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Period functions of half-integral weight modular forms

par Dohoon CHOI, Subong LIM et Wissam RAJI

Résumé. Dans cet article, we étudions la cohomologie d'Eichler associée aux formes cupsidales de poids demi-entiers en utilisant la fonction zêta de Dedekind $\eta(z)$. Nous montrons que la η-multiplication (resp. θ-multiplication) induit un isomorphisme entre l'espace des formes cupsidales de poids demi-entiers et le groupe de cohomologie associé à l'espace $\eta\mathcal{P}$ (resp. $\theta\mathcal{P}$). Nous montrons aussi qu'il existe un isomorphisme entre la somme directe de deux espaces de telles formes et le groupe de cohomologie.

Abstract. In this paper, we study the Eichler cohomology associated with half-integral weight cusp forms using the Dedekind eta function $\eta(z)$ and the theta function $\theta(z)$. We prove that η-multiplication (resp. θ-multiplication) gives an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group associated with the space $\eta\mathcal{P}$ (resp. $\theta\mathcal{P}$). We also show that there is an isomorphism between the direct sum of two spaces of cusp forms of half-integral weights and the cohomology group.

1. Introduction

The Eichler cohomoogy theory is a cohomology theory for Fuchsian groups, introduced by Eichler [4]. This gives an isomorphism between the direct sum of two spaces of cusp forms of integral weights with the cohomology group associated with the polynomial space. Since then, the Eichler cohomology theory has been extensively studied [5, 6, 11, 13] and it has many applications in number theory in connection with periods and period functions (for example, see [1, 10, 12]).

For half-integral weights Knopp [8] defined a bigger space $\mathcal{P}$, which is the space of holomorphic functions $g(z)$ on the upper half plane $\mathbb{H}$ satisfying

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the following growth condition

$$|g(z)| < K(|z|^\rho + y^{-\sigma})$$

for some positive constants $K,\ \rho$ and σ, where $z = x + iy \in \mathbb{H}$. In [8] and [9] Knopp and Mawi proved that the space of cusp forms of a half-integral weight is isomorphic to the cohomology group associated with $\mathcal{P}$.

In this paper, we study more about Eichler cohomology and period functions associated with half-integral weight cusp forms. For this we use two important half-integral weight modular forms: the Dedekind eta function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

and the theta function

$$\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2},$$

where $q = e^{2\pi i z}$. We prove that η-multiplication gives an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group associated with $\eta\mathcal{P}$. It gives the following commutative diagram in which all arrows are isomorphisms

$$\begin{array}{ccc} S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z})) & \longrightarrow & \tilde{H}^1_{2-k,\overline{\chi\chi_\eta}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{P}) \\ \times\eta\uparrow & & \downarrow\times\eta \\ S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) & \longrightarrow & \tilde{H}^1_{\frac{5}{2}-k,\bar{\chi}}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P}). \end{array}$$

We also prove the similar result for $\Gamma_0(4)$ using $\theta(z)$ instead of $\eta(z)$. Moreover, we show that there is an isomorphism between the direct sum of two spaces of cusp forms of half-integral weights and the cohomology group.

For an integer k let $M_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ (resp. $S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ be the space of holomorphic modular forms (resp. cusp forms) of weight $k - \frac{1}{2}$ with a multiplier system χ on $\mathrm{SL}_2(\mathbb{Z})$. Let χ_η be the eta-multiplier system of weight $\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$, i.e.,

$$\chi_\eta(\gamma) = \frac{\eta(\gamma z)}{\eta(z)(cz+d)^{\frac{1}{2}}}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Precisely, the multiplier system χ_η is determined by (for example, see [7, Section 2.8])

$$\chi_\eta(\gamma) = \begin{cases} e(b/24) & \text{if } \gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \\ e\left(\frac{a+d-3c}{24c} - \frac{1}{2}s(d,c)\right) & \text{if } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } c > 0, \end{cases} \tag{1.1}$$

where $e(x) = e^{2\pi i x}$. Here $s(d,c)$ denotes the Dedekind sum defined by

$$s(d,c) := \sum_{0\le n<c} \frac{n}{c}\psi\left(\frac{dn}{c}\right),$$

where $\psi(x) = x - [x] - \frac{1}{2}$.

It is known that the space $\mathcal{P}$ is preserved under the slash operator $|_{r,\psi}\gamma$ for any real r, multiplier system ψ of weight r and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (for example, see [8, Section 4]). Using the $\mathrm{SL}_2(\mathbb{Z})$-module $\mathcal{P}$, we consider another $\mathrm{SL}_2(\mathbb{Z})$-module

$$\eta\mathcal{P} = \{\eta(z)f(z) |\ f(z) \in \mathcal{P}\}.$$

In this case, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\eta\mathcal{P}$ by

$$\left(\left(\eta(z)f\right)\Big|_{\frac{5}{2}-k,\psi\chi_\eta}\gamma\right)(z) = \eta(z)(f|_{2-k,\psi}\gamma)(z) \tag{1.2}$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where ψ is a multiplier system of weight $2-k$. Let $\tilde{H}^1_{\frac{5}{2}-k,\psi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P})$ be the parabolic Eichler cohomology group of weight $\frac{5}{2}-k$ with a multiplier system $\psi\chi_\eta$ on $\mathrm{SL}_2(\mathbb{Z})$ associated with the module $\eta\mathcal{P}$ (for the precise definition see Section 3). Then we have an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group.

Theorem 1.1. *Let k be an integer and χ be a multiplier system of weight $k-\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$. Then the space $S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to the cohomology group $\tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P})$.*

Note that if χ is a multiplier system then $\chi\bar{\chi}$ is a trivial multiplier system so that it can be a multiplier system of any even integral weight. Let k be an integer with $k \geq 2$. For a multiplier system χ of weight $k-\frac{1}{2}$ we let $\chi^*_\eta = \chi\chi^2_\eta$. Note that $\overline{\chi^*_\eta}$ is a multiplier system of weight $k-\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$ because $\chi^*_\eta\overline{\chi^*_\eta}$ is a trivial multiplier system and it can be a multiplier system of weight $2k$. Let P_{k-2} be the space of polynomials of z of degree $\leq k-2$. Note that it is a $\mathrm{SL}_2(\mathbb{Z})$-module for which the action of $\mathrm{SL}_2(\mathbb{Z})$ on P_{k-2} is given by the slash operator of weight $2-k$

$$(f|_{2-k,\psi}\gamma)(z) := (cz+d)^{k-2}\overline{\psi(\gamma)}f(\gamma z)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where ψ is a multiplier system of weight $2-k$ on $\mathrm{SL}_2(\mathbb{Z})$. Then ηP_{k-2} is a $\mathrm{SL}_2(\mathbb{Z})$-module by the slash operator $|_{\frac{5}{2}-k,\psi\chi_\eta}$ as in (1.2). Let $\tilde{H}^1_{\frac{5}{2}-k,\psi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}), \eta P_{k-2})$ be the parabolic Eichler cohomology group of weight $\frac{5}{2}-k$ with a multiplier system $\psi\chi_\eta$ on $\mathrm{SL}_2(\mathbb{Z})$ associated with the module ηP_{k-2} (for the precise definition see Section 3). Then we have an isomorphism between the direct sum of two spaces of cusp forms of half-integral weights and the cohomology group.

Theorem 1.2. *Let k be an integer with $k \geq 2$ and χ be a multiplier system of weight $k-\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$. Then the space $S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \oplus S_{k-\frac{1}{2},\overline{\chi^*_\eta}}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to $\tilde{H}^1_{\frac{5}{2}-k,\chi^*_\eta}(\mathrm{SL}_2(\mathbb{Z}), \eta P_{k-2})$.*

On the other hand, if we use the theta function $\theta(z)$, we can obtain similar results for $\Gamma_0(4)$. Let χ_θ be the theta-multiplier system of weight $\frac{1}{2}$ on $\Gamma_0(4)$, i.e.,

$$\chi_\theta(\gamma) = \frac{\theta(\gamma z)}{\theta(z)(cz+d)^{\frac{1}{2}}}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. Note that χ_θ is given by $\chi_\theta(\gamma) = \left(\frac{c}{d}\right)\bar{\epsilon}_d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ (for example, see [7, Section 2.8]), where $\left(\frac{c}{d}\right)$ denotes the extended quadratic residue symbol and

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv -1 \pmod 4. \end{cases}$$

Then we have the following result, which is analogous to Theorem 1.1.

Theorem 1.3. *Let k be an integer and $\chi = \chi_\theta^{2k-1}$ be a multiplier system of weight $k-\frac{1}{2}$ on $\Gamma_0(4)$. Then the space $S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$ is isomorphic to $\tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}(\Gamma_0(4), \theta\mathcal{P})$.*

If we let $\chi_\theta^* := \chi\chi_\theta^2$ for a multiplier system χ of weight $k-\frac{1}{2}$, then we see that $\overline{\chi_\theta^*}$ is also a multiplier system of weight $k-\frac{1}{2}$ on $\Gamma_0(4)$. As in the case of $\mathrm{SL}_2(\mathbb{Z})$ we obtain the following theorem.

Theorem 1.4. *Let k be an integer with $k \geq 2$ and $\chi = \chi_\theta^{2k-1}$ be a multiplier system of weight $k-\frac{1}{2}$ on $\Gamma_0(4)$. Then the space $S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \oplus S_{k-\frac{1}{2},\overline{\chi_\theta^*}}(\Gamma_0(4))$ is isomorphic to $\tilde{H}^1_{\frac{5}{2}-k,\chi_\theta^*}(\Gamma_0(4), \theta P_{k-2})$.*

Remark. Knopp and Mawi also proved that there is an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group. More precisely, in [9] Knopp and Mawi proved that

$$S_{k-\frac{1}{2},\chi}(\Gamma) \cong \tilde{H}^1_{\frac{5}{2}-k}(\Gamma, \mathcal{P}),$$

where Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which is a Fuchsian group of the first kind with at least one parabolic class. If we combine this result with Theorem 1.1 and 1.3, then in the cases of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(4)$ we obtain

$$\tilde{H}^1_{\frac{5}{2}-k}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{P}) \cong \tilde{H}^1_{\frac{5}{2}-k}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P})$$

and

$$\tilde{H}^1_{\frac{5}{2}-k}(\Gamma_0(4), \mathcal{P}) \cong \tilde{H}^1_{\frac{5}{2}-k}(\Gamma_0(4), \theta\mathcal{P}).$$

Note that $\mathcal{P}$ is the set of holomorphic functions which have polynomial growth at the boundary of $\mathbb{H}$. But elements of $\eta\mathcal{P}$ and $\theta\mathcal{P}$ should vanish at some boundary points because $\eta(z)$ and $\theta(z)$ decrease exponentially at some cusps. This implies that $\eta\mathcal{P}$ and $\theta\mathcal{P}$ are strictly smaller than $\mathcal{P}$.

Nevertheless, they give the cohomology groups isomorphic to the cohomology group associated with $\mathcal{P}$. It can be useful to analyze because we have smaller modules.

Now we look at some examples of period functions of a half-integral weight, which lies in the module of Theorem 1.2 or 1.4.

Example. (1) Suppose that k is an integer with $k \geq 2$. Let $f(z)$ be a cusp form in $S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ with the Fourier expansion

$$\sum_{m=0}^{\infty} a(m)e^{2\pi i(m+\kappa)z},$$

where κ is a real number in $[0,1)$ such that $\chi(T) = e^{2\pi i\kappa}$ for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We consider the period function of $f(z)$ with respect to the $\mathrm{SL}_2(\mathbb{Z})$-module ηP_{k-2}

$$F(z) := \int_0^{i\infty} f(\tau)\eta(\tau)\eta(z)(\tau - z)^{k-2} d\tau.$$

If we compute this integral, then we have

$$F(z) = \eta(z) \sum_{n=0}^{k-2} (-1)^n z^n i^{k-n-1} \int_0^{\infty} \sum_{n=0}^{\infty} c(n) e^{-2\pi(n+\kappa')v} v^{k-2-n} dv,$$

where $c(n) = \sum_{m=0}^{n} a(m)b(n-m)$ and $b(n)$ is defined by

$$b(n) := \left|\left\{ l \in \mathbb{Z} \middle| \ \frac{3l^2 - l}{2} = n \text{ and } l \equiv 0 \pmod 2 \right\}\right| - \left|\left\{ l \in \mathbb{Z} \middle| \ \frac{3l^2 - l}{2} = n \text{ and } l \equiv 1 \pmod 2 \right\}\right|$$

and $\kappa' = \kappa + \frac{1}{24}$. If we continue to compute $F(z)$, then we have

$$F(z) = \eta(z) \sum_{n=0}^{k-2} (-1)^n \left(\frac{i}{2\pi} \right)^{k-1-n} \Gamma(k-n-1) L(f\eta, k-n-1) z^n,$$

where $L(f\eta, s) = \sum_{m=0}^{\infty} \frac{c(m)}{(m+\kappa')^s}$ is the L-function associated with the cusp form $f(z)\eta(z)$. Therefore, $F(z)$ is a product of $\eta(z)$ and a polynomial of degree $k-2$. If $f(z)\eta(z)$ is a newform, then this polynomial satisfies a good property, i.e., a ratio of coefficients of even (resp. odd) degrees are rational over the algebraic number field $\mathbb{Q}(c(0), \ldots, c(n), \ldots)$ by the Periods Theorem [12].

(2) Suppose that k is an integer with $k \geq 2$. Let

$$f(z) = \sum_{m=1}^{\infty} a(m) e^{2\pi imz} \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4)),$$

where $\chi = \chi_\theta^{2k-1}$. We consider the period function of $f(z)$ with respect to the $\Gamma_0(4)$-module θP_{k-2}

$$G(z) := \int_0^{i\infty} f(\tau)\theta(\tau)\theta(z)(\tau - z)^{k-2} d\tau.$$

Then this integral can be divided into two parts

$$G(z) = G_1(z) + G_2(z),$$

where

$$G_1(z) := \int_0^{i\infty} f(\tau)\theta(z)(\tau - z)^{k-2} d\tau$$

and

$$G_2(z) := \int_0^{i\infty} f(\tau)\Big(2\sum_{m=1}^{\infty} e^{2\pi i m^2\tau}\Big)\theta(z)(\tau - z)^{k-2} d\tau.$$

After some computations we see that

$$G_1(z) = \theta(z)\sum_{n=0}^{k-2}(-1)^n\Big(\frac{i}{2\pi}\Big)^{k-n-1}\Gamma(k-n-1)L(f, k-n-1)z^n$$

and

$$G_2(z) = 2\theta(z)\sum_{n=0}^{k-2}(-1)^n\Big(\frac{i}{2\pi}\Big)^{k-n-1}\Gamma(k-n-1)L(f', k-n-1)z^n.$$

Here, the function $f'(z)$ is defined by

$$f'(z) = \sum_{m=1}^{\infty} b(m)e^{2\pi i m z},$$

where $b(m) = \sum_{t<\sqrt{m}} a(m - t^2)$. This type of Fourier coefficients appeared in the papers [2] and [14].

The remainder of this paper is organized as follows. In Section 2 we recall the definition of modular forms and in Section 3 we introduce the Eichler cohomology group. Finally, in Section 4 we prove the main results: Theorem 1.1, 1.2, 1.3 and 1.4.

2. Modular forms

In this section we review the definition of modular forms and cusp forms. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $k \in \frac{1}{2}\mathbb{Z}$ and χ be a (unitary) multiplier system in weight k on Γ. Thus $\chi(\gamma)$ is a complex number independent of z such that

(1) $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma$,

(2) χ satisfies the consistency condition

$$\chi(\gamma_3)(c_3z+d_3)^k = \chi(\gamma_1)\chi(\gamma_2)(c_1\gamma_2 z+d_1)^k(c_2z+d_2)^k,$$

where $\gamma_3=\gamma_1\gamma_2$ and $\gamma_i = \left(\begin{smallmatrix} * & * \\ c_i & d_i \end{smallmatrix}\right), i=1,2$ and 3,

(3) χ satisfies the non-triviality condition $\chi(-I)=e^{\pi i k}$ if $-I\in\Gamma$.

We have the slash operator defined by

$$(f|_{k,\chi}\gamma)(z) := \overline{\chi(\gamma)}(cz+d)^{-k}f(\gamma z)$$

for any function $f(z)$ on $\mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Let $T = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right),\ \lambda>0$, generate the subgroup Γ_∞ of translations in Γ. If $f(z)$ satisfies $(f|_{k,\chi}T)(z) = f(z)$, then

$$f(z+\lambda) = \chi(T)f(z) = e^{2\pi i\kappa}f(z),$$

with $0\le\kappa<1$. Thus, if $f(z)$ is holomorphic in $\mathbb{H}$, then $f(z)$ has a Fourier expansion at $i\infty$

$$f(z) = \sum_{n=-\infty}^{\infty} a(n)e^{[2\pi i(n+\kappa)z]/\lambda}. \tag{2.1}$$

Suppose that in addition to $i\infty$, Γ has $t\ge 0$ inequivalent parabolic classes. Each of these classes corresponds to a cyclic subgroup of parabolic elements in Γ leaving fixed a parabolic cusp on the boundary of $\mathbb{H}$. Such a parabolic cusp lies on the real axis. Let $q_1,\cdots,q_t$ be the inequivalent parabolic cusps other than $i\infty$ on the boundary of $\mathbb{H}$ and let Γ_j be the cyclic subgroup of Γ fixing $q_j,\ 1\le j\le t$. Let $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $A_j(q_j)=\infty$. Then $\lambda_j>0$ is chosen so that

$$Q_j := A_j^{-1}\begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix}A_j$$

generates Γ_j. For $1\le j\le t$, put $\chi(Q_j)=e^{2\pi i\kappa_j},\ 0\le\kappa_j<1$. If a holomorphic function $f(z)$ satisfies $(f|_{k,\chi}Q_j)(z)=f(z)$, then $f(z)$ has the following Fourier expansion at q_j

$$(c_jz+d_j)^{-k}f(A_j^{-1}z) = \sum_{n=-\infty}^{\infty} a_j(n)e^{[2\pi i(n+\kappa_j)z]/\lambda_j}. \tag{2.2}$$

Here, λ_j is a positive real number called the width of the cusp q_j. Using the above notations, we review the following definitions.

Definition. Suppose $f(z)$ is holomorphic in $\mathbb{H}$ and satisfies the functional equation

$$(f|_{k,\chi}\gamma)(z) = f(z)$$

for all $\gamma\in\Gamma$.

(1) If $f(z)$ has only terms with $n+\kappa\ge 0$ in (2.1) and $n+\kappa_j\ge 0,\ 1\le j\le t$, in (2.2), then $f(z)$ is called a holomorphic modular form. The set of holomorphic modular forms is denoted by $M_{k,\chi}(\Gamma)$.

(2) If $f(z) \in M_{k,\chi}(\Gamma)$ and has only terms with $n+\kappa > 0,\ n+\kappa_j > 0$ in the expansions (2.1), (2.2), respectively, then $f(z)$ is called a cusp form. The collection of cusp forms in $M_{k,\chi}(\Gamma)$ is denoted by $S_{k,\chi}(\Gamma)$.

3. The Eichler cohomology

In this section, we define the Eichler cohomology group. Let $k \in \frac{1}{2}\mathbb{Z}$ and Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let χ be a multiplier system of weight k on Γ. If $\mathcal{M}$ is a vector space of functions on $\mathbb{H}$ and is preserved under the slash operator $|_{k,\chi}$, we can form the Eichler cohomology group associated with $\mathcal{M}$ as follows.

A collection $\{p_\gamma(z) |\ \gamma \in \Gamma\}$ of elements of $\mathcal{M}$ is called a cocycle if it satisfies the following cocycle condition

$$p_{\gamma_1\gamma_2}(z) = p_{\gamma_2}(z) + (p_{\gamma_1}|_{k,\chi}\gamma_2)(z) \tag{3.1}$$

for $\gamma_1, \gamma_2 \in \Gamma$. Then a coboundary is a collection $\{p_\gamma(z) |\ \gamma \in \Gamma\}$ such that

$$p_\gamma(z) = (p|_{k,\chi}\gamma)(z) - p(z)$$

for $\gamma \in \Gamma$ with $p(z)$ a fixed element of $\mathcal{M}$. Furthermore, a parabolic cocycle $\{p_\gamma(z) |\ \gamma \in \Gamma\}$ is a collection of elements of $\mathcal{M}$ satisfying (3.1), in which for every parabolic element B in Γ there exists a fixed element $Q_B(z) \in \mathcal{M}$ such that

$$p_B(z) = (Q_B|_{k,\chi}B)(z) - Q_B(z).$$

Note that coboundaries are parabolic cocycles. The parabolic Eichler cohomology group $\tilde{H}^1_{k,\chi}(\Gamma, \mathcal{M})$ is defined as the vector space obtained by forming the quotient of the parabolic cocycles by the coboundaries.

4. Proof of Theorem 1.1, 1.2, 1.3 and 1.4

In this section we prove the main results of this paper. To prove Theorem 1.1 and 1.2 we need the following lemma.

Lemma 4.1. *Let k be an integer with $k \geq 2$ and χ be a multiplier system of weight $k + \frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$. Then we have*

$$S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \cong S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z})).$$

Proof of Lemma 4.1. First we define a map

$$\phi_\eta : S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \to S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}))$$

by $\phi_\eta(f)(z) = f(z)\eta(z)$ for $f(z) \in S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$. One can see that this map is injective. For surjectivity we note that

$$\eta(z) = q^{\frac{1}{24}} + \cdots$$

and $\eta(z)$ has no zero in $\mathbb{H}$.

TABLE 4.1

Multiplier system	Fourier expansion
χ_0	$a(1)q^1 + a(2)q^2 + a(3)q^3 + \cdots$
χ_1	$a(0)q^{\frac{1}{12}} + a(1)q^{1+\frac{1}{12}} + a(2)q^{2+\frac{1}{12}} + \cdots$
χ_2	$a(0)q^{\frac{2}{12}} + a(1)q^{1+\frac{2}{12}} + a(2)q^{2+\frac{2}{12}} + \cdots$
χ_3	$a(0)q^{\frac{3}{12}} + a(1)q^{1+\frac{3}{12}} + a(2)q^{2+\frac{3}{12}} + \cdots$
χ_4	$a(0)q^{\frac{4}{12}} + a(1)q^{1+\frac{4}{12}} + a(2)q^{2+\frac{4}{12}} + \cdots$
χ_5	$a(0)q^{\frac{5}{12}} + a(1)q^{1+\frac{5}{12}} + a(2)q^{2+\frac{5}{12}} + \cdots$
χ_6	$a(0)q^{\frac{6}{12}} + a(1)q^{1+\frac{6}{12}} + a(2)q^{2+\frac{6}{12}} + \cdots$
χ_7	$a(0)q^{\frac{7}{12}} + a(1)q^{1+\frac{7}{12}} + a(2)q^{2+\frac{7}{12}} + \cdots$
χ_8	$a(0)q^{\frac{8}{12}} + a(1)q^{1+\frac{8}{12}} + a(2)q^{2+\frac{8}{12}} + \cdots$
χ_9	$a(0)q^{\frac{9}{12}} + a(1)q^{1+\frac{9}{12}} + a(2)q^{2+\frac{9}{12}} + \cdots$
χ_{10}	$a(0)q^{\frac{10}{12}} + a(1)q^{1+\frac{10}{12}} + a(2)q^{2+\frac{10}{12}} + \cdots$
χ_{11}	$a(0)q^{\frac{11}{12}} + a(1)q^{1+\frac{11}{12}} + a(2)q^{2+\frac{11}{12}} + \cdots$

For the full modular group $\mathrm{SL}_2(\mathbb{Z})$ all multiplier systems occur in one family $i \mapsto \chi_i$ with parameter $i \in \mathbb{R} \bmod 12\mathbb{Z}$ since it is the order of the abelianization of $\mathrm{SL}_2(\mathbb{Z})$, where $\chi_i = (\chi_\eta)^{2i}$. It is also known that the multiplier system is suitable for weight $k \equiv i \pmod 2$ (for example, see [3, Section 2]). Therefore, we have that if $g(z) \in S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}))$ then $\chi\chi_\eta$ should be χ_i for $i \equiv k \pmod 2$. If $g(z)$ has a Fourier expansion of the form

$$g(z) = \sum_{n+\kappa>0} a(n)e^{2\pi i(n+\kappa)z},$$

then the form of Fourier expansion of $g(z)$ can be obtained by the Table 4.1 according to the multiplier system χ_i. It comes from the fact that $\kappa = \chi_i(T)$ and $\chi_i(T) = e^{2\pi i\frac{i}{24}}$ (for example, see [7, Section 2.8]). Therefore, $\frac{g(z)}{\eta(z)}$ is a cusp form in $S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ and hence the map ϕ_η is surjective. □

Proof of Theorem 1.1. We define a map

$$\mu : S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \to \tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}\Big(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P}\Big)$$

as follows. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we define a non-holomorphic period function

$$p^*_\gamma(g; z) := \overline{\int_{i\infty}^{\gamma^{-1}(i\infty)} g(\tau)\eta(\tau)\overline{\eta(z)}(\tau - \bar{z})^{k-2}d\tau}$$

for $g(z) \in S_{k-\frac{1}{2},\overline{\chi}}(\mathrm{SL}_2(\mathbb{Z}))$. Actually, this function is real-analytic on $\mathbb{H}$. Then for $g(z) \in S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ we define $\mu(g)$ as the cohomology class in $\tilde{H}^1_{\frac{5}{2}-k,\bar{\chi}}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P})$ containing a cocycle $\{p^*_\gamma(g; z) | \ \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$.

By Lemma 4.1 the Dedekind eta function $\eta(z)$ induces an isomorphism

$$\phi_\eta : S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \to S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z})).$$

By the Eichler cohomology theorem for arbitrary real weight modular forms (see Theorem 2.1 in [9]), the space $S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to the cohomology group $\tilde{H}^1_{2-k,\overline{\chi\chi_\eta}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{P})$. Moreover one can see that cohomology groups $\tilde{H}^1_{2-k,\overline{\chi\chi_\eta}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{P})$ and $\tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}(\mathrm{SL}_2(\mathbb{Z}), \eta\mathcal{P})$ are isomorphic by the mapping

$$\{p_\gamma(z) | \ \gamma \in \mathrm{SL}_2(\mathbb{Z})\} \mapsto \{\eta(z)p_\gamma(z) | \ \gamma \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{4.1}$$

Note that this isomorphism is formal and is not related to some specific property of $\eta(z)$. Since the map μ is a composition of the above three isomorphisms, the map μ is also an isomorphism. □

Proof of Theorem 1.2. We define a map

$$\tilde{\mu} : S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \oplus S_{k-\frac{1}{2},\overline{\chi^*_\eta}}(\mathrm{SL}_2(\mathbb{Z})) \to \tilde{H}^1_{\frac{5}{2}-k,\chi^*_\eta}\Big(\mathrm{SL}_2(\mathbb{Z}), \eta P_{k-2}\Big)$$

as follows by using the period functions associated with cusp forms of half-integral weights. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we define a holomorphic period function

$$p_\gamma(f; z) := \int_{i\infty}^{\gamma^{-1}(i\infty)} f(\tau)\eta(\tau)\eta(z)(\tau - z)^{k-2} d\tau$$

for $f(z) \in S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$. Then for $f(z) \in S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z}))$ and $g(z) \in S_{k-\frac{1}{2},\overline{\chi^*_\eta}}(\mathrm{SL}_2(\mathbb{Z}))$ we give a map

$$\tilde{\mu}(f, g) = \alpha(f) + \beta(g),$$

where $\alpha(f)$ is the cohomology class in $\tilde{H}^1_{\frac{5}{2}-k,\chi^*_\eta}(\mathrm{SL}_2(\mathbb{Z}), \eta P_{k-2})$ containing a cocycle $\{p_\gamma(f; z) | \ \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ and $\beta(g)$ is the cohomology class containing a cocycle $\{p^*_\gamma(g; z) | \ \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$.

By Lemma 4.1 we have an isomorphism

$$S_{k-\frac{1}{2},\chi}(\mathrm{SL}_2(\mathbb{Z})) \oplus S_{k-\frac{1}{2},\overline{\chi^*_\eta}}(\mathrm{SL}_2(\mathbb{Z})) \cong S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z})) \oplus S_{k,\overline{\chi\chi_\eta}}(\mathrm{SL}_2(\mathbb{Z})).$$

By the Eichler cohomology theorem for integral weight modular forms (for example, see Corollary 1 in [6]), the space $S_{k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z})) \oplus S_{k,\overline{\chi\chi_\eta}}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to $\tilde{H}^1_{2-k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}), P_{k-2})$. Moreover one can see that cohomology groups $\tilde{H}^1_{2-k,\chi\chi_\eta}(\mathrm{SL}_2(\mathbb{Z}), P_{k-2})$ and $\tilde{H}^1_{\frac{5}{2}-k,\chi^*_\eta}(\mathrm{SL}_2(\mathbb{Z}), \eta P_{k-2})$ are isomorphic by the same map as in (4.1). Combining the above three isomorphisms, one can see that the map $\tilde{\mu}$ is also an isomorphism. □

Table 4.2

Multiplier system	Fourier expansion at 1/2
ψ_1	$b_{\frac{1}{2}}(0)q^{\frac{1}{2}} + b_{\frac{1}{2}}(1)q^{\frac{3}{2}} + \cdots$
ψ_2	$b_{\frac{1}{2}}(1)q^{1} + b_{\frac{1}{2}}(2)q^{2} + \cdots$

As in the proof of Theorem 1.1 and 1.2, we need the following lemma to prove Theorem 1.3 and 1.4.

Lemma 4.2. *Let k be an integer with $k \geq 2$ and $\chi = \chi_\theta^{2k-1}$ be a multiplier system of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$. Then we have*

$$S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \cong S_{k,\chi\chi_\theta}(\Gamma_0(4)).$$

Proof of Lemma 4.2. We define a map

$$\phi_\theta : S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \to S_{k,\chi\chi_\theta}(\Gamma_0(4))$$

by $\phi_\theta(f)(z) = f(z)\theta(z)$ for $f(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$. Then we can see that this map is injective. To show that it is surjective, we use the same argument in the proof of Lemma 4.1. Note that $\Gamma_0(4)$ has three inequivalent cusps $i\infty$, 0 and $\frac{1}{2}$ and $\theta(z)$ has no zero in $\mathbb{H} \cup \{i\infty, 0\}$. At the cusp $\frac{1}{2}$, the theta function $\theta(z)$ has a Fourier expansion of the form

$$\sum_{n=0}^{\infty} a_{\frac{1}{2}}(n)e^{2\pi i(n+\frac{1}{4})z} = a_{\frac{1}{2}}(0)q^{\frac{1}{4}} + a_{\frac{1}{2}}(1)q^{\frac{5}{4}} + \ldots \tag{4.2}$$

Let $\gamma_{\frac{1}{2}}$ be the generator of $\Gamma_{\frac{1}{2}}$ of the form $\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \lambda_{\frac{1}{2}} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$, where $\lambda_{\frac{1}{2}}$ is a positive integer. Then (4.2) comes from the fact that $\kappa_{\frac{1}{2}} = \chi_\theta(\gamma_{\frac{1}{2}})$, $\gamma_{\frac{1}{2}} = \left(\begin{smallmatrix} -1 & 1 \\ -4 & 3 \end{smallmatrix}\right)$ and $\lambda_{\frac{1}{2}} = 1$. If $\chi\chi_\theta = \chi_\theta^{2k}$, then we have $\chi\chi_\theta = \psi_i$ for $i \equiv k \pmod 2$, where $\psi_i = (\chi_\theta)^{2i}$. If $g(z) \in S_{k,\psi_i}(\Gamma_0(4))$ has a Fourier expansion of the form at the cusp $\frac{1}{2}$

$$g(z) = \sum_{n+\kappa>0} b_{\frac{1}{2}}(n)e^{2\pi i(n+\kappa_{\frac{1}{2}})z},$$

then the Fourier expansion of $g(z)$ is given by Table 4.2 because we see that $\kappa_{\frac{1}{2}} = \frac{i}{2}$ and $\lambda_{\frac{1}{2}} = 1$ from (4.2). Therefore, a function $\frac{g(z)}{\theta(z)}$ is a cusp form in $S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$ and the map ϕ_θ is surjective, which completes the proof. □

Proof of Theorem 1.3. The proof is similar with the proof of Theorem 1.1. First we give a map

$$\nu : S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \to \tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}\Big(\Gamma_0(4), \theta\mathcal{P}\Big)$$

by using the period function associated with cusp forms of a half-integral weight. For $\gamma \in \Gamma_0(4)$ and $g(z) \in S_{k-\frac{1}{2},\bar{\chi}}(\Gamma_0(4))$ we put

$$q^*_\gamma(g;z) := \overline{\int_{i\infty}^{\gamma^{-1}(i\infty)} g(\tau)\theta(\tau)\overline{\theta(z)}(\tau-\bar{z})^{k-2}d\tau}.$$

Then for $g(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$ we define $\nu(g)$ as the cohomology class in $\tilde{H}^1_{\frac{5}{2}-k,\bar{\chi}}(\mathrm{SL}_2(\mathbb{Z}), \theta\mathcal{P})$ containing a cocycle $\{q^*_\gamma(g;z)|\ \gamma \in \Gamma_0(4)\}$.

By Lemma 4.2 the theta function $\theta(z)$ induces an isomorphism between $S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$ and $S_{k,\chi\chi_\theta}(\Gamma_0(4))$. We also have an isomorphism between the space $S_{k,\chi\chi_\theta}(\Gamma_0(4))$ and the cohomology group $\tilde{H}^1_{2-k,\overline{\chi\chi_\theta}}(\Gamma_0(4), \mathcal{P})$ by Theorem 2.1 in [9]. Moreover one can see that $\tilde{H}^1_{2-k,\overline{\chi\chi_\theta}}(\Gamma_0(4), \mathcal{P})$ is isomorphic to $\tilde{H}^1_{\frac{5}{2}-k,\overline{\chi}}(\Gamma_0(4), \theta\mathcal{P})$ by the mapping

$$\{p_\gamma(z)|\ \gamma \in \Gamma_0(4)\} \mapsto \{\theta(z)p_\gamma(z)|\ \gamma \in \Gamma_0(4)\}. \tag{4.3}$$

Then we obtain that the map ν is an isomorphism since the map ν is a composition of the above three isomorphisms, which completes the proof. □

Proof of Theorem 1.4. The proof is similar with the proof of Theorem 1.2. We give a map

$$\tilde{\nu} : S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \oplus S_{k-\frac{1}{2},\overline{\chi^*_\theta}}(\Gamma_0(4)) \to \tilde{H}^1_{\frac{5}{2}-k,\chi^*_\theta}(\Gamma_0(4), \theta P_{k-2})$$

as follows. For $\gamma \in \Gamma_0(4)$ we define

$$q_\gamma(f;z) := \int_{i\infty}^{\gamma^{-1}(i\infty)} f(\tau)\theta(\tau)\theta(z)(\tau-z)^{k-2}d\tau$$

for $f(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$. Then for $f(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4))$ and $g(z) \in S_{k-\frac{1}{2},\overline{\chi^*_\theta}}(\Gamma_0(4))$ we give a map

$$\tilde{\nu}(f,g) = \alpha(f) + \beta(g),$$

where $\alpha(f)$ is the cohomology class in $\tilde{H}^1_{\frac{5}{2}-k,\chi^*_\theta}(\Gamma_0(4), \theta P_{k-2})$ containing a cocycle $\{q_\gamma(f;z)|\ \gamma \in \Gamma_0(4)\}$ and $\beta(g)$ is the cohomology class containing a cocycle $\{q^*_\gamma(g;z)|\ \gamma \in \Gamma_0(4)\}$.

Lemma 4.2 implies that the function $\theta(z)$ induces the following isomorphism

$$S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \oplus S_{k-\frac{1}{2},\overline{\chi^*_\theta}}(\Gamma_0(4)) \cong S_{k,\chi\chi_\theta}(\Gamma_0(4)) \oplus S_{k,\overline{\chi\chi_\theta}}(\Gamma_0(4)).$$

By Corollary 1 in [6] the space $S_{k,\chi\chi_\theta}(\Gamma_0(4)) \oplus S_{k,\overline{\chi\chi_\theta}}(\Gamma_0(4))$ is isomorphic to $\tilde{H}^1_{2-k,\chi\chi_\theta}(\Gamma_0(4), P_{k-2})$. Moreover one can see that $\tilde{H}^1_{2-k,\chi\chi_\theta}(\Gamma_0(4), P_{k-2})$ is isomorphic to $\tilde{H}^1_{\frac{5}{2}-k,\chi^*_\theta}(\Gamma_0(4), \theta P_{k-2})$ by the same mapping as in (4.3).

Then since the map $\tilde{\nu}$ is a composition of the above three isomorphisms, the map $\tilde{\nu}$ is also an isomorphism. $\square$

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