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Résumé. Soient E une série d'Eisenstein et f une forme modulaire parabolique, de même niveau N. Supposons que E et f soient vecteurs propres pour les opérateurs de Hecke, et qu'ils soient tous les deux normalisés de sorte que $a_1(f) = a_1(E) = 1$. Le résultat principal de cet article est le suivant : si E et f sont congruents modulo un idéal premier $\mathfrak{p} \mid p$, alors les valeurs spéciales des fonctions $L(E, \chi, j)$ et $L(f, \chi, j)$ sont également congruentes modulo $\mathfrak{p}$. Plus précisement, on montre que

$$\frac{\tau(\bar{\chi})L(f,\chi,j)}{(2\pi i)^{j-1}\Omega_f^{\mathrm{sgn}(E)}} \equiv \frac{\tau(\bar{\chi})L(E,\chi,j)}{(2\pi i)^j\Omega_E} \pmod{\mathfrak{p}}$$

où le signe $\mathrm{sgn}(E)$ est ± 1 et ne dépend que de E, et $\Omega_f^{\mathrm{sgn}(E)}$ est la période canonique de f. Ici χ désigne un caractère primitif de Dirichlet de conducteur m, $\tau(\bar{\chi})$ une somme de Gauss, et j un entier tel que $0 < j < k$ et $(-1)^{j-1} \cdot \chi(-1) = \mathrm{sgn}(E)$. Enfin, Ω_E est une unité $\mathfrak{p}$-adique indépendante de χ et de j. Ce résultat est une généralisation des travaux de Stevens et Vatsal en poids $k = 2$.

Dans cet article on construit le symbole modulaire de E, et on calcule les valeurs spéciales. La dernière section conclut avec des exemples numériques du théorème principal.

Abstract. Let E and f be an Eisenstein series and a cusp form, respectively, of the same weight $k \geq 2$ and of the same level N, both eigenfunctions of the Hecke operators, and both normalized so that $a_1(f) = a_1(E) = 1$. The main result we prove is that when E and f are congruent mod a prime $\mathfrak{p}$ (which we take in this paper to be a prime of $\overline{\mathbb{Q}}$ lying over a rational prime $p > 2$), the algebraic parts of the special values $L(E, \chi, j)$ and $L(f, \chi, j)$ satisfy congruences mod the same prime. More explicitly, we prove that, under certain conditions,

$$\frac{\tau(\bar{\chi})L(f,\chi,j)}{(2\pi i)^{j-1}\Omega_f^{\mathrm{sgn}(E)}} \equiv \frac{\tau(\bar{\chi})L(E,\chi,j)}{(2\pi i)^j\Omega_E} \pmod{\mathfrak{p}}$$

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where the sign of E is ± 1 depending on E, and $\Omega_f^{\mathrm{sgn}(E)}$ is the corresponding canonical period for f. Also, χ is a primitive Dirichlet character of conductor m, $\tau(\bar{\chi})$ is a Gauss sum, and j is an integer with $0 < j < k$ such that $(-1)^{j-1} \cdot \chi(-1) = \mathrm{sgn}(E)$. Finally, Ω_E is a $\mathfrak{p}$-adic unit which is independent of χ and j. This is a generalization of earlier results of Stevens and Vatsal for weight $k = 2$.

In this paper we construct the modular symbol attached to an Eisenstein series, and compute the special values. We give numerical examples of the congruence theorem stated above, and in the penultimate section we give the proof of the congruence theorem.

1. Introduction

The idea that congruences between modular forms should carry over to congruences in the special values of their L-functions began with the work of Mazur, who studied the case of prime level and the congruences between Eisenstein series and cusp forms. His results were subsequently generalized by Stevens [16], and then refined by the second of the present authors, who also treated the case of congruences between cusp forms [18]. The paper [18] contains other congruence theorems for cusp forms of higher weight, but the case of congruences between higher-weight cusp forms and higher-weight Eisenstein series was left open, as was the case of congruences between cusp forms at primes for which the corresponding Galois representation is Eisenstein (reducible), and it is these gaps that we propose to close.

To explain our results, let us introduce some notation. Let Γ denote a congruence subgroup of $SL_2(\mathbf{Z})$ and let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of degree zero divisors on the rational cusps of the upper half-plane. If A is any Γ-module, an A-valued modular symbol over Γ is a Γ-equivariant homomorphism $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to A$. An A-valued boundary symbol over Γ is a Γ-equivariant homomorphism $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q})) \to A$, where $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ is the group of all divisors on the rational cusps. Let $S_\Gamma(A)$ and $B_\Gamma(A)$ denote the groups of A-valued modular symbols and boundary symbols over Γ respectively. Then according to [5], Section 4, there is an exact sequence as follows:

$$0 \to B_\Gamma(A) \to S_\Gamma(A) \to H^1_p(\Gamma, A) \to A$$

where $H^1_p(\Gamma, A)$ denotes the parabolic coholomogy group of Eichler and Shimura. If f is a cusp form on Γ, then there is a standard modular symbol M_f associated to f given by integration of a (vector-valued) differential form associated to f, and the values of this modular symbol are related to special values of L-functions.

The main idea (which goes back to Mazur) for proving congruences between the special values of L-functions of cusp forms runs as follows, and may be easily explained in the case of cusp forms of weight 2 and rational coefficients, when the coefficient module is simply $\mathbf{C}$ and the action of Γ is trivial. One knows (by work of Shimura) that one can write $M_f = \Omega_f^+ N_f^+ + \Omega_f^- N_f^-$, where $\Omega_f^\pm$ are certain complex numbers, and $N_f^\pm$ are modular symbols with values in $\mathbf{Z} \subset \mathbf{C}$. If g is another cusp form, one gets in the same manner another pair of modular symbols $N_g^\pm$, again with values in $\mathbf{Z}$. Since everything is $\mathbf{Z}$-valued, one can simply reduce modulo p to obtain modular symbols $\overline{N}_f^\pm$ with values in $\mathbf{F}_p$, and similarly for g. Now one can apply the exact sequence above with $A = \mathbf{F}_p$. Then one observes that the modular symbols $\overline{N}_f^\pm$ and $\overline{N}_g^\pm$ both map to $H_p^1(\Gamma, \mathbf{F}_p)^\pm$. By choosing the scalars $\Omega_f^\pm$ and $\Omega_g^\pm$ appropriately, we can arrange for the images of these elements to be nonzero modulo p.

Note now that all these elements land inside the subspace of the cohomology group where the action of the Hecke algebra is given by the Hecke eigenvalues of f and g respectively. In particular, when there is a congruence between f and g, the eigenvalues are the *same* modulo p. Thus, if we know that the corresponding eigenspace of $H_p^1(\Gamma, \mathbf{F}_p)^\pm$ has dimension 1 ("multiplicity one") then the images of $\overline{N}_f^\pm$ and $\overline{N}_g^\pm$ are equal up to scaling by a fixed nonzero constant. In other words, the images of the modular symbols of f and g are themselves congruent, if a suitable scaling is applied!

Since the values of the modular symbols of f and g are related to their L-values, one would like to conclude that the special values of f and g are equal, modulo p. This is essentially correct, but notice that there is a delicate point here that must be addressed. The values in question are computed as values of the modular symbols $\overline{N}_f^\pm$ and $\overline{N}_g^\pm$, which lie in $S_\Gamma(\mathbf{F}_p)$. However, the multiplicity one theorem is only valid for the cohomology group $H_p^1(\Gamma, \mathbf{F}_p)$, and one cannot conclude (and indeed it may not even be true in general) that the modular symbols $\overline{N}_f^\pm$ and $\overline{N}_g^\pm$ are equal up to constant: one can only conclude that they are equal up to constant and some unknown element that maps to zero in cohomology, namely, up to some $\mathbf{F}_p$- valued boundary symbol. This appears to be rather unfortunate, but happily the situation can be salvaged by observing that the boundary symbol actually evaluates to zero on the divisors of interest, since, as is well known, the divisors in question are homologically trivial under a mild hypothesis, and therefore have trivial boundary in the completed modular curve X_Γ. Observe also that one cannot circumvent this issue by splitting the boundary exact sequence above, since in general we will be dealing with Eisenstein primes, and that the sequence has no splitting at such primes (Manin-Drinfeld).

Our task is therefore to generalize this procedure to Eisenstein series and forms of higher weight. However, we run in to two main problems that have to be solved. The first of these is to produce a modular symbol for a higher weight Eisenstein series that has suitable rationality and integrality properties. Thus we first construct a modular symbol attached to an Eisenstein series, assuming the presence of a congruence modulo p with a cuspform. The modular symbol we define takes values in a certain module over $\mathbf{F}_p[\Gamma]$, and is built on the maps used in [17], Example 6.4(a). With the modular symbol in hand, one can attempt to imitate the argument based on multiplicity one that we have sketched above. Here again one must deal with the fact that the map from modular symbols to cohomology does not split. In the higher weight case there is a further complication coming from the fact that we are dealing with non-constant coefficients, so that an arbitrary boundary symbol does not necessarily evaluate to zero on a divisor with boundary zero in the completed modular curve. We get around this issue by showing that all boundary symbols nevertheless have the property that their twisted special values are zero (Theorem 4.5). While this result is rather simple and the proof entirely elementary, it is nevertheless rather important, and seems not to have been remarked previously.

The organization of this paper is as follows. The material in Section 2 is mostly a review of results already in the literature: first we define some functions related to special values of L-functions attached to modular forms, and we state some basic properties of those functions. In Section 3 we define the modular symbol attached to an Eisenstein series, and we prove some basic properties of it in more generality than just the weight 2 case. In Section 4 we calculate the special values of this modular symbol, and relate them to the character twists of the corresponding L-functions. In Section 5, we give the proof of the congruence theorem for the special values of character twists of a cusp form and a congruent Eisenstein series. In Section 6, we give some numerical examples.

The authors would like to thank Samit Dasgupta for some useful suggestions. We remark also that result similar to our main congruence theorem has recently been announced by Y. Hirano [8]; his result is more general than ours in that it includes characters whose conductor may be divisible by p. However, the principal idea is very similar. We also thank Hirano for pointing out a mistake in an earlier version of the paper.

2. Preliminary Results

2.1. Functions Connected to Special Values. Before we can state the main results later on, we need to define some functions connected to special values of L-functions and prove some results about them. The results in this section hold for general modular forms; throughout this section let f be a modular form of weight $k \geq 2$ and any level.

Let A be a ring, and let $L_n(A)$ (for a nonnegative integer n) be the symmetric polynomial algebra over A of degree n. (Thus the elements of $L_n(A)$ are homogeneous polynomials of degree n with coefficients in A.) Throughout what follows, we will always take A to be a subring of $\mathbb{C}$.

$L_n(\mathbb{C})$, for any nonnegative integer n, admits a left action of $\mathrm{GL}_2^+(\mathbb{Q})$: if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, and $P(X,Y) \in L_n(\mathbb{C})$, then

$$\alpha | P(X,Y) = \det(\alpha)^{-n} \cdot P(aX + cY, bX + dY)$$

We will make frequent use of this action below.

Throughout what follows we will always put $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Also, let

$$D_f(s) := \int_0^\infty \tilde{f}(z) y^{s-1} dz$$

where the tilde means that we subtract $a_0(f)$, and y is the imaginary part of z. This integral will converge whenever we take s with real part big enough (it depends on the weight). The main point is:

Proposition 2.1. *In the region of convergence of the integral,*

$$D_f(s) = i \cdot \Gamma(s) \cdot (2\pi)^{-s} \cdot L(f,s)$$

This identity links the above integral to the L-function of f. For the proof, see [12], p. I-5.

The next few results stated below are generalizations of weight 2 results that can be found in [16], Chapter 2—in particular see Propositions 2.1.2 and 2.2.2. The proofs in higher weight are adaptations of the proofs used there; the modifications are simple and we omit them.

Proposition 2.2. *Within the region of convergence of the integral, we have the formula*

$$D_f(s) = \int_i^\infty \tilde{f}(z) y^{s-1} dz + i^k \int_i^\infty (\widetilde{f|\sigma})(z) y^{k-1-s} dz$$
$$-i \left(\frac{a_0(f)}{s} + i^k \frac{a_0(f|\sigma)}{k-s} \right)$$

Furthermore, this formula defines a meromorphic continuation of $D_f(s)$ to the entire complex plane, with functional equation

$$D_f(s) = i^k D_{f|\sigma}(k-s)$$

With this in mind, let us now define a new function that takes values in $L_n(\mathbb{C})$:

$$F_f(s) := \int_0^\infty \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz$$

This integral will clearly converge for large enough values of s.

Proposition 2.3. *Within the region of convergence of the integral, we have the formula*

$$F_f(s) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot D_f(s+j) \cdot X^j Y^{k-2-j}$$

Furthermore, this defines a meromorphic continuation of $F_f(s)$ to the entire complex plane.

Corollary 2.4. *We have*

$$F_f(s) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^{j+1} \cdot \Gamma(s+j) \cdot \tfrac{1}{(2\pi)^{s+j}} \cdot L(f, s+j) \cdot X^j Y^{k-2-j}$$

and this holds for all s.

The corollary shows that

$$F_f(1) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^{j+1} \cdot (j!) \cdot \tfrac{1}{(2\pi)^{j+1}} \cdot L(f, j+1) \cdot X^j Y^{k-2-j}$$

In other words, $F_f(1)$ is a polynomial whose coefficients encode all the special values of the L-function of f at the so-called *critical integers*, namely those strictly between 0 and k.

Now we return to Proposition 2.3. There is another formula as well, and we state it here as a separate result.

Proposition 2.5. *We have the formula*

$$F_f(s) = \int_i^\infty \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz - \int_i^\infty (\widetilde{f|\sigma})(z)[\sigma|(zX+Y)^{k-2}]y^{1-s}dz$$

$$+ia_0(f|\sigma) \cdot \sigma| \left(\sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot \tfrac{1}{2-s+j} \right)$$

$$-ia_0(f) \cdot \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot \tfrac{1}{s+j}$$

Corollary 2.6. *For an arbitrary base point z_0 in the upper half-plane,*

$$F_f(1) = \int_{z_0}^{\infty} \tilde{f}(z)(zX+Y)^{k-2}dz - \int_{z_0}^{\infty} (\widetilde{f|\sigma})(z)[\sigma|(zX+Y)^{k-2}]dz$$
$$-a_0(f)\cdot\int_0^{z_0}(zX+Y)^{k-2}dz + a_0(f|\sigma)\cdot\int_0^{z_0}\sigma|(zX+Y)^{k-2}dz$$
$$-\int_{z_0}^{\sigma z_0} f(z)(zX+Y)^{k-2}dz$$

We now state two more lemmas which will be used in later sections.

Lemma 2.7. *Put $\alpha := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with rational entries and positive determinant. Then*

$$a_0(f|\alpha) = \frac{a^{k-1}}{d}\cdot a_0(f)$$

Lemma 2.8. *Put $\alpha := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, all with rational entries and positive determinant. Then*

$$F_{f|\alpha\tau}(1) = \tau^{-1}|F_{f|\alpha}(1)$$

Finally, we state need one last lemma, which is a generalization of [16], Lemma 3.1.1. Once again, the proof is standard.

Lemma 2.9. *Let f be a modular form of weight $k \geq 2$, and let χ be a primitive Dirichlet character of conductor m. Then*

$$\tau(\bar{\chi})D_{f\otimes\chi}(s) = m^{1-s}\sum_{a=0}^{m-1}\bar{\chi}(a)D_{f|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(s)$$

2.2. Modular Symbols and Cusp Forms. In this section, we let f be a normalized (meaning $a_1 = 1$) cuspidal eigenform of even weight $k \geq 2$ and level Γ for some congruence subgroup Γ. Our goal in this section is to define a modular symbol M_f attached to f and show a link between M_f and the algebraic parts of special values $L(f,\chi,j)$ for a primitive character χ and an integer j with $1 \leq j \leq k-1$. (The meaning of "algebraic part" will be explained below.) The discussion in this section will closely follow that of [5], Section 4.

We first state the definition of a modular symbol, along with two other definitions that we will use below:

Definition 2.10. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of degree zero divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z})\cap \mathrm{GL}_2^+(\mathbb{Q})]$-module (with the matrices acting on the left). We refer to a map as an A-valued *modular symbol* over a congruence subgroup Γ if the map is a Γ-homomorphism from degree zero divisors to elements of A.

Definition 2.11. Let f be as above. Then the *standard weight k modular symbol* M_f is the $L_{k-2}(\mathbb{C})$-valued modular symbol defined as follows: on divisors $\{b\}-\{a\}$ (with $a,b\in\mathbb{P}^1(\mathbb{Q})$),

$$M_f(\{b\}-\{a\}) := 2\pi i\int_a^b f(z)(zX+Y)^{k-2}dz$$

Define M_f on all other degree-zero divisors by linearity.

Definition 2.12. Let χ be a primitive Dirichlet character of conductor m. Let Φ by any A-valued modular symbol. The operator R_χ, called the *twist operator*, is defined as follows: for any degree-zero divisor D,

$$(\Phi|R_\chi)(D) := \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}^{-1}|\Phi(\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}D)$$

Remark 2.13. This operator also appears in Section 4 of [5], but our definition is slightly different. This is because our matrix action (of $\mathrm{GL}_2(\mathbb{Q}$ on $L_{k-2}(\mathbb{C})$ is a left action. If we switch to a right action by inverting, this is the same action that Greenberg and Stevens use. So here we adjust to the fact that we are using a left action by using $\bar{\chi}$ in the definition instead of χ, and changing from a right action of $\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}$ to a left action of the inverse.

Now we can state a result concerning the special values of the L-function of f:

Theorem 2.14. *Let f be as above, and let χ be a primitive Dirichlet character of conductor m. Then*

$$(M_f|R_\chi)(\{\infty\}-\{0\}) = 2\pi i\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{f\otimes\chi}(1+j)\cdot X^jY^{k-2-j}$$

This theorem is equivalent to [5], Theorem 4.14 (after adjusting the notation).

Corollary 2.15. *With f and χ as in the above theorem,*

$$(M_f|R_\chi)(\{\infty\}-\{0\}) = \sum_{j=0}^{k-2}(-1)^{j+1}\binom{k-2}{j}\cdot j!\cdot m^j\cdot\tfrac{\tau(\bar{\chi})L(f,\chi,1+j)}{(2\pi i)^j}\cdot X^jY^{k-2-j}$$

Proof. Combine the theorem with Proposition 2.1. □

The above corollary gives a connection between M_f and $L(f, \chi, j)$; but so far we do not have any assurances of any algebraicity properties of either. To show how we get algebraic numbers from the modular symbol M_f, we define an involution on modular symbols induced by the action of the matrix $\iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which sends

$$M_f(\{b\} - \{a\}) \mapsto \iota | M_f(\{-b\} - \{-a\})$$

(the left action of ι on polynomials simply sends $Y \mapsto -Y$). Now choose a "sign" $\pm$—meaning the $+1$ or -1 eigenspace of this involution—and project the modular symbol M_f to one of these eigenspaces. We obtain a new modular symbol which we will denote $M_f^{\pm}$ (for one choice of sign). It is a theorem of Shimura (proved in [6], or also see [5], [9], or [18]) that there exist transcendental numbers $\Omega_f^{\pm}$, called *periods*, such that the modular symbols $\frac{1}{\Omega_f^{\pm}} M_f^{\pm}$ both give values in $L_{k-2}(K)$, where K is the algebraic field extension generated over $\mathbb{Q}$ by the Hecke eigenvalues of f. Shimura's theorem even tells us, for a primitive character χ and a critical integer j, which sign to choose so that the number

$$\frac{\tau(\bar{\chi}) L(f, \chi, j)}{(2\pi i)^{j-1} \Omega_f^{\pm}}$$

is algebraic. (The choice of sign is $(-1)^{j-1}\mathrm{sgn}(\chi)$.) For that choice of sign, the above expression is called the *algebraic part* of $L(f, \chi, j)$. *A priori*, this algebraic part is only defined up to a factor in $K^{\times}$. However, since $O_{K,\mathfrak{p}}$ is a discrete valuation ring, we may normalize up to a unit in $O_{K,\mathfrak{p}}^{\times}$ by requiring that the algebraic part of $L(f, \chi, j)$ lies in $O_{K,\mathfrak{p}}$ for all χ and j , and that at least one algebraic part lies in $O_{K,\mathfrak{p}}^{\times}$. Of course, if O_K is a principal ideal domain, then we can normalize up to a factor in $O_K^{\times}$.

3. Modular Symbols and Eisenstein Series

3.1. A Basis of Eisenstein Series. We begin with a definition of our basic Eisenstein series ϕ_{k,x_1,x_2}. Following [17], Section 6, pick a positive integer $k > 2$ (unlike in that paper, here we do not assume k is even) and let $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$, and define

$$G_{k,x_1,x_2}(z) := \frac{(k-1)!}{(2\pi i)^k} \sum_{\substack{(a_1,a_2) \in \mathbb{Q} - (0,0) \\ (a_1,a_2) \equiv (x_1,x_2) \pmod{\mathbb{Z}}}} (a_1 z + a_2)^{-k}$$

This series converges absolutely and defines a holomorphic Eisenstein series of weight k. Define ϕ_{k,x_1,x_2} as follows. Let N be the least common denominator of x_1 and x_2 and consider the map

$$\psi_{x_1,x_2} : (\tfrac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \to \mathbb{C}^\times$$

defined by

$$\psi_{x_1,x_2}(\tfrac{a_1}{N}, \tfrac{a_2}{N}) = e^{2\pi i(a_2x_1 - a_1x_2)}$$

Then define

$$\phi_{k,x_1,x_2}(z) = \sum_{(a_1,a_2)\in(\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2} \psi_{x_1,x_2}(a_1,a_2)G_{k,a_1,a_2}(z)$$

ϕ_{k,x_1,x_2} can also be understood as the Fourier transform of the distribution $f \mapsto G_f$, in the sense of [17], Definition 3.6 (and the beginning of Section 4). A basic fact is that ϕ_{k,x_1,x_2} is a modular form of weight k and level $\Gamma(N)$.

Our first goal is to study the special values of the L-functions attached to these Eisenstein series. First we quickly define the periodified Bernoulli functions. Let $\tilde{B}_k(x)$ be the k-th Bernoulli polynomial for $k \geq 0$. If $\lfloor\cdot\rfloor$ is the greatest integer function on real numbers, then define

$$B_k(x) = \tilde{B}_k(x - \lfloor x \rfloor)$$

Now we can state our first result:

Proposition 3.1. *ϕ_{k,x_1,x_2} has Fourier expansion*

$$\phi_{k,x_1,x_2}(z) = \frac{B_k(x_1)}{k} - J(k,x_1,x_2;z) - (-1)^k J(k,-x_1,-x_2;z)$$

where

$$J(k,a,b;z) := \sum_{\substack{\kappa\equiv a \pmod 1\\ \kappa\in\mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} e^{2\pi i z m\kappa} e^{2\pi i m b}$$

For the proof, see [14]. (It is useful to notice that $J(k,a,b;z)$ is uniformly convergent for z in the upper half-plane.)

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with all integer entries and positive determinant; then we can use the distribution law given in ([17], equation 3.9) to conclude

$$\phi_{k,x_1,x_2}|\alpha^{-1} = \det(\alpha)^{-k+2} \sum_{\substack{y=(y_1,y_2)\in(\mathbb{Q}/\mathbb{Z})^2\\ y\alpha\equiv x \pmod{\mathbb{Z}}}} \phi_{k,y_1,y_2} \tag{3.1}$$

If α has determinant 1, this specializes to

$$\phi_{k,x_1,x_2}|\alpha = \phi_{k,ax_1+cx_2,bx_1+dx_2}$$

Remark 3.2. Any element of $\mathrm{GL}_2^+(\mathbb{Q})$ can be written as a scalar matrix times the inverse of a matrix with integral entries, so this also shows how to evaluate $\phi_{k,x_1,x_2}|\alpha$ for any matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. As an example, we will compute, for a general ϕ_{k,x_1,x_2}, the action of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. To begin, we write the matrix as

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

By the definition of matrices acting on modular forms, the action of the scalar matrix is simply to multiply by d^{k-2}. Now we can use (3.1) directly on the second matrix:

$$\phi_{k,x_1,x_2}| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} = d^{-k+2} \sum_{\nu=0}^{d-1} \phi_{k,\frac{x_1+\nu}{d},x_2}$$

Since we also have a scalar multiple of d^{k-2} from the action of the scalar matrix, the final result is

$$\phi_{k,x_1,x_2}| \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \sum_{\nu=0}^{d-1} \phi_{k,\frac{x_1+\nu}{d},x_2}$$

This example will be used later.

Define, for $x \in \mathbb{Q}/\mathbb{Z}$,

$$Z(s,x) := \sum_{n=1}^{\infty} e^{2\pi i n x} n^{-s}$$

and

$$\zeta(s,x) := \sum_{\substack{m \equiv x \pmod 1 \\ m \in \mathbb{Q}^+}} m^{-s}$$

(Clearly these functions are well-defined for $x \in \mathbb{Q}/\mathbb{Z}$.) Now we have:

Proposition 3.3. *$L(\phi_{k,x_1,x_2}, s) = -\zeta(1-(k-s), x_1)Z(s,x_2) - (-1)^k\zeta(1-(k-s), -x_1)Z(s,-x_2)$.*

Proof. Consider $J(k,a,b;z)$, defined above. If we put $q = e^{2\pi i z}$, then

$$J(k,a,b;z) = \sum_{\substack{\kappa \equiv a \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} q^{m\kappa} e^{2\pi i m b}$$

So its L-function is

$$\sum_{\substack{\kappa\equiv a \pmod 1\\ \kappa\in\mathbb{Q}^+}} \kappa^{k-1}\cdot\sum_{m=1}^{\infty}(m\kappa)^{-s}e^{2\pi imb}$$

$$= \sum_{\substack{\kappa\equiv a \pmod 1\\ \kappa\in\mathbb{Q}^+}} \kappa^{k-1-s}\cdot\sum_{m=1}^{\infty}m^{-s}e^{2\pi imb}$$

$$= \zeta(1-(k-s),a)Z(s,b)$$

Now the result follows from the above proposition. □

We will need the following three properties of the two functions defined above:

Proposition 3.4. *For any positive integer n and any x as above (and B_n as above),*

$$\zeta(1-n,x) = -\frac{B_n(x)}{n}$$

This is a well-known property of the Hurwitz zeta function. See, for example, [11], p. 341.

Proposition 3.5. *With n and x as above, unless $x\in\mathbb{Z}$ and $n=1$,*

$$\zeta(1-n,-x) = (-1)^n\zeta(1-n,x)$$

This follows from a well-known property of the Bernoulli polynomials. See, for example, [11], equations B.10 and B.13.

Proposition 3.6. *With n and x as above, unless $x\in\mathbb{Z}$ and $n=1$,*

$$Z(n,x)+(-1)^nZ(n,-x) = -i^n\cdot(2\pi)^n\cdot\Gamma(n)^{-1}\cdot\frac{B_n(x)}{n}$$

This follows from the definition of $Z(n,x)$ and from the Fourier expansions of the periodified Bernoulli polynomials (which can be found in [13], p. 16).

Now we will use these facts to prove a result about the polynomial $F_E(1)$ when E is of the form ϕ_{k,x_1,x_2}.

Proposition 3.7. *Let E be equal to ϕ_{k,x_1,x_2} for some integer $k>2$ and some $x_1,x_2\in(\mathbb{Q}/\mathbb{Z})^2$. Then for any integer j with $0\le j\le k-2$, the coefficient of the X^jY^{k-2-j} term in $F_E(1)$ is*

$$\binom{k-2}{j}(-1)^j\frac{B_{k-j-1}(x_1)}{k-j-1}\cdot\frac{B_{j+1}(x_2)}{j+1}$$

except in the following cases. When $x_1 = 0$, and k is even, the coefficient of X^{k-2} will be

$$i^{k+1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}}$$

and when $x_1 = 0$ and k is odd, the coefficient of X^{k-2} is

$$i^{k+2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n x_2)}{n^{k-1}}.$$

When $x_2 = 0$ and k is even, the coefficient of Y^{k-2} will be

$$i^{k+3}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_1))}{n^{k-1}}$$

and when $x_2 = 0$ and k is odd, the coefficient of Y^{k-2} will be

$$i^{k}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(-x_1))}{n^{k-1}}$$

Remark 3.8. We have excluded the case $k = 2$ from the result above, but it is treated in [16], Section 2.5. The non-exceptional cases yield the same formula as the above proposition when $k = 2$, but the exceptional cases are different (and when $k = 2$ we exclude the case $x_1 = x_2 = 0$ entirely).

Proof. First we will deal with the exceptional cases. The first is when k is even, $j = k - 2$ and $x_1 = 0$. In this case, we will have

$$L(E, k-1) = -\tfrac{1}{2}(Z(k-1, x_2) + Z(k-1, -x_2)) = -\sum_{n=1}^{\infty} \cos(2\pi n x_2) n^{1-k}$$

and the claim follows. (We are computing the coefficient using Corollary 2.4). If k is odd instead of even, the second exceptional case is proved using a similar calculation.

The last two cases are when $j = 0$ and $x_2 = 0$. Here we can simply use the functional equation at the end of Proposition 2.2 and then this reduces to the same computations as in the first case.

It only remains to show the general case. We are looking to compute $L(E, j+1)$ for $0 \le j \le k-2$, and all three of the above identities apply. Starting from Proposition 3.3, we begin by applying Proposition 3.5 to the Hurwitz zeta functions. Then we take out a factor of $-\zeta(1-(k-j-1), x_1)$, which is equal to $\frac{B_{k-j-1}(x_1)}{k-j-1}$ by Proposition 3.4. Finally we apply Proposition 3.6 to the sum or difference of $Z(j+1, x_2)$ and $Z(j+1, -x_2)$ terms. When we combine the results with the formula in Corollary 2.4, the factors of 2π and the gamma factors cancel; collecting all the powers of i, we obtain exactly the desired result. □

An important fact that is immediately implied by the result above is the following:

Corollary 3.9. *Let E be of the form ϕ_{k,x_1,x_2} as above. Then the real part of $F_E(1)$ is rational.*

Before we continue, we introduce one last definition. Given any number field K, we let $\mathcal{E}_k(K)$ be the K-span of the Eisenstein series ϕ_{k,x_1,x_2} for all $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$.

3.2. The Map S_E.

3.2.1. *Definition and Basic Properties.* Given a field K and an Eisenstein series $E \in \mathcal{E}_k(K)$, our immediate goal is to define a map which takes as input an element of $GL_2^+(\mathbb{Q})$ and outputs an element of $L_{k-2}(K)$. The purpose of this map will be to help us define a modular symbol attached to E in terms of outputs of this map, as we will show later. To that end we define the map S_E, which does not quite give a K-rational polynomial in all cases, but after proving some basic properties of S_E we will be able to define a new map which does give K-rational polynomials.

Corollary 2.6, proved above, leads us to the following definition:

Definition 3.10. Define $S_E : \mathrm{GL}_2^+(\mathbb{Q}) \to L_n(\mathbb{C})$ by

$$S_E(\alpha) := \int_{z_0}^{\alpha z_0} E(z)(zX+Y)^{k-2}dz$$

$$+a_0(E) \cdot \int_0^{z_0} (zX+Y)^{k-2}dz - a_0(E|\alpha) \cdot \int_0^{z_0} \alpha|(zX+Y)^{k-2}dz$$

$$-\int_{z_0}^{\infty} \tilde{E}(z)(zX+Y)^{k-2}dz + \int_{z_0}^{\infty} (\widetilde{E|\alpha})(z)[\alpha|(zX+Y)^{k-2}]dz$$

Notice that this is well-defined because, as we can easily check, the derivative with respect to z_0 is 0, so this definition does not depend on the choice of z_0. Notice also (directly from the definition) that we can write E as a linear combination of Eisenstein series of the form ϕ_{k,x_1,x_2} and the map S_E will respect the linearity.

It is an immediate consequence of Corollary 2.6 that

$$S_E(\sigma) = -F_E(1) \tag{3.2}$$

We now prove some basic properties of S_E. Both of the next two results were proved for the case $k = 2$ in Proposition 2.3.3 of [16].

Proposition 3.11. *S_E satisfies the relation*

$$S_E(\alpha\beta) = S_E(\alpha) + \alpha|S_{E|\alpha}(\beta)$$

Proof. If we consider the last four terms in the definition of S_E, it is a simple calculation to show that grouped together without the first term, they satisfy the relation. (We need to use the fact that the action of α on the polynomials inside the integrals commutes with integration, which we know since the integrals are absolutely convergent.) But the first term also satisfies this relation; to see this it suffices to show the identity

$$\int_{z_0}^{\beta z_0} (E|\alpha)(z)[\alpha|(zX+Y)^{k-2}]dz = \int_{\alpha z_0}^{\alpha\beta z_0} E(z)(zX+Y)^{k-2}dz$$

This is a straightforward calculation using the substitution $u = \alpha z$ on the left-hand integral (along with the definition of $\alpha^{-1}|(uX+Y)^{k-2}$). ☐

Theorem 3.12. *Put* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *(with all rational entries and positive determinant) and* $M_\alpha = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}$. *Then* S_E *satisfies the following formula: if* $c = 0$, *then*

$$S_E(\alpha) = a_0(E) \int_0^{b/d} (tX+Y)^{k-2}dt$$

If $c > 0$, *then*

$$S_E(\alpha) = a_0(E) \int_0^{a/c} (tX+Y)^{k-2}dt$$
$$+a_0(E|\alpha) \int_{-d/c}^{0} \alpha|(tX+Y)^{k-2}dt$$
$$-M_\alpha|F_{E|M_\alpha}(1)$$

Proof. We begin with the case $c = 0$. Starting from the definition of $S_E(\alpha)$, we split the first integral and obtain

$$S_E(\alpha) = \int_{z_0}^{\alpha z_0} \tilde{E}(z)(zX+Y)^{k-2}dz + a_0(E) \int_{z_0}^{\alpha z_0} (zX+Y)^{k-2}dz$$
$$+a_0(E) \cdot \int_0^{z_0} (zX+Y)^{k-2}dz - a_0(E|\alpha) \cdot \int_0^{z_0} \alpha|(zX+Y)^{k-2}dz$$
$$-\int_{z_0}^{\infty} \tilde{E}(z)(zX+Y)^{k-2}dz + \int_{z_0}^{\infty} (\widetilde{E|\alpha})(z)[\alpha|(zX+Y)^{k-2}]dz$$

Now since this does not depend on z_0, as explained above, we let $z_0 \to i\infty$. Since $c = 0$, this means $\alpha z_0 \to i\infty$ as well. So all the integrals that converge in this case—namely, the first one and the last two—will vanish, and we only need to treat the other three. The goal is to show that they combine to give a polynomial not dependent on z_0, and that this polynomial is the one given above.

We can combine the first two remaining integrals to conclude that the expression we need to find the limit of is

$$a_0(E)\int_0^{\alpha z_0}(zX+Y)^{k-2}dz - a_0(E|\alpha)\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz$$

To prove the formula in this case, it suffices to show that

$$-a_0(E|\alpha)\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz = a_0(E)\int_{\alpha z_0}^{b/d}(uX+Y)^{k-2}du$$

for then we could combine the two integrals and obtain the desired result immediately. But this is simple to show: firstly, we use Lemma 2.7 to replace $a_0(E|\alpha)$ with $\frac{a^{k-1}}{d}a_0(E)$. Then we use the definition of $\alpha|(zX+Y)^{k-2}$ (along with the fact that the determinant of α is ad) along with the substitution $u = \alpha z = \frac{az+b}{d}$. From there an elementary calculation shows the desired result.

We now turn to the case $c > 0$, having already proved the $c = 0$ case (which we will use below). We begin from the identity

$$\alpha = \begin{pmatrix} 1/c & 0 \\ 0 & 1/c \end{pmatrix}\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

where $\delta = ad - bc$ is just the determinant of α.

The first step in evaluating $S_E(\alpha)$ is to use Proposition 3.11 to separate the diagonal matrix from the other three. Then, notice that S_E evaluated on the diagonal matrix is 0 (just by using the $c = 0$ case). Furthermore, the action of it on polynomials is simply to multiply everything by the scalar c^{k-2} (using the definition and homogeneity). However, by definition of the action of a matrix on a modular form, we see that

$$E|\begin{pmatrix} 1/c & 0 \\ 0 & 1/c \end{pmatrix}(z) = c^{-k+2}E(z)$$

so the scalar multiples cancel when applying Proposition 3.11.

The above argument implies, using Proposition 3.11 repeatedly, that we now have

$$S_E(\alpha) = S_E(\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix})$$

$$+ \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}|S_{E|\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma)$$

$$+\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}\sigma|S_{E|\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}_\sigma}(\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix})$$

Considering the three terms separately will show the final result.

We begin by treating the first term, $S_E\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}$. This is simply an application of the $c = 0$ case above; we obtain

$$a_0(E)\int_0^{a/c}(tX+Y)^{k-2}dt$$

Next, consider the final term, which (after some matrix multiplication) is equal to

$$\alpha\cdot\begin{pmatrix}1 & -d\\ 0 & c\end{pmatrix}|S_{E|\alpha\cdot\begin{pmatrix}1 & -d\\ 0 & c\end{pmatrix}}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix}$$

Before we consider the matrix actions at all, we use the $c = 0$ case to evaluate S_E. We conclude that this term is equal to

$$\alpha\cdot\begin{pmatrix}1 & -d\\ 0 & c\end{pmatrix}|a_0\left(E|\alpha\cdot\begin{pmatrix}1 & -d\\ 0 & c\end{pmatrix}\right)\int_0^d(tX+Y)^{k-2}dt$$

If we pull out the constants and use Lemma 2.7, we get

$$\tfrac{1}{c}\cdot a_0(E|\alpha)\cdot\alpha\cdot\begin{pmatrix}1 & -d\\ 0 & c\end{pmatrix}|\int_0^d(tX+Y)^{k-2}dt$$

Using the substitution $u = \frac{t-d}{c}$ and computing the definition of the action of the matrix just before the integral, an elementary calculation confirms that the above expression is equal to

$$a_0(E|\alpha)\int_{-d/c}^0\alpha|(uX+Y)^{k-2}du$$

Now we consider the middle term, which Proposition 3.11 tells us is

$$\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}|S_{E|\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}}(\sigma)$$

After we apply (3.2), this is equal to

$$-\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}|F_{E|\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}}(1)$$

From here, consider that

$$\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}=\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}$$

This means we can apply Lemma 2.8 and the expression becomes

$$-\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}|F_{E|\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}}(1)$$

Putting all the terms together, this gives the above formula and completes the proof. □

Remark 3.13. It would appear at first glance that we have not covered the case $c < 0$. However, it is clear that the matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ evaluates to 0 under S_E. Also, when we take into account the action on polynomials and the action on modular forms, the combination of the two actions will always be trivial whether the weight is odd or even. So to evaluate S_E in the $c < 0$ case we simply change the signs of all the entries in the matrix and use the $c > 0$ case:

$$S_E(\begin{pmatrix}a & b\\ c & d\end{pmatrix}) = S_E(\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix}) = a_0(E)\int_0^{a/c}(tX+Y)^{k-2}dt$$

$$+a_0(E|\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix})\int_{-d/c}^{0}\alpha|(tX+Y)^{k-2}dt-\begin{pmatrix}1 & -a\\ 0 & -c\end{pmatrix}|F_{E|\begin{pmatrix}1 & -a\\ 0 & -c\end{pmatrix}}(1)$$

3.2.2. ***The Involution*** ι**.** Let $\iota=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, a matrix of determinant -1. We define the action of ι on an Eisenstein series as follows: first define

$$\phi_{k,x_1,x_2}|\iota = (-1)^k\phi_{k,x_1,-x_2}$$

and then extend by linearity to general Eisenstein series.

It is elementary to check that for any numbers a and c, the following holds:

$$\iota \begin{pmatrix} 1 & -a \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \iota \tag{3.3}$$

We will make frequent use of this fact below.

Now for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, define the map

$$S_E^\iota(\alpha) := (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota)$$

We will need the following lemma:

Lemma 3.14. *S_E^ι satisfies the relation*

$$S_E^\iota(\alpha\beta) = S_E^\iota(\alpha) + \alpha|S_{E|\alpha}^\iota(\beta)$$

Proof. We compute directly from the definition and Proposition 3.11:

$$\begin{aligned} S_E^\iota(\alpha\beta) &= (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\beta\iota) = (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota\iota^{-1}\beta\iota) \\ &= (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota) + (-1)^{k-1}\iota^{-1}\alpha\iota|S_{E|\alpha\iota}(\iota^{-1}\beta\iota) \\ &= (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota) + (-1)^{k-1}\alpha|\iota|S_{(E|\alpha)|\iota}(\iota^{-1}\beta\iota) \\ &= S_E^\iota(\alpha) + \alpha|S_{E|\alpha}^\iota(\beta) \end{aligned}$$

which is the desired result. □

3.2.3. *The Map ξ_E and Rationality.*

Define the map

$$\xi_E := \tfrac{1}{2}(S_E + S_E^\iota)$$

The main result about ξ_E is the following:

Proposition 3.15. *For any number field K, any E in $\mathcal{E}_k(K)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, $\xi_E(\alpha) \in L_{k-2}(K)$.*

Proof. We will show this for an arbitrary Eisenstein series of the form ϕ_{k,x_1,x_2} and then the result will follow by linearity. Put $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Without loss of generality, we can assume the matrix has integer entries and $c \geq 0$ (because it is equal to such a matrix times a scalar matrix). First notice that

$$\iota^{-1}\alpha\iota = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

Now we will use Theorem 3.12 (and the subsequent remark), as well as the notation M_α from the statement of the theorem.

First we see that if $c = 0$, the result is clear (since both summands in the definition of ξ_E are clearly K-rational in this case). So we may assume $c > 0$. Now (using Equation (3.3)) it suffices to show that the expression

$$M_\alpha | F_{E|M_\alpha}(1) + (-1)^{k-1} M_\alpha \iota | F_{E|M_\alpha \iota}(1)$$

gives a K-rational polynomial.

We first compute using the distribution law:

$$E|M_\alpha = \sum_{\nu=0}^{c-1} \phi_{k, \frac{x_1+\nu}{c}, x_2 + a\frac{x_1+\nu}{c}}$$

(To see this more clearly, separate the matrix into the product

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and for the action of the former, refer to Remark 3.2.) This means that $F_{E|M_\alpha}(1)$ will equal

$$\sum_{j=0}^{k-2} \sum_{\nu=0}^{c-1} \binom{k-2}{j} i^{j+1} (j!) \tfrac{1}{(2\pi)^{j+1}} \cdot L(\phi_{k, \frac{x_1+\nu}{c}, x_2 + a\frac{x_1+\nu}{c}}, j+1) \cdot X^j Y^{k-2-j}$$

Similarly, we can compute $F_{E|M_\alpha \iota}$ using

$$E|M_\alpha \iota = (-1)^k \sum_{\nu=0}^{c-1} \phi_{k, \frac{x_1+\nu}{c}, -x_2 - a\frac{x_1+\nu}{c}}$$

By Proposition 3.7, $F_{E|M_\alpha}(1)$ and $F_{E|M_\alpha \iota}(1)$ are rational polynomials, unless one of the terms $\frac{x_1+\nu}{c}$ or $x_2 + a\frac{x_1+\nu}{c}$ is zero. There are now 4 cases to consider, based on whether one of these vanishes and whether k is even or odd. Suppose we are in the first case, i.e. that k is even and one of the terms $\frac{x_1+\nu}{c}$ vanishes. Then

$$F_{E|M_\alpha}(1) = X^{k-2} \cdot i^{k+1} (k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}} + G(X, Y)$$

and

$$F_{E|M_\alpha \iota}(1) = X^{k-2} \cdot i^{k+1} (k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_2))}{n^{k-1}} + H(X, Y)$$

for some rational polynomials $G, H \in L_{k-2}(K)$. Since the cosine function is even, and the action of ι on polynomials is trivial on X, it is clear that

$$F_{E|M_\alpha}(1) + (-1)^{k-1} \iota | F_{E|M_\alpha \iota}(1) = G(X, Y) - \iota | H(X, Y)$$

which is a rational polynomial. The other three cases are similar, keeping track of the action of ι, the parity of k, and whether we are using the cosine function, which is even, or the sine function, which is odd. □

3.2.4. *The Map ξ'_E.* Our definition and the subsequent computation with ξ_E leads us to consider another map

$$\xi'_E := \tfrac{1}{2i}(S_E - S^{\iota}_E)$$

Under this definition, we have

$$S_E = \xi_E + i\xi'_E$$

We wish to do a similar computation as in the previous section, using the explicit formula to compute $\xi'_E(\alpha)$ for a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and positive determinant (as before, we can extend to rational entries by using multiplication by a scalar matrix). As in the above computation, we start by letting $E = \phi_{k,x_1,x_2}$ for some $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$ and then we can extend by linearity.

By definition, and by the remark following Theorem 3.12,

$$S_E(\alpha) - S^{\iota}_E(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = S_E(\alpha) - (-1)^{k-1}\iota|S_{E|\iota}(\begin{pmatrix} -a & b \\ c & -d \end{pmatrix})$$

If $c = 0$, the two terms cancel. This is because the terms $a_0(E)$ are the same in both (the action of ι does not change it in the second term when k is even, and multiplies it by -1 when k is odd) which makes the sum equal to

$$a_0(E) \cdot \left[\int_0^{b/d} (tX+Y)^{k-2}dt + (-1)^k \int_0^{-b/d} (tX-Y)^{k-2}dt\right]$$

Now it is clear, using the transformation $t \mapsto -t$ in the second integral, that the two terms must cancel.

Now we suppose $c > 0$ (as before, we can reduce the $c < 0$ case to this case). The explicit formula has three terms. Breaking up the computation term-by-term, the first term will be

$$a_0(E) \cdot \left[\int_0^{a/c} (tX+Y)^{k-2}dt + (-1)^k \int_0^{-a/c} (tX-Y)^{k-2}dt\right]$$

which, similarly to the above, is 0.

The corresponding calculation for the second term will also be zero; here we need to know that $a_0(E|\alpha) = (-1)^k a_0(E|\alpha\iota)$, which is clear when E is of the form ϕ_{k,x_1,x_2}, since the constant term only depends on x_1, which is unchanged by the ι-action.

That leaves the third term; so (using Equation (3.3)) it remains to compute

$$F_{E|\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}}^{(1)} - (-1)^{k-1}\iota|F_{E|\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}\iota}^{(1)}$$

and then apply the action of $\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}$ to get the final result.

As in the computation for ξ_E, the distribution law tells us that

$$E|\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix} = \sum_{\nu=0}^{c-1} \phi_{k,\frac{x_1+\nu}{c},x_2+a\frac{x_1+\nu}{c}}$$

Now we will carry out the rest of the computation using Proposition 3.7. The proposition tells us that for each summand above, the X^jY^{k-2-j}-term is

$$\binom{k-2}{j}(-1)^j\frac{B_{k-j-1}(\frac{x_1+\nu}{c})}{k-j-1}\cdot\frac{B_{j+1}(x_2+a\frac{x_1+\nu}{c})}{j+1}X^jY^{k-2-j}$$

except for the exceptional cases which we will deal with below. For now let us treat the non-exceptional cases. When we act by ι on the above Eisenstein series and the polynomial part before using Proposition 3.7, we end up with terms corresponding to the above, of the form

$$\binom{k-2}{j}(-1)^j\frac{B_{k-j-1}(\frac{x_1+\nu}{c})}{k-j-1}\cdot\frac{B_{j+1}(-(x_2+a\frac{x_1+\nu}{c}))}{j+1}X^j(-Y)^{k-2-j}$$

In general, we have $B_n(-x) = (-1)^nB_n(x)$, and if $k-2-j$ is even, then $j+1$ is odd, and vice versa (since k is even). This means that in all cases, the two corresponding terms will be equal and will cancel when we subtract them.

So it remains to compute the two exceptional cases: the X^{k-2} term when the first subscript is zero, and the Y^{k-2} term when the second subscript is zero. We show the result for the even weight case; the odd weight case is exactly the same. Looking at the terms from the distribution law above, and again using Proposition 3.7, the Y^{k-2} term will be

$$\sum_{\nu=0}^{c-1} Y^{k-2}\cdot\delta_{x_2+a\frac{x_1+\nu}{c}}\cdot i^{k-2}(k-2)!\cdot(2\pi)^{-(k-1)}\cdot\sum_{n=1}^{\infty}\frac{\cos(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

(where here δ means 1 if the subscript is an integer and 0 otherwise). By similar logic, the X^{k-2} term is

$$-\sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(x_2))}{n^{k-1}}$$

This means we have shown the following:

Proposition 3.16. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ of even weight, and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ as above, $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is*

$$(aX + cY)^{k-2} \cdot \sum_{\nu=0}^{c-1} \delta_{x_2 + a\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

$$-\sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(x_2))}{n^{k-1}}$$

If instead E has odd weight, then $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is

$$(aX + cY)^{k-2} \cdot \sum_{\nu=0}^{c-1} \delta_{x_2 + a\frac{x_1+\nu}{c}} \cdot i^{k-1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

$$-\sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(x_2))}{n^{k-1}}$$

There is a more succinct way to phrase the above formula. If we put, for $E = \phi_{k,x_1,x_2}$ and k even,

$$\mathcal{C}(E) := i^{k-2} \cdot X^{k-2} \cdot \delta_{x_1} \cdot (k-2)! \cdot (2\pi)^{-(k-1)} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}}$$

and for k odd,

$$\mathcal{C}(E) := i^{k-1} \cdot X^{k-2} \cdot \delta_{x_1} \cdot (k-2)! \cdot (2\pi)^{-(k-1)} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x_2)}{n^{k-1}}$$

and extend the definition by linearity to other Eisenstein series, we can state:

Corollary 3.17. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ as above, $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is*

$$\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} | \left(\sigma | \mathcal{C}(E| \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \sigma) - \mathcal{C}(E| \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}) \right)$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This result has another corollary which will be useful below:

Corollary 3.18. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$\xi'_E(\alpha) = \alpha | \mathcal{C}(E|\alpha) - \mathcal{C}(E)$$

Proof. We first check this property on generators of $\mathrm{SL}_2(\mathbb{Z})$. For $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the result follows directly from the formula in the above corollary (with $a = 0$ and $c = 1$). For $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we wish to show that ξ'_E evaluates to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} | \mathcal{C}(\phi_{k,x_1,x_1+x_2}) - \mathcal{C}(\phi_{k,x_1,x_2})$$

Now if $x_1 \neq 0$, both terms above vanish by the definition of $\mathcal{C}(E)$. But if $x_1 = 0$, then the infinite sum in both terms is identical, so since the polynomial term on the left is also unchanged, the two terms will cancel. Since the above corollary states that $\xi'_E(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ is in fact 0, this shows the desired result for the two generators of $\mathrm{SL}_2(\mathbb{Z})$.

To show the result in general, we use the fact that

$$\xi'_E(\alpha\beta) = \xi'_E(\alpha) + \alpha | \xi'_{E|\alpha}(\beta)$$

for any $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$. (This is a consequence of the definition, Proposition 3.11, and Lemma 3.14.) If the desired property is satisfied for the two matrices α and β, then

$$\begin{aligned}\xi'_E(\alpha\beta) &= \alpha|\mathcal{C}(E|\alpha) - \mathcal{C}(E) + \alpha|[\beta|\mathcal{C}(E|\alpha\beta) - \mathcal{C}(E|\alpha)] \\ &= \alpha\beta|\mathcal{C}(E|\alpha\beta) - \mathcal{C}(E)\end{aligned}$$

This shows the desired result for all of $\mathrm{SL}_2(\mathbb{Z})$. □

3.3. Primes Dividing the Denominators of Values of ξ_E. In this section we wish to prove the following:

Lemma 3.19. *Suppose E is an Eisenstein series of the form $\phi_{k,x/N,y/N}$. Then for any $\alpha \in SL_2(\mathbb{Z})$, $\xi_E(\alpha)$ is a polynomial whose coefficients' denominators are divisible only by primes dividing N and primes less than or equal to $k+1$.*

Proof. We begin by showing that this is true when α is one of the two generators of $\mathrm{SL}_2(\mathbb{Z})$. We begin by using Theorem 3.12 and Proposition 3.1 to compute

$$\xi_E(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \frac{B_k(x/N)}{k} \int_0^1 (tX+Y)^{k-2} dt$$

Since the Bernoulli polynomial's coefficients only have denominators divisible by primes at most $k+1$ (a fact that follows from, for example, the Von Staudt-Clausen Theorem), it is clear that our claim holds for this generator.

The other generator is $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. To evaluate ξ_E on this generator we use the equation $S_E = \xi_E + i\xi'_E$, the equation $S_E(\sigma) = -F_E(1)$, Proposition 3.7, and the computation in the proof of Proposition 3.16. The result is that $\xi_E(\sigma)$ is a rational polynomial whose coefficients are 0 or are given by the values of Bernoulli polynomials evaluated on x/N and y/N divided by positive integers less than k. So the denominators of these coefficients must be divisible only by primes dividing N and primes less than k. This proves the claim for both generators.

Now to complete the proof, we use the fact that for any $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$,

$$\xi_E(\alpha\beta) = \xi_E(\alpha) + \alpha|\xi_{E|\alpha}(\beta) \tag{3.4}$$

(which follows from Proposition 3.11, Lemma 3.14, and the definitions). We need to know that $E|\alpha$ satisfies the same hypotheses as E, which is clearly true; and we need to know that the matrix action of α does not introduce any new denominators, which is clearly true since α has integer entries and determinant 1. This proves the desired result. □

Remark 3.20. Going carefully through the steps of the proof, we see that if $k+1$ is prime, and relatively prime to N, then the only place this appears as a factor of any of the denominators is from the constant term of the Bernoulli polynomial B_k. We will use this fact below.

3.4. The Eisenstein Series Associated to a Pair of Dirichlet Characters.

Definition 3.21. Let ε_1 and ε_2 be two Dirichlet characters mod N_1 and N_2 respectively; we do not assume they are primitive, but we assume that N_1 and N_2 are not both 1 and that the product of the two characters is odd when k is odd and even when k is even. Then for any integer $k \geq 2$, we define the Eisenstein series

$$E(k, \varepsilon_1, \varepsilon_2; z) := \sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x)\bar{\varepsilon_1}(y)\phi_{(k,\frac{x}{N_2},\frac{y}{N_1})}(N_2 z)$$

Let K be the field generated over $\mathbb{Q}$ by the values of the two characters. Then $E(k, \varepsilon_1, \varepsilon_2) \in \mathcal{E}_k(K)$. In this section we compute the Fourier expansion and the L-function of $E(k, \varepsilon_1, \varepsilon_2)$.

Define, for any Dirichlet character ψ mod m,

$$\hat{\psi}(n) := \sum_{a=0}^{m-1} \psi(a) e^{2\pi i a n/m}$$

Recall the definition

$$J(k, a, b; z) := \sum_{\substack{\kappa \equiv a \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} e^{2\pi i z m \kappa} e^{2\pi i m b}$$

We used this definition earlier (in Proposition 3.1) to state the Fourier expansion of ϕ_{k,x_1,x_2}. Now, to help compute the Fourier expansion of $E(k, \varepsilon_1, \varepsilon_2)$, we compute

$$\sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x)\bar{\varepsilon_1}(y) J(k, x/N_2, y/N_1; N_2 z)$$

$$= \sum_{m=1}^{\infty} \left(\sum_{y=0}^{N_1-1} \bar{\varepsilon_1}(y) e^{2\pi i m (y/N_1)} \right) \left(\sum_{x=0}^{N_2-1} \sum_{\substack{\kappa \equiv x/N_2 \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \varepsilon_2(x) e^{2\pi i m N_2 z \kappa} \right)$$

$$= \sum_{m=1}^{\infty} \hat{\bar{\varepsilon_1}}(m) \cdot \sum_{x=0}^{N_2-1} \sum_{\substack{h \equiv x \pmod{N_2} \\ h \in \mathbb{Z}^+}} h^{k-1} \cdot \frac{1}{N_2^{k-1}} \cdot \varepsilon_2(x) e^{2\pi i m z h}$$

$$= N_2^{1-k} \sum_{m=1}^{\infty} \hat{\bar{\varepsilon_1}}(m) \sum_{h=1}^{\infty} h^{k-1} \varepsilon_2(h) e^{2\pi i z m h}$$

$$= N_2^{1-k} \sum_{n=1}^{\infty} \left(\sum_{\substack{mh=n \\ m,h \in \mathbb{Z}^+}} \hat{\bar{\varepsilon}}_1(m) h^{k-1} \varepsilon_2(h) \right) q^n$$

where in the last line, $q = e^{2\pi i z}$.

Since $\bar{\varepsilon_1}\varepsilon_2$ has the same sign as $(-1)^k$, it follows that the corresponding sum for $(-1)^k J(k, -x/N_2, -y/N_1)$ will be the same as for $J(k, x/N_2, y/N_1)$. Using Proposition 3.1, this proves

Proposition 3.22. *Let $E = E(\varepsilon_1, \varepsilon_2)$ as above. Then*

$$L(E, s) = -2N_2^{1-k} L(\hat{\bar{\varepsilon}}_1, s) L(\varepsilon_2, s - k + 1)$$

where the L-functions on the right are Dirichlet L-functions.

When ε_1 is primitive, this construction is the same (up to scaling) as the usual construction of the Eisenstein series associated to a pair of Dirichlet characters, such as the one that can be found in [10].

3.5. Modular Symbols Attached to Eisenstein Series. >From now on we will add the following hypotheses on our Eisenstein series E. We assume that it is of the form $-\frac{N_2^{k-1}}{2} E(k, \varepsilon_1, \varepsilon_2)$ (in light of the above computation for the Fourier expansion, the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We recall the definition of a modular symbol. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of degree zero divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})]$-module. We refer to a map as an A-valued *modular symbol* over a congruence subgroup Γ if the map is a Γ-homomorphism from degree zero divisors to elements of A.

Suppose that r is the greatest integer such that $\mathfrak{p}^r$ divides all the constant terms at the cusps of E. By assumption, r is positive. Now we define a map

$$M_E : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to L_{k-2}(K^+/\mathfrak{p}^r \mathcal{O}_{K,\mathfrak{p}})$$

where K^+ means we are thinking of K as an additive group only, and $\mathcal{O}_{K,\mathfrak{p}}$ is the localization of $\mathcal{O}_K$ at the prime ideal $\mathfrak{p}$. The map is defined as follows:

$$M_E(\{b\} - \{a\}) = \xi_E(\gamma_b) - \xi_E(\gamma_a)$$

where γ_b and γ_a are elements of $\mathrm{SL}_2(\mathbb{Z})$ that map the cusp at infinity to the cusps b and a, respectively.

Our goal is to show:

Theorem 3.23. *Let E, K, $\mathcal{O}_K$, $\mathfrak{p}$, $\mathcal{O}_{K,\mathfrak{p}}$, p, and r be as above. Let N_1 and N_2 be the moduli of ε_1 and ε_2, respectively, and suppose that p is relatively prime to N, the least common multiple of N_1 and N_2. Then M_E is a modular symbol, over the same congruence subgroup Γ for which E is modular, taking values in $L_{k-2}(\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}^r\mathcal{O}_{K,\mathfrak{p}})$.*

Proof. To begin with, we must show that the map is well-defined. In other words, the matrices γ_b and γ_a are defined only up to multiplication by the stabilizer of ∞ on the right, and the stabilizer of the specific cusp on the left. We will show that for any cusp, choosing a different γ does not change the value of $\xi_E(\gamma)$ mod $\mathfrak{p}^r$.

Let γ_a be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ that sends ∞ to a cusp a. Let α be another such matrix that stabilizes a, so that $\alpha\gamma_a$ also sends ∞ to a. But $\alpha\gamma_a = \gamma_a(\gamma_a^{-1}\alpha\gamma_a)$, and $\gamma_a^{-1}\alpha\gamma_a$ stabilizes ∞. So now it suffices to show that for any choice of integer n, $\gamma_a \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ gives the same values mod $\mathfrak{p}^r$ as γ_a does when evaluating ξ_E on them. Now by Equation (3.4) (and the definition of ξ_E),

$$\xi_E(\gamma_a \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}) - \xi_E(\gamma_a) = \gamma_a|\xi_{E|\gamma_a}(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix})$$

$$= a_0(E|\gamma_a)\int_0^n \gamma_a|(tX+Y)^{k-2}dt$$

where for the last line we have used Theorem 3.12 and the definition of ξ_E. By choice of r, it is now clear (since $p > k$ and the action of γ_a introduces no denominators) that the coefficients of the last expression are $\mathfrak{p}$-adic integers divisible by $\mathfrak{p}^r$, which suffices to show that M_E is well-defined.

The next step is to show that this is a Γ-homomorphism. This is a simple computation using Equation (3.4): if a and b are any cusps, and $\gamma \in \Gamma$,

$$M_E(\gamma(\{b\} - \{a\})) = \xi_E(\gamma\gamma_b) - \xi_E(\gamma\gamma_a)$$

$$= \gamma|\xi_E(\gamma_b) - \gamma|\xi_E(\gamma_a)$$

which is the desired result. (We have used the fact that $E|\gamma = E$.)

The last step is to show that M_E takes values with coefficients not just in K, but with denominators not divisible by $\mathfrak{p}$. This is a consequence of the fact that E is a half-integer multiple of an algebraic integer linear combination of Eisenstein series satisfying Lemma 3.19, and the fact that $p > k$ and p and N are relatively prime. This shows the theorem when $p \neq k+1$.

To finish the proof, we must show this still holds in the case $p = k+1$, when p appears exactly once in the denominator of each of the constant coefficients of $B_k(x/N_2)$ (for $0 \leq x \leq N_2 - 1$). But now we use the definition of $E(k, \varepsilon_1, \varepsilon_2)$ to see that

$$a_0(E) = \sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x)\bar{\varepsilon_1}(y)B_k(x/N_2)/k$$

and so the constant coefficients will cancel when we take the sum of character values. A similar calculation shows the same result for $a_0(E|\alpha)$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. This shows that p does not appear in the denominators of $\xi_E(\alpha)$ even when $p = k + 1$, completing the proof. $\square$

3.6. Hecke Operators and Modular Symbols. In this section E will be an Eisenstein series satisfying the same assumptions as in the above theorem. So far we have not discussed the action of Hecke operators on modular symbols. In this section, now that we have defined the modular symbol M_E, we prove a result concerning the action of the Hecke operators on it.

First we need the general definition of the Hecke operators. Following [5], we define them using double coset operators. Let g be a matrix with positive determinant and integer entries, and let $\tilde{\Gamma}$ be any congruence subgroup. The double coset $\tilde{\Gamma}g\tilde{\Gamma}$ can be written as a finite disjoint union of right cosets of the form $\tilde{\Gamma}g_j$. We now write, for any modular symbol Φ,

$$\Phi|T(g) = \sum_j \Phi|g_j$$

where for a degree-zero divisor D, $(\Phi|g_j)(D) = g_j^{-1}|\Phi(gD)$ (as we did before, we change the definition in [5] to account for the fact that our matrix action on polynomials is a left action). For any prime ℓ, the Hecke operator T_ℓ arises from the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$. We also have, for any positive integer d, the diamond operator $\langle d \rangle$ which arises from any element of $\Gamma_0(N)$ whose lower right entry is congruent to d mod N. (Here N is the level of the modular form we are acting on.)

This shows that the Hecke operators act on modular symbols in a similar way to modular forms: as a sum of actions by matrices. For a cusp form f and the corresponding modular symbol M_f, it is a fact stated in [5] (and easily proved from the definitions) that for a matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$M_f|\alpha = M_{f|\alpha}$$

We will now prove the corresponding result for an Eisenstein series E:

Lemma 3.24. *Let E be an Eisenstein series as in the above theorem, let M_E be the associated modular symbol, and let $\alpha \in GL_2^+(\mathbb{Q})$. Then for any* *degree-zero divisor of the form* $\{b\} - \{a\}$,

$$(M_E|\alpha)(\{b\} - \{a\}) = \xi_{E|\alpha}(\gamma_b) - \xi_{E|\alpha}(\gamma_a)$$

where γ_b *and* γ_a *are elements of* $SL_2(\mathbb{Z})$ *that map the cusp at infinity to the cusps* b *and* a*, respectively.*

Proof. This is a straightforward computation from the definitions (and also Proposition 3.11 and Lemma 3.14):

$$\begin{aligned}(M_E|\alpha)(\{b\} - \{a\}) &= \alpha^{-1}|M_E(\{\alpha b\} - \{\alpha a\}) \\ &= \alpha^{-1}|[\xi_E(\alpha\gamma_b) - \xi_E(\alpha\gamma_a)] \\ &= \alpha^{-1}|[\xi_E(\alpha) + \alpha|\xi_{E|\alpha}(\gamma_b) - \xi_E(\alpha) - \alpha|\xi_{E|\alpha}(\gamma_a)] \\ &= \xi_{E|\alpha}(\gamma_b) - \xi_{E|\alpha}(\gamma_a)\end{aligned}$$

□

Combining the above lemma with the fact that the Hecke operators can be expressed as the sum of right actions of matrices, we arrive at the following:

Corollary 3.25. *Let* E *be an Eisenstein series as above, and suppose that* E *is a simultaneous eigenfunction for the Hecke operators* T_ℓ *(*ℓ *prime) and* $\langle d\rangle$*. Then* M_E *is also a simultaneous eigenfunction for the Hecke operators with the same eigenvalues as* E*.*

Proof. As implied above, this follows from the lemma and the definition of the Hecke operators. We also use the fact that the ξ_E map respects summing different Eisenstein series (in the sense that for two Eisenstein series $E_1, E_2 \in \mathcal{E}_k(K)$, $\xi_{E_1} + \xi_{E_2} = \xi_{E_1+E_2}$) and also scalar multiplication. These facts show that for any Hecke operator T with eigenvalue a_T,

$$a_T\xi_E = \xi_{E|T} = \sum_j \xi_{E|g_j}$$

and then the definitions and the lemma show that for any cusps a and b,

$$(M_E|T)(\{b\} - \{a\}) = \sum_j (\xi_{E|g_j}(\gamma_b) - \xi_{E|g_j}(\gamma_a)) = a_T M_E(\{b\} - \{a\})$$

□

4. Twisted Special Values

In this section we keep the same assumptions on E as we had at the end of the previous section, which we restate here. We assume that it is of the form $-\frac{N_2^{k-1}}{2}E(k, \varepsilon_1, \varepsilon_2)$ (the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We know from the previous section that we can associate to E a modular symbol M_E. Let χ be a primitive Dirichlet character of conductor m. We recall the definition of the *twist operator* R_χ on modular symbols. If Φ is any modular symbol, then for any degree-zero divisor D,

$$(\Phi|R_\chi)(D) := \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}^{-1} |\Phi(\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} D)$$

As in [5], we refer to the "special values" attached to a modular symbol as the evaluation of that modular symbol on the divisor $\{\infty\} - \{0\}$.

4.1. Twisted Special Values on Boundary Symbols.

Definition 4.1. Let $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})]$-module. An A-valued *boundary symbol* over a congruence subgroup Γ is a Γ-equivariant homomorphism $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q})) \to A$.

Comparing this definition with that of a modular symbol, it is clear that all boundary symbols are modular symbols. Therefore we can apply the twist operator to a boundary symbol when we restrict the boundary symbol to degree-zero divisors. In this section we will let $A = L_{k-2}(S)$ where S is any ring in which $(k-2)!$ is invertible. The goal of this section is to show that for any A-valued boundary symbol B, and any primitive character χ of conductor m, we have $(B|R_\chi)(\{\infty\} - \{0\}) = 0$, at least for the group $\Gamma_1(N)$ with $(N, pm) = 1$. In practice, we will use the cases where $S = \mathbf{C}$ or where S is a finite ring of characteristic p.

Our first result classifies polynomials fixed by certain elements stabilizing the cusp at infinity.

Lemma 4.2. *Let P be a homogeneous polynomial in X and Y of degree $k-2$ with coefficients in S. Suppose P is fixed under the $SL_2(\mathbb{Z})$-action of the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Suppose that n and $(k-2)!$ are invertible in S. The P is of the form CX^{k-2} for some constant $C \in S$.*

Proof. In general, P is of the form $CX^{k-2} + a_{k-3}X^{k-3}Y + \cdots + a_1 XY^{k-3} + a_0 Y^{k-2}$. Meanwhile, we are assuming P is fixed by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The action of a matrix of this form fixes X and sends $Y \mapsto nX + Y$. So the action on P gives a polynomial of the form

$$CX^{k-2} + a_{k-3}X^{k-3}(nX+Y) + \cdots + a_1 X(nX+Y)^{k-3} + a_0(nX+Y)^{k-2}$$

Collecting all the X^{k-2} terms together and using our hypothesis that P is fixed under this action, we obtain the equation

$$a_{k-3}n + a_{k-4}n^2 + \cdots + a_1 n^{k-3} + a_0 n^{k-2} = 0$$

However, if this equation is true for one nonzero n, then by iterating the matrix action, it is also true for all multiples of n. Choosing the multiples $n, 2n, \dots (k-2)n$, we obtain a system of $k-2$ equations in $a_{k-3}, a_{k-2}, \dots, a_1$, a_0 whose coefficient matrix is invertible (since its determinant will be a power of n times a Vandermonde determinant of $1, 2, \dots, k-2$). This is enough to show that $a_{k-3}, a_{k-2}, \dots, a_1, a_0$ are all zero, which completes the proof. □

Observe that this lemma is much easier in characteristic zero than in characteristic p, since we do not need any hypothesis on invertibility of n or $(k-2)!$.

Next we want an analogue of the lemma above that deals with the stabilizer of general cusp. To deal with this case, we need to introduce some notation. Thus let s denote a cusp of Γ. If $\gamma \in SL_2(\mathbf{Z})$ is such that $\gamma(\infty) = s$, then define a positive integer $n = n_s$ to be the smallest positive integer such that $\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma^{-1}$ lies in Γ. Note that such a matrix necessarily lies in the stabilizer of s. The integer n is sometimes called the *width* of the cusp s, at least for regular cusps of Γ. We note that, once again, this piece of notation is only relevant in dealing with characteristic p.

Lemma 4.3. *Let P be a homogeneous polynomial in X and Y of degree $k-2$ with coefficients in S. Suppose P is fixed under the action of the stabilizer in Γ of the cusp at a rational number a/c in lowest terms. Suppose that n and $(k-2)!$ are invertible in S, where $n = n_s$ as above. Then P is of the form $C(aX+cY)^{k-2}$ for some constant C.*

Proof. We have $s = a/c$ in lowest terms, so that we may take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be a matrix with $\gamma(\infty) = s$, where $ad - bc = 1$. By hypothesis, Γ contains an element of the form $\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma^{-1}$, and this matrix lies in the stabilizer of s in Γ. Now for a matrix of this form to fix P, we must have

$\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma^{-1}|P = P$, which implies

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma^{-1}|P = \gamma^{-1}|P.$$

This means that $P' = \gamma^{-1}|P$ and the integer n satisfy the hypotheses of Lemma 4.2. Thus P' is of the form CX^{k-2} for a constant C and,

$$\begin{aligned} P &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} |CX^{k-2} \\ &= C(aX + cY)^{k-2} \end{aligned}$$

which is the desired result. □

Now we wish to specialize to the case where $\Gamma = \Gamma_1(N)$. The following result is elementary.

Lemma 4.4. *Suppose $\Gamma = \Gamma_1(N)$, where $(N, p) = 1$. Let $s = a/c$ in lowest terms be a rational cusp of Γ where $c \neq 0$ and $(c, N) = 1$. Then the following statements hold:*

- *if $s' = a'/c$ with the same c, the cusps s' and s are Γ-equivalent, and*
- *we have $n = n_s = N$, so that n is relatively prime to p.*

Proof. For the first statement, we refer the reader to [3], Example 9.1.3, page 76. For the second, we compute directly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 - acn & a^2 n \\ -c^2 n & 1 + acn \end{pmatrix}$$

and this lies in $\Gamma_1(N)$ if and only if n is divisible by N, since $(c, N) = 1$ by assumption. □

With all these results in hand, we can state the following useful result.

Theorem 4.5. *Let B be an $L_{k-2}(S)$-valued boundary symbol for $\Gamma = \Gamma_1(N)$, where S is any ring in which N and $(k-2)!$ are invertible. Let χ be a primitive Dirichlet character of conductor m relatively prime to N. Then $(B|R_\chi)(\{\infty\} - \{0\}) = 0$.*

Proof. From the definitions, we have

$$(B|R_\chi)(\{\infty\} - \{0\}) = \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} (\{\infty\} - \{0\}))$$

$$= \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\{\infty\}) - \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\{\tfrac{a}{m}\})$$

where we may split the sum because B is a boundary symbol.

The key observation now is that if γ is a matrix in Γ that stabilizes a cusp α, then it must also fix the polynomial $B(\alpha)$ under the matrix action. (This follows directly from the Γ-homomorphism property.) So we may treat each of the sums above using Lemmas 4.2 and 4.3.

For the first sum, we use Lemma 4.2 to conclude that $B(\{\infty\})$ is of the form $C_1 X^{k-2}$. That polynomial is fixed under the action of $\begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix}$, so the first sum is a constant times X^{k-2} times a sum of character values, and thus is zero. For the second sum, we use Lemma 4.3 together with Lemma 4.4, and argue in a similar manner. The point is that the constant C provided by Lemma 4.3 is independent of a, since B is Γ-equivariant, and all such cusps are Γ-equivalent according to Lemma 4.4. We need to know also that the widths are invertible, which was checked in Lemma 4.4. □

4.2. Twisted Special Values Associated to *E*. Now we are able to connect the modular symbol M_E with the special values of $L(E, \chi, j)$, for a primitive Dirichlet character χ, at the critical integers.

Theorem 4.6. *Let E, k, ε_1, ε_2, N_1, N_2, $\mathfrak{p}$, p, and r be as above, and let χ be a primitive Dirichlet character of conductor m, with m relatively prime to both p and N, the least common multiple of N_1 and N_2. Then*

$$(M_E|R_\chi)(\{\infty\} - \{0\}) \equiv \tau(\bar{\chi}) \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot m^j \cdot D_{E\otimes\chi}(1+j) \cdot X^j Y^{k-2-j}$$

where the equivalence is mod $\mathfrak{p}^r$ where $\mathfrak{p}$ is understood to be an ideal of the ring of integers of $K[\chi]$ localized at a prime above $\mathfrak{p}$.

Proof. We begin by computing from the definitions:

$$(M_E|R_\chi)(\{\infty\} - \{0\}) = \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |M_E(\{\infty\} - \{\tfrac{a}{m}\})$$

$$= \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/m & 0 \\ 0 & 1/m \end{pmatrix} |[\xi_E(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \xi_E(\gamma_{a/m})]$$

where $\gamma_{a/m}$ is a matrix of the form $\begin{pmatrix} a & b_a \\ m & d_a \end{pmatrix}$ of determinant 1, i.e. it is an element of $\mathrm{SL}_2(\mathbb{Z})$ that carries ∞ to $\frac{a}{m}$. Since $\xi_E(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 0$, the sum is equal to

$$-m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|\xi_E(\gamma_{a/m})$$

Now we claim that

$$-\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|\xi'_E(\gamma_{a/m}) = 0$$

To prove the claim, we define a map B_E on divisors on cusps: for a cusp α,

$$B_E(\alpha) := \gamma_\alpha|\mathcal{C}(E|\gamma_\alpha)$$

where $\mathcal{C}(E)$ is as defined in the previous section, and we extend linearly to other divisors. It is elementary to check that this map is a well-defined Γ-homomorphism, where Γ is the congruence subgroup that stabilizes E; so now we can use Theorem 4.5 to conclude that

$$(B_E|R_\chi)(\{\infty\} - \{0\}) = 0$$

But computing from the definitions and Corollary 3.18, we see that

$$0 = (B_E|R_\chi)(\{\infty\} - \{0\}) = \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|[B_E(\{\infty\}) - B_E(\{\tfrac{a}{m}\})]$$

$$= \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|[B_E(\{\infty\}) - B_E(\{\tfrac{a}{m}\})]$$

$$= \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|[\mathcal{C}(E) - \gamma_{a/m}|\mathcal{C}(E|\gamma_{a/m})]$$

$$= -\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|\xi'_E(\gamma_{a/m})$$

which proves the claim.

The claim shows that

$$-m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|\xi_E(\gamma_{a/m})$$

$$= -m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}|S_E(\gamma_{a/m})$$

From here we carry out the computation using Theorem 3.12, which says that

$$S_E(\gamma_{a/m}) = a_0(E)\int_0^{a/m}(tX+Y)^{k-2}dt$$

$$+a_0(E|\gamma_{a/m})\int_{-d_a/m}^{0}\gamma_{a/m}|(tX+Y)^{k-2}dt$$

$$-\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}|F_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1)$$

The first two terms of the sum are $\mathfrak{p}$-integral and divisible by $\mathfrak{p}^r$, by our assumption on E. (In order to know this, we need to know that the integrals contain no terms with negative p-adic valuation. That is implied by the hypotheses that p and m are relatively prime and also that $p > k-1$.) The matrix action in the definition of the twist operator will not change these facts since it introduces no denominators.

We now wish to treat the sum

$$m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}|F_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1)$$

$$= m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}|\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot D_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1+j)\cdot X^jY^{k-2-j}$$

by Proposition 2.3. Then we apply the definition of matrices acting on polynomials and switch the order of summation to obtain

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot\sum_a\bar{\chi}(a)D_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1+j)\cdot X^jY^{k-2-j}$$

Now we use Lemma 2.9 with $s = j+1$; the resulting sum is

$$\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{E\otimes\chi}(1+j)X^jY^{k-2-j}$$

Since the first two terms reduce to 0 mod $\mathfrak{p}^r$, this is exactly the desired result. ☐

Corollary 4.7. *With all notation the same as in the above theorem,*

$$(M_E|R_\chi)(\{\infty\}-\{0\})$$
$$\equiv \sum_{j=0}^{k-2}(-1)^{j+1}\begin{pmatrix}k-2\\ j\end{pmatrix}\cdot j!\cdot m^j\cdot \tfrac{\tau(\bar{\chi})L(E,\chi,1+j)}{(2\pi i)^{j+1}}\cdot X^jY^{k-2-j}$$

where the equivalence is mod $\mathfrak{p}^r$ where $\mathfrak{p}$ is understood to be an ideal of the ring of integers of $K[\chi]$ localized at a prime above $\mathfrak{p}$.

Proof. Combine the above theorem with Proposition 2.1. ☐

4.3. The Sign of *E* and the Action of *ι*. We keep the same assumptions on E as earlier in this section. Recall the matrix $\iota=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and consider the degree-zero divisor

$$\Lambda_E(\chi):=\sum_{a=0}^{m-1}\bar{\chi}(a)(\{\infty\}-\{\tfrac{a}{m}\})$$

For any $a,b\in\mathbb{P}^1(\mathbb{Q})$, the action of ι on a degree-zero divisor $\{b\}-\{a\}$ is

$$(\{b\}-\{a\})^\iota=\{-b\}-\{-a\}$$

(see, for example, [2] or [17]). We are going to obtain identities involving the twisted special values computed above by considering the polynomial $M_E(\Lambda_E(\chi)^\iota)$ for an arbitrary primitive character χ.

By the above, we have

$$\Lambda_E(\chi)^\iota=\sum_a\bar{\chi}(a)(\{\infty\}-\{-\tfrac{a}{m}\})$$

and so

$$\Lambda_E(\chi)^\iota=\chi(-1)\sum_a\bar{\chi}(-a)(\{\infty\}-\{-\tfrac{a}{m}\})$$

This means

$$M_E(\Lambda_E(\chi)^\iota)=\mathrm{sgn}(\chi)M_E(\Lambda_E(\chi))$$

Now recall that E is of the form $E(k,\varepsilon_1,\varepsilon_2)$. We define the *sign* of E to be $-\varepsilon_1(-1)$. (This is the same definition as in [16]. Note that it does not change with the weight of E.)

Lemma 4.8. *For any E as above and any degree-zero divisor d,*

$$M_E(d^\iota)=sgn(E)\cdot\iota|M_E(d)$$

Proof. If $\{\alpha\}$ is a cusp, ι sends it to $\{-\alpha\}$. If we let $\gamma_\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ mapping $\{\infty\}$ to $\{\alpha\}$, then an element mapping $\{\infty\}$ to $\{-\alpha\}$ is $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \iota^{-1}\gamma_\alpha\iota$. We will show that

$$\xi_E(\iota^{-1}\gamma_\alpha\iota) = \mathrm{sgn}(E)\cdot\iota|\xi_E(\gamma_\alpha)$$

and then the definition of M_E will imply the lemma. To show the identity claimed above, we simply compute directly:

$$\xi_E(\iota^{-1}\gamma_\alpha\iota) = \tfrac{1}{2}(S_E(\iota^{-1}\gamma_\alpha\iota) + S_E^\iota(\iota^{-1}\gamma_\alpha\iota))$$

$$= \tfrac{1}{2}(S_E(\iota^{-1}\gamma_\alpha\iota) + (-1)^{k-1}\iota|S_{E|\iota}(\gamma_\alpha))$$

Because the action of ι on Eisenstein series and on polynomials is an involution, this is equal to

$$\tfrac{1}{2}\iota|(\iota|S_{(E|\iota)|\iota}(\iota^{-1}\gamma_\alpha\iota) + (-1)^{k-1}S_{E|\iota}(\gamma_\alpha))$$

$$= (-1)^{k-1}\tfrac{1}{2}\iota|[S_{E|\iota}(\gamma_\alpha) + (-1)^{k-1}\iota|S_{(E|\iota)|\iota}(\iota^{-1}\gamma_\alpha\iota)]$$

$$= (-1)^{k-1}\iota|\xi_{E|\iota}(\gamma_\alpha)$$

Now using the fact that $E = E(k,\varepsilon_1,\varepsilon_2)$ and the definition of the ι-action on E, we see that $E|\iota = (-1)^k\varepsilon_1(-1)E$, so this shows the claim and hence the lemma. □

If we combine the lemma with the equation immediately before it, we see that we now have two different ways of computing $M_E(\Lambda_E(\chi)^\iota)$, so the results are equal:

$$\mathrm{sgn}(\chi)M_E(\Lambda_E(\chi)) = \mathrm{sgn}(E)\cdot\iota|M_E(\Lambda_E(\chi))$$

On the right-hand side, ι acts as $(-1)^k$ times the involution on polynomials $Y \mapsto -Y$. So any term with the power of Y having the same parity as the weight will be fixed by the involution, and any term with the power of Y having opposite parity as the weight will be negated by it. This shows the following:

Proposition 4.9. *Let E and χ be as above, and consider the polynomial $M_E(\Lambda_E(\chi))$. If sgn(E) = sgn(χ), then the coefficients of the terms X^jY^{k-2-j} with $0 \le j \le k-2$ with j odd are all zero. If sgn$(E) \neq$ sgn(χ), then the coefficients of the terms X^jY^{k-2-j} with $0 \le j \le k-2$ with j even are all zero.*

Remark 4.10. In weight 2, where there is only one term, a constant times X^0Y^0, this proposition implies that $M_E(\Lambda_E(\chi))$ can be nonzero only if $\mathrm{sgn}(E) = \mathrm{sgn}(\chi)$, and is always zero when the signs do not match. This was already known in weight 2—see, for example, [16].

5. The Congruence Theorem

The foregoing results were obtained in order to show congruence results concerning the special values of the L-functions of a cusp form and a congruent Eisenstein series. (We will explain below what it means for two modular forms to be congruent mod a prime.)

In this section we keep the same assumptions on E as we had at the end of the previous section, which we restate here. We assume that it is of the form $-\frac{N_2^{k-1}}{2}E(k,\varepsilon_1,\varepsilon_2)$ (the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We now begin with the following:

Theorem 5.1. *Let E be as above. Then the modular symbol M_E is not identically zero mod $\mathfrak{p}$.*

Proof. We will show that there exists a character χ such that the χ-twisted special value of M_E is indivisible by $\mathfrak{p}$. To do this, we shall use the calculation of special values above, together with a result of Friedman and Washington on indivisibility of twisted Bernoulli numbers. Let ℓ denote an odd prime which we will specify later, and let χ denote a primitive Dirichlet character of ℓ-power conductor ℓ^n with n large. We assume that $\chi = \chi_t \cdot \chi_w$ where χ_t is some *fixed* character (independent of n) and χ_w has order ℓ^n (for large n). The character χ_t will have parity depending on the sign of E.

Then we may apply Proposition 6.1 and Corollary 4.7 to determine the $\mathfrak{p}$-adic divisibility of the coefficients of the χ-twisted special value of M_E. Combining these results shows that the coefficient of X^jY^{k-j-2} is the product of three types of term: elementary explicit constants, Euler products over primes $q|N_1N_2$, and twisted Bernoulli numbers $B_{j+1}(\overline{\tilde{\varepsilon_1}\chi})$ and $B_{k-j-1}(\tilde{\varepsilon_2}\chi)$. We want to show that for suitable ℓ and large n, that all these quantities are $\mathfrak{p}$-adic units. This is obvious in the case of the elementary constants for all $\ell \neq p$ and any n, since $p > k$. In the case of the Euler factors, it is evident that, since the ℓ-power roots of unity are distinct modulo $\mathfrak{p}$, then for any given ℓ there are only finitely many integers n such that terms of the form $1 - \frac{\tilde{\varepsilon_1}(q)\chi(q)}{q^{j+1}}$ or $1 - \tilde{\varepsilon_2}(q)\chi(q)q^{k-j}$ are congruent to zero modulo $\mathfrak{p}$. It remains to deal with the Bernoulli numbers, and this we do by appealing to a theorem of Friedman and Washington.

The main idea is very simple. We want to show that we can choose χ such that $B_j(\overline{\tilde{\varepsilon_1}\chi})$ and $B_{k-j-1}(\tilde{\varepsilon_2}\chi)$ are both units, for any convenient choice of j. This is a well-known fact, but unfortunately the precise statement has many cases, because of trivial vanishing of twisted Bernoulli numbers if the parities are not exactly correct ($B_j(\nu) = 0$ if ν and j have opposite parity).

Let us start with the case of even weight, and the sub-case the characters ϵ_1 and ϵ_2 are both odd. In this case the result is most easily phrased for the coefficient of Y^{k-2}, so that $j = 0$. Then we are dealing with $L(E, \chi, 1)$, and the Bernoulli numbers are $B_1(\overline{\tilde{\varepsilon_1}\chi}) \cdot B_{k-1}(\tilde{\varepsilon_2}\chi)$. We want to know that these are $\mathfrak{p}$-adic units for suitable ℓ and n and χ. We take χ_t (and hence χ) to be even. Then the required result follows directly from [4], especially the remark at the end of the proof of Lemma 3 on page 432, and a brief translation of the notation from section 1 of that paper.

It remains to treat the sub-case where ϵ_1 and ϵ_2 are even, and the case of odd weight and the corresponding subcases of that, depending on the (opposite) parities of the ϵ_i. The arguments are entirely similar and we omit them. For instance, in the remaining case of even weight, we take χ_t and hence χ to be odd (since the sign of E is minus, in this case), and again $j = 0$ works as before. □

Now let f be a normalized (meaning $a_1 = 1$) cuspidal eigenform of the same weight and level as E. We will also assume that f is congruent to E mod $\mathfrak{p}^r$ in the following sense. If $f = \sum a_n q^n$ and $E = \sum b_n q^n$ are given by the standard Fourier expansions in terms of $q = e^{2\pi i z}$, we say that E and f are congruent modulo $\mathfrak{p}^r$ if $a_n \equiv b_n \pmod{\mathfrak{p}^r}$ for all $n \geq 1$ and $\mathfrak{p}^r$ divides the constant terms of the Fourier expansions of E at all cusps. Here we understand that $\mathfrak{p}$ is a prime of residue characteristic p in the ring of integers of a number field K containing the Fourier coefficients of E and f.

With this definition, we can finally state our main result:

Theorem 5.2. *Let f and E be a cusp form and an Eisenstein series respectively, of the same weight $k \geq 2$ and level N, with $E \equiv f \pmod{\mathfrak{p}}$, and $(N, p) = 1$, and $p > k$. Then there exists a canonical period $\Omega_f^{sgn(E)}$ for f and a $\mathfrak{p}$-adic unit Ω_E such that the following statement holds:*

Let χ be a primitive Dirichlet character of conductor m, with m prime to both N and p. Then for all positive integers $j < k$ with $(-1)^{j-1} \cdot \chi(-1) = sgn(E)$, we will have

$$\frac{\tau(\bar{\chi})L(f, \chi, j)}{(2\pi i)^{j-1}\Omega_f^{sgn(E)}} \equiv -\frac{\tau(\bar{\chi})L(E, \chi, j)}{(2\pi i)^j \Omega_E} \pmod{\mathfrak{p}^r}.$$

Proof. Let M_f denote the modular symbol associated to f in Section 2. Let $M_f = M_f^+ + M_f^-$ denote the decomposition of M_f into eigenspaces for the involution ι, and write $M_f^\pm = N_f\Omega_f^\pm$, where $\Omega_f^\pm$ are periods of f selected so that the modular symbols $N_f^\pm$ are K-rational. The precise normalization of the periods is delicate, and we proceed as follows. If Γ is any congruence subgroup and A is a Γ-module, then there is a map which we denote $\delta = \delta_A$ from the space of A-valued modular symbols for Γ to the cohomology group $H^1(\Gamma, A)$, as explained in [5]. The image of this map is the parabolic cohomology group, and the kernel is the group of A-valued boundary symbols.

In the case at hand, we select the periods $\Omega_f^\pm$ so that the cohomology classes $\delta(N_f^\pm) = \delta(N_f)^\pm$ lie in $H^1(\Gamma, L_{k-2}(\mathcal{O}_{K,\mathfrak{p}}))$, where $\mathcal{O}$ denotes the ring of integers of K, and such that $\delta(N_f)^\pm$ are nonzero modulo $\mathfrak{p}$. We remark here that this definition is rendered somewhat complicated by the fact that the group $H^1(\Gamma, L_{k-2}(\mathcal{O}_{K,\mathfrak{p}}))$ may have nontrivial torsion and may not be a lattice inside the rational cohomology. In fact there is no torsion to worry about: since we have $(N, p) = 1$ and we are dealing with an Eisenstein prime $\mathfrak{p}$ which is therefore ordinary, we may apply of the results of Hida (see for instance [7], Lemma 4.6), which state that the ordinary part of the integral cohomology is torsion-free and therefore forms a lattice inside the rational cohomology. We note also that Hida assumes $p \geq 5$; this assumption holds in our case since $p > k \geq 3$. (One could also address the issue of torsion via a multiplicity one theorem, as in equation (5.1) below and the discussion at the end of the present proof, but an elementary argument via Hida theory seems to be preferable.) Thus this definition makes sense, and is consistent with the definition of canonical periods given in [18].

With this normalization the periods $\Omega_f^\pm$ are determined up to some $\mathfrak{p}$-adic unit. We caution the reader however that it is *not* apparent whether or not the modular symbols $N_f^\pm$ themselves are integral.

Now let M_E denote the modular symbol on $\Gamma_1(N)$ with values in $L_{k-2}(\mathcal{O}/\mathfrak{p}^r)$ associated to E that was constructed in Section 3. According to Theorem 5.1, M_E is nonzero. Furthermore, we have proven in Section 4 that M_E is an eigenvector for ι with eigenvalue given by the sign of E. Thus we have cohomology classes $\overline{\delta(N_f)}^{\mathrm{sgn}(E)}$ and $\delta(M_E)$ with values in $L_{k-2}(\mathcal{O}/\mathfrak{p}^r)$, where the bar denotes reduction modulo $\mathfrak{p}^r$. Then we claim that

$$c \cdot \overline{\delta(N_f)}^{\mathrm{sgn}(E)} = \delta(M_E) \tag{5.1}$$

where c is a unit in $\mathcal{O}/\mathfrak{p}^r$.

Let us admit this claim for the moment and see how to complete the proof. We wish to lift the cohomology classes $\delta(M_E)$ and $\overline{\delta(N_f)}^{\mathrm{sgn}(E)}$ to modular symbols over $\mathcal{O}/\mathfrak{p}^r$. In the case of the Eisenstein class, we have the obvious lift M_E itself. However, as we have remarked above, it is not clear that N_f is integral, so we cannot simply lift $\overline{\delta(N_f)}^{\mathrm{sgn}(E)}$ to $\overline{N_f}^{\mathrm{sgn}(E)}$, since $\overline{N_f}^{\mathrm{sgn}(E)}$ does not make sense. We argue instead as follows. The map from integral modular symbols to integral cohomology classes is surjective. Since N_f was defined such that $\delta(N_f)^{\pm}$ lies in the integral cohomology group, there exists *some* integral modular symbol $A_f^{\pm}$ such that $\delta(A_f)^{\pm} = \delta(N_f)^{\pm}$. So we may write $N_f^{\pm} = A_f^{\pm} + B_f^{\pm}$ where B_f is a rational boundary symbol. By definition $\delta(A_f)^{\pm} = \delta(N_f)^{\pm}$, so we may lift the cohomology class $\overline{\delta(N_f)}^{\mathrm{sgn}(E)}$ to the modular symbol $\overline{A_f}^{\mathrm{sgn}(E)}$.

Since we have $\overline{\delta(A_f)}^{\mathrm{sgn}(E)} = \overline{\delta(N_f)}^{\mathrm{sgn}(E)}$, the claim (5.1) implies that $M_E - c \cdot \overline{A_f}^{\mathrm{sgn}(E)}$ is a boundary symbol. Applying Theorem 4.5 in characteristic p, we find that the special values of M_E coincide with the special values of $\overline{A_f}^{\mathrm{sgn}(E)}$. The former special values have already been computed in terms of the special values of E, so it remains to compute the special values of $\overline{A_f}^{\mathrm{sgn}(E)}$. But since we have $N_f = A_f + B_f$ where B_f is a rational boundary symbol, we may apply Theorem 4.5 again, this time in characteristic zero, to conclude that the special values of A_f coincide with those of N_f. Since the special values of N_f are L-values of f, our theorem follows if we take Ω_E to be a fixed lift of the unit c to $\mathcal{O}$.

It remains therefore to prove the claim. In the case of weight two, it turns out that $H^1_p(\Gamma, \mathcal{O}/\mathfrak{p}^r)[\mathfrak{m}]$ is isomorphic to a subgroup of the étale part of a certain group scheme occurring as a subgroup of $J_1(N,p)[p^\infty]$, and the claim is equivalent to a multiplicity one statement for this subgroup, which is proved in [19], Theorem 2.1 (and see also [18], Theorem 2.7). The case of weight k may be reduced to that of weight 2 and $J_1(Np)$ by a routine application of Hida theory as developed in [7], Section 4. We omit the details of the reduction to weight 2, since they are entirely standard. It is relevant however to point out that the computed value of $\mathrm{sgn}(E)$ is crucial to distinguish the étale and multiplicative parts of the subgroup schemes in question. □

Remark 5.3. A different and self-contained proof of the multiplicity one statement which holds in all weights has recently been given by Hirano [8]

Remark 5.4. As should be clear from the somewhat contorted proof of the congruence theorem above, the precise normalization of the periods of the cusp form f is a somewhat delicate matter. It would be natural, for instance, to normalize the modular symbol of f so that it is integral and nonzero modulo $\mathfrak{p}$; however as we have already remarked, it is not clear in this case whether or not the modular symbol maps to zero in cohomology. In principle, this normalization may be different from the one given above, but it turns out in fact that both normalizations are the same: as we have recently learned, this follows from a result of Bellaïche and Dasgupta [1], Proposition 2.9, which states, roughly speaking, that the boundary symbols lie in the opposite eigenspace for the involution ι from the modular symbols of interest. One can use this result to simplify the arguments to a certain extent, since all issues of lifting cohomology classes to modular symbols are eliminated, but we have preferred to give a more self-contained treatment.

Remark 5.5. One could also consider a variant of the congruence theorem where we have a pair of congruent cusp forms, and the congruence prime is Eisenstein. This case was excluded in [18] for technical reasons. However it is clear from the proof of Theorem 5.2 that an entirely analogous result holds in this case too. We leave it to the reader to formulate the statement.

6. Computing Special Values

At the end of this section we will exhibit some computed examples of twisted special values attached to cusp forms and Eisenstein series. But first we must explain how these computations were done.

In order to compute the special values

$$\frac{\tau(\overline{\chi})L(f,\chi,j)}{(2\pi i)^{j-1}\Omega_f^{\pm}}$$

we used the computational method of modular symbols outlined in [15], Chapter 8. The details are standard, and we omit them. The only point we wish to make is to clarify how the periods were normalized: they are selected so that the special values are all integral, and at least one special value is a unit (see Remark [1]. In the cases tabulated below, the forms all have rational Fourier coefficients, so the normalization is particularly simple.

As for the Eisenstein series, we use a standard closed form arising from the connection between $L(E,\chi,s)$ and classical Dirichlet L-functions, since it is well known that when E is an eigenfunction, that we can find two Dirichlet characters ε_1 and ε_2 (not necessarily nontrivial or primitive) such that

$$L(E,s) = L(\varepsilon_1,s)L(\varepsilon_2,s-k+1)$$

(For full details, see [10], chapter 4, in particular section 4.7.)

Now let χ be a nontrivial primitive Dirichlet character. Since χ is totally multiplicative, we have

$$L(E,\chi,s) = L(\chi\varepsilon_1,s)L(\chi\varepsilon_2,s-k+1)$$

Then we can evaluate $L(E,\chi,s)$ at the critical integers simply by evaluating the Dirichlet L-functions on the right-hand side of the above equation. This can be done using standard formulas for Dirichlet L-functions (which can be found in [11], chapter 10, or Section 3.3 of [16]). The final result is stated below.

Proposition 6.1. *Let E, ε_1, and ε_2 be as above, and let N_1 and N_2 be the conductors of ε_1 and ε_2 (though the characters need not be primitive). Let χ be a primitive Dirichlet character of conductor m. Let $\tilde{\varepsilon_1}$ and $\tilde{\varepsilon_2}$ be the primitive characters that induce ε_1 and ε_2 respectively, and let $\tilde{N}_i$ denote the corresponding conductors. Let j be an integer strictly between 0 and the weight of E such that $\varepsilon_1\chi$ has the same sign as $(-1)^j$. Then*

$$\tau(\bar{\chi})L(E,\chi,j)/(2\pi i)^j$$
$$= (-i)^j\tau(\bar{\chi})\tau(\tilde{\varepsilon_1}\chi)\cdot C_j\cdot B_j(\overline{\tilde{\varepsilon_1}\chi})\cdot B_{k-j}(\tilde{\varepsilon_2}\chi)$$
$$\cdot \prod_{q|mN_1}(1-\tfrac{\tilde{\varepsilon_1}(q)\chi(q)}{q^j})\cdot\prod_{q|mN_2}(1-\tilde{\varepsilon_2}(q)\chi(q)q^{k-j-1})$$

where if j is odd,

$$C_j = (-i)\cdot 2^{-j}\cdot(m\tilde{N}_1)^{1-j}\cdot(\tfrac{j-1}{2})!^{-1}\cdot[(\tfrac{-1}{2})(\tfrac{-3}{2})\cdots(\tfrac{2-j}{2})]^{-1}\cdot\tfrac{1}{j(k-j)}$$

and if j is even,

$$C_j = 2^{-j}\cdot(m\tilde{N}_1)^{-j}\cdot(\tfrac{j}{2}-1)!^{-1}\cdot[(\tfrac{-1}{2})(\tfrac{-3}{2})\cdots(\tfrac{1-j}{2})]^{-1}\cdot\tfrac{1}{j(k-j)}$$

Remark 6.2. For critical integers not meeting the condition that $\varepsilon_1\chi$ has the same sign as $(-1)^j$, $L(E,\chi,j)$ is zero due to a trivial zero arising from the Dirichlet L-functions.

Corollary 6.3. *With all notation as in the above proposition, including the definition of C_j, we have $D_{E\otimes\chi}(j)$ equal to $i\cdot j!\cdot\tau(\tilde{\varepsilon_1}\chi)\cdot C_j$ times*

$$B_j(\overline{\tilde{\varepsilon_1}\chi})\cdot B_{k-j}(\tilde{\varepsilon_2}\chi)\cdot\prod_{q|mN_1}(1-\tfrac{\tilde{\varepsilon_1}(q)\chi(q)}{q^j})\cdot\prod_{q|mN_2}(1-\tilde{\varepsilon_2}(q)\chi(q)q^{k-j-1})$$

Proof. Use the above proposition and Proposition 2.1. □

With this in hand, we can give specific examples of a cusp form and a congruent Eisenstein series (mod a prime p specified below).

In the first 3 tables, f is the unique newform of weight 4 and level $\Gamma_0(5)$, and E is a congruent Eisenstein series whose L-function is the product of Dirichlet L-functions $L(\epsilon_5,s)L(\epsilon_1,s-3)$, where ϵ_j refers to the principal character mod j. In the tables, m refers to the conductor of a primitive quadratic character. In the first and third tables, the character is odd; in the second table it is even. In the last column, p refers to the prime such that $E\equiv f \pmod p$; in this case $p=13$ (and we exclude characters with $13|m$). The "ratio mod p" is the ratio of the second column to the third column. We have stated the results in this way in order to exhibit the the unit Ω_E in the theorem.

m	$\frac{\tau(\overline{\chi})L(f,\chi,1)}{\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,1)}{2\pi i}$	Ratio mod p
3	100	-2/45	12
4	-100	-1/10	12
7	300	-48/35	12
8	800	-9/5	12
11	-2400	-12/5	12
15	-400	-16	12
19	-8800	-44/5	12
20	-1400	-30	12
23	5900	-432/5	12
24	-10800	-184/5	12

m	$\frac{\tau(\overline{\chi})L(f,\chi,2)}{2\pi i\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,2)}{(2\pi i)^2}$	Ratio mod p
8	0	13/200	N/A
12	0	13/75	N/A
17	0	208/425	N/A
21	-300/7	64/175	12
24	-50/3	18/25	12
28	0	-208/175	N/A
29	-400/29	432/725	12
33	0	624/275	N/A
37	0	52/37	N/A

m	$\frac{\tau(\overline{\chi})L(f,\chi,3)}{(2\pi i)^2\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,3)}{(2\pi i)^3}$	Ratio mod p
3	-10/9	-7/3375	12
4	-5/8	-31/8000	12
7	-30/49	-72/6125	12
8	-5/4	-189/16000	12
11	-240/121	-186/15125	12
15	-8/9	-8/225	12
19	-880/361	-682/45125	12
20	-7/4	-3/80	12
23	-590/529	-4536/66125	12
24	-15/8	-713/18000	12

In these last 3 tables, f is the unique newform of weight 4 and level $\Gamma_0(7)$. E is the Eisenstein series whose L-function is given by $L(\epsilon_7, s)$ $L(\epsilon_1, s-3)$. This time $p = 5$ (and we exclude characters with $5|m$).

m	$\frac{\tau(\overline{\chi})L(f,\chi,1)}{\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,1)}{2\pi i}$	Ratio mod p
3	49	-2/63	4
4	-147	-1/7	4
7	49	-8/7	4
8	-539	-12/7	4
11	-1568	-24/7	4
19	6713	-66/7	4
23	-6272	-576/7	4
24	11368	-276/7	4

m	$\frac{\tau(\overline{\chi})L(f,\chi,2)}{2\pi i\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,2)}{(2\pi i)^2}$	Ratio mod p
8	49/8	3/49	4
12	0	25/147	N/A
13	0	100/637	N/A
17	0	400/833	N/A
21	56/3	8/21	4
24	0	75/98	N/A
28	7/2	8/7	4
29	784/29	864/1421	4
33	0	1200/539	N/A
37	0	2400/1813	N/A

m	$\frac{\tau(\overline{\chi})L(f,\chi,3)}{(2\pi i)^2\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,3)}{(2\pi i)^3}$	Ratio mod p
3	7/18	-19/9261	4
4	21/32	-43/10976	4
7	1/2	-4/343	4
8	77/128	-129/10976	4
11	112/121	-516/41503	4
19	959/722	-99/6517	4
23	448/529	-12384/181447	4
24	203/144	-437/10976	4

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