphi_2)$-orbit of $(P_1,P_2)$ is Zariski dense in $A_1\times A_2$, after which Lemma 3.3(a) tells us that $\mathcal{O}_{\varphi_1}(P_1)$ is Zariski dense in $A_1$ and $\mathcal{O}_{\varphi_2}(P_2)$ is Zariski dense in $A_2$.

Under the assumption that $\overline{\mathcal{O}_{\varphi_1}(P_1)}=A_1$ and $\overline{\mathcal{O}_{\varphi_2}(P_2)}=A_2$, we are going to prove the following result.

**Claim 4.2.**

$$\alpha(\varphi_1,P_1)=\delta(\varphi_1)\quad\textit{and}\quad \alpha(\varphi_2,P_2)=\delta(\varphi_2), \tag{4.10}$$

Assuming this claim, the following computation completes the proof of Theorem 1.2:

$$\begin{aligned}\alpha(\varphi,P) &= \alpha(\varphi,P_1+P_2) && \text{from (4.9)},\\ &= \alpha\bigl(\varphi_1\times\varphi_2,(P_1,P_2)\bigr) && \text{from (4.6)},\\ &= \max\{\alpha(\varphi_1,P_1),\alpha(\varphi_2,P_2)\} && \text{from (4.8)},\\ &= \max\{\delta(\varphi_1),\delta(\varphi_2)\} && \text{from (4.10)},\\ &= \delta(\varphi_1\times\varphi_2) && \text{from (4.7)},\\ &= \delta(\varphi) && \text{from (4.6)}.\end{aligned}$$

We now prove Claim 4.2. We note that if $R \in A$ is in the kernel of the isogeny $f-1$, then

$$\rho R = \big(G_1(f)(f-1)^r + G_2(f)F_2(f)\big)R = G_2(f)F_2(f)R \in A_2.$$

Hence

$$R \in A_1 \cap \ker(f-1) \implies \rho R \in A_1 \cap A_2 \subset A[\rho] \implies R \in A[\rho^2].$$

This proves that the group endomorphism

$$f_1 - 1 : A_1 \longrightarrow A_1$$

has finite kernel, so it is surjective. In particular, the point $Q_1 \in A_1$ is in the image of $f_1 - 1$, so $\alpha_{\varphi_1}(P_1) = \delta_{\varphi_1}$ from the special case of the theorem with which we started the proof. This proves the first statement in Claim 4.2.

For the second statement in Claim 4.2, we will show that both $\alpha(\varphi_1, P_2)$ and $\delta(\varphi_2)$ are equal to 1. We use the following elementary result.

**Lemma 4.3.** *Fix $r \ge 1$. There are polynomials $c_{r,j}(T) \in \mathbb{Z}[T]$ of degree at most $r-1$ so that for all $n \ge 0$ we have*

$$X^n \equiv \sum_{j=0}^{r-1} c_{r,j}(n)X^j \pmod{(X-1)^r}.$$

*Proof.* We compute

$$\begin{aligned}
X^n &= \big((X-1)+1\big)^n \\
&= \sum_{k=0}^{n} \binom{n}{k}(X-1)^k \\
&\equiv \sum_{k=0}^{r-1} \binom{n}{k}(X-1)^k \pmod{(X-1)^r} \\
&\equiv \sum_{k=0}^{r-1} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}X^j \pmod{(X-1)^r} \\
&\equiv \sum_{j=0}^{r-1} \left[\sum_{k=j}^{r-1} (-1)^{k-j}\binom{k}{j}\binom{n}{k}\right] X^j \pmod{(X-1)^r}.
\end{aligned}$$

The quantity in braces is $c_{r,j}(n)$. □

We now observe that

$$(f-1)^r A_2 = (f-1)^r F_2(f) A_2 = F(f)A_2 = 0,$$

since $F(f) = 0$, so we see that $(f_2 - 1)^r$ kills $A_2$. So using Lemma 4.3, we find that the action of the iterates of $f_2$ on $A_2$ is given by

$$f_2^n = \sum_{j=0}^{r-1} c_{r,j}(n) f_2^j \in \operatorname{End}(A_2).$$

Note that the polynomials $c_{r,j}$ have degree at most $r-1$ and do not depend on $n$.

Let $H_2$ be an ample symmetric divisor on $A_2$, and let $d_2 = \dim(A_2)$. Then

$$\begin{aligned}
((f_2^n)^* H_2) \cdot H_2^{d_2-1} &= \left( \sum_{j=0}^{r-1} c_{r,j}(n) f_2^j \right)^* H_2 \cdot H_2^{d_2-1} \\
&= \sum_{j=0}^{r-1} \bigl(c_{r,j}(n) f_2^j\bigr)^* H_2 \cdot H_2^{d_2-1} \\
&= \sum_{j=0}^{r-1} c_{r,j}(n)^2 (f_2^j)^* H_2 \cdot H_2^{d_2-1} \\
&\le C(A_2, H_2, f_2) n^{2r-2},
\end{aligned}$$

since the $c_{r,j}$ polynomials have degree at most $r-1$. (We have also used the fact that since $H_2$ is symmetric, we have $m^* H_2 \sim m^2 H_2$ for any integer $m$.) This allow us to compute

$$\delta(f_2) = \lim_{n\to\infty} \Bigl( ((f_2^n)^* H_2) \cdot H_2^{d_2-1} \Bigr)^{1/n} \le \lim_{n\to\infty} \bigl( C(A_2, H_2, f_2) n^{2r-2} \bigr)^{1/n} = 1,$$

which shows that $\delta(f_2) = 1$.

We next do a similar height calculation. To ease notation, we write

$$\|R\| = \sqrt{\hat{h}_{A_2,H_2}(R)}$$

for the norm associated to the $H_2$-canonical height on $A_2$. (See [7, §B5] or [11, Chapter 5] for basic properties of canonical heights on abelian varieties.) Then

$$\begin{aligned}
\|\varphi_2^n(P_2)\| &= \left\| f_2^n(P_2) + \sum_{i=0}^{n-1} f_2^i(Q_2) \right\| \\
&= \left\| \sum_{j=0}^{r-1} c_{r,j}(n) f_2^j(P_2) + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} c_{r,j}(i) f_2^j(Q_2) \right\| \\
&\le (r + nr) \max_{\substack{0\le j<r \\ 0\le i\le n}} |c_{r,j}(i)| \cdot \max_{0\le j<r} \|f_2^j(Q_2)\| \\
&\le C'(r, f_2, Q_2) n^r.
\end{aligned}$$

This allows us to compute

$$\begin{aligned}\alpha(\varphi_2, P_2) &= \lim_{n\to\infty} \hat{h}_{H_2}\big(\varphi_2^n(P_2)\big)^{1/n} \\ &\le \lim_{n\to\infty} \big(C'(r, f_2, Q_2)n^r\big)^{2/n} \\ &= 1.\end{aligned}$$

Hence $\alpha(\varphi_2, P_2) = 1$, which is also equal to $\delta(\varphi_2)$. This completes the proof of the second part of Claim 4.2, and with it, the proof of Theorem 1.2. ☐

## 5. An Auxiliary Lemma

In this final section we prove a lemma that is a bit stronger than is needed to justify formula (3.1), which we used in the proof of Lemma 3.2.

**Lemma 5.1.**

(a) *Let $X, Y, Z$ be non-singular varieties, let $\lambda : Y \to Z$ be a morphism, and let $\varphi : X \dashrightarrow Y$ be a rational map. Then*

$$(\lambda\circ\varphi)^* = \varphi^*\circ\lambda^* \quad \textit{as maps } \operatorname{Pic}(Z)\to\operatorname{Pic}(X).$$

(b) *Let $W, X, Y$ be non-singular varieties, let $\lambda : W \to X$ be a finite morphism, and let $\varphi : X \dashrightarrow Y$ be a rational map. Then*

$$(\varphi\circ\lambda)^* = \lambda^*\circ\varphi^* \quad \textit{as maps } \operatorname{Pic}(Y)\to\operatorname{Pic}(W).$$

*Proof.*

(a) — We blow up $\pi : \tilde{X} \to X$ to resolve the map $\varphi$, so we have a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] \arrow[dr, "\tilde{\varphi}"] & & \\ X \arrow[r, dashed, "\varphi"] & Y \arrow[r, "\lambda"] & Z \end{tikzcd}$$

where $\pi$ is a birational map and $\tilde{\varphi}$ is a morphism. Let $D \in \operatorname{Pic}(Z)$. The map $\lambda\circ\tilde{\varphi}$ is a morphism resolving the rational map $\lambda\circ\varphi$, so

$$\begin{aligned}(\lambda\circ\varphi)^*D &= \pi_*(\lambda\circ\tilde{\varphi})^*D && \text{by definition of pull-back,}\\ &= \pi_*\circ(\tilde{\varphi}^*\circ\lambda)^*D && \text{since } \tilde{\varphi} \text{ and } \lambda \text{ are morphisms,}\\ &= (\pi_*\circ\tilde{\varphi}^*)\circ\lambda^*D && \\ &= \varphi^*\circ\lambda^*D && \text{by definition of pull-back.}\end{aligned}$$

(b) — We blow up $\pi : \tilde{X} \to X$ to resolve the map $\varphi$, and then we blow up $W$ to resolve the map $\pi^{-1} \circ \lambda$. This gives a commutative diagram

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{\tilde{\lambda}} & \tilde{X} & & \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\pi} & \searrow^{\tilde{\varphi}} & \\ W & \xrightarrow{\lambda} & X & \overset{\varphi}{\dashrightarrow} & Y \end{array}$$

Here $\mu$ and $\pi$ are birational morphisms and $\tilde{\lambda}$ and $\tilde{\varphi}$ are morphisms. We claim that

$$\lambda^* \circ \pi_* = \mu_* \circ \tilde{\lambda}^*. \tag{5.1}$$

Assuming the validity of (5.1), we compute

$$\begin{aligned} \lambda^* \circ \varphi^* D &= \lambda^* \circ \pi_* \circ \tilde{\varphi}^* D && \text{by definition of pull-back,} \\ &= \mu_* \circ \tilde{\lambda}^* \circ \tilde{\varphi}^* D && \text{from (5.1),} \\ &= \mu_* \circ (\tilde{\varphi} \circ \tilde{\lambda})^* D && \text{since } \tilde{\varphi} \text{ and } \tilde{\lambda} \text{ are morphisms,} \\ &= (\varphi \circ \lambda)^* D && \text{by definition of pull-back,} \end{aligned}$$

where for the last line we have used the fact that $\tilde{\varphi} \circ \tilde{\lambda}$ is a morphism that resolves the rational map $\varphi \circ \lambda$. It remains to verify (5.1).[3]

Let $D \in \mathrm{Div}(\tilde{X})$ be an irreducible divisor, and let $|D|$ denote the support of $D$. There are two cases. First suppose that $D$ is an exceptional divisor, so $\pi_* D = 0$. This means that $\dim \pi(|D|) \le \dim(X) - 2$, and since $\tilde{\lambda}$ is surjective, we have $\pi(|D|) = \lambda \circ \mu \circ \tilde{\lambda}^{-1}(|D|)$. Hence

$$\dim \lambda \circ \mu \circ \tilde{\lambda}^{-1}(|D|) \le \dim(X) - 2.$$

We now use the fact that $\lambda$ is a finite map to deduce that

$$\dim \mu \circ \tilde{\lambda}^{-1}(|D|) \le \dim(W) - 2.$$

It follows that

$$\mu_* \circ \tilde{\lambda}^* D = 0.$$

Next suppose that $D$ is a horizontal divisor relative to $\pi$, so $D = \pi^* \circ \pi_* D$. This allows us to compute

$$\begin{aligned} \mu_* \circ \tilde{\lambda}^* D &= \mu_* \circ \tilde{\lambda}^* \circ \pi^* \circ \pi_* D && \text{using } D = \pi^* \circ \pi_* D, \\ &= \mu_* \circ (\pi \circ \tilde{\lambda})^* \circ \pi_* D && \text{since } \tilde{\lambda} \text{ and } \pi \text{ are morphisms,} \\ &= \mu_* \circ (\lambda \circ \mu)^* \circ \pi_* D && \text{commutativity of the diagram} \\ &= \mu_* \circ \mu^* \circ \lambda^* \circ \pi_* D && \text{since } \lambda \text{ and } \mu \text{ are morphisms,} \\ &= \lambda^* \circ \pi_* D && \text{since } \mu_* \circ \mu^* = \mathrm{id}_W^*. \end{aligned}$$

[3] We remark that (5.1) requires $\lambda$ be a finite map. It is not true for morphisms, even birational morphisms. For example, let $W = \tilde{X}$ and $\lambda = \pi$ and $\mu = \mathrm{id}_W$, then $\lambda^* \circ \pi_* = \pi^* \circ \pi_*$ kills exceptional divisors, while $\mu_* \circ \tilde{\lambda}^*$ is the identity map.

This shows in both cases that $\mu_* \circ \tilde{\lambda}^* = \lambda^* \circ \pi_*$, which completes the proof of (5.1). $\square$

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