JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 24, n° 2 (2012), p. 307-337.

<http://jtnb.cedram.org/item?id=JTNB_2012__24_2_307_0>

Journal de Théorie des Nombres
de Bordeaux **24** (2012), 307-337

Relations among arithmetical functions, automatic sequences, and sum of digits functions induced by certain Gray codes

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Résumé. Dans l'étude de la somme des chiffres S_2 en base deux, la fonction arithmétique u définie par $u(0)=0$ et $u(n)=(-1)^{n-1}$ pour $n \geq 1$ joue un rôle de première importance. Dans cet article, nous commençons par généraliser la relation entre S_2 et u en introduisant une permutation sur l'ensemble des suites à valeurs complexes, nulles en 0. Comme application, certaines relations impliquant la fonction somme des chiffres $S_{\mathcal{G}}$ associée à un code binaire infini $\mathcal{G}$ de type Gray sont mises en vidence. En particulier nous montrons que la différence $n \mapsto S_{\mathcal{G}}(n) - S_{\mathcal{G}}(n-1)$ s'obtient par un automate. La formule sommatoire de P. Flajolet et L. Ramshaw pour la somme des chiffres associée au classique code refléchi de Gray est aussi généralisée. La méthode est analytique ; elle utilise la tranformée de Mellin et la formule de Perron pour les séries de Dirichlet.

Abstract. In the study of the 2-adic sum of digits function $S_2(n)$, the arithmetical function $u(0)=0$ and $u(n)=(-1)^{n-1}$ for $n \geq 1$ plays a very important role. In this paper, we firstly generalize the relation between $S_2(n)$ and $u(n)$ to a bijective relation between arithmetical functions. And as an application, we investigate some aspects of the sum of digits functions $S_{\mathcal{G}}(n)$ induced by binary infinite Gray codes $\mathcal{G}$. We can show that the difference of the sum of digits function, $S_{\mathcal{G}}(n) - S_{\mathcal{G}}(n-1)$, is realized by an automaton. And the summation formula of the sum of digits function for reflected binary code, proved by P. Flajolet and L. Ramshaw, is also generalized. Here we use analytic tools such as Mellin transform and Perron's formula for Dirichlet series.

Manuscrit reçu le 1er février 2011.
Mots clefs. arithmetical function, sum of digits function, Gray code, automatic sequence, Delange's theorem.
Classification math. 11A25, 11B85.

1. Introduction

Let $q \geq 2$ be an integer, n be a non-negative integer, and

$$n = \sum_{k=0}^{\infty} a_k(n) q^k, \quad 0 \leq a_k(n) \leq q-1,$$

be its q-adic expansion. The q-adic sum of digits function $S_q(n)$ is defined by

$$S_q(n) = \sum_{k=0}^{\infty} a_k(n).$$

On the average of $S_q(n)$, in 1975, H. Delange [3] obtained the following famous result.

Delange's Theorem. *For any positive integer N,*

$$\frac{1}{N} \sum_{n=0}^{N-1} S_q(n) = \frac{q-1}{2\log q} \log N + F\Big(\frac{\log N}{\log q}\Big),$$

where the function $F(x)$ is defined by either of the following two ways (I) *and* (II), *and $F(x)$ is periodic with period* 1, *continuous, and nowhere differentiable*:

(I) $$F(x) = \frac{q-1}{2}(1 + \lfloor x \rfloor - x) + q^{1+\lfloor x \rfloor - x} \sum_{r=0}^{\infty} q^{-r} \int_0^{q^r(q^{x-\lfloor x \rfloor -1})} \Big(\lfloor qt \rfloor - q\lfloor t \rfloor - \frac{q-1}{2}\Big) dt,$$

where $\lfloor x \rfloor$ denotes the integer part of x,

(II) $$F(x) = \sum_{k \in \mathbf{Z}} C_k e^{2\pi i k x}$$

whose Fourier coefficients are given by

$$\begin{cases} C_0 = \dfrac{q-1}{2\log q}(\log(2\pi) - 1) - \dfrac{q+1}{4}, & \\ C_k = i\dfrac{q-1}{2\pi k} \dfrac{\zeta\big(\frac{2\pi i k}{\log q}\big)}{1 + \frac{2\pi i k}{\log q}}, & k \neq 0, \end{cases}$$

where $\zeta(s)$ denotes the Riemann zeta-function.

From now on, we are specially interested in the case $q = 2$.

Let $u : \mathbf{N} \cup \{0\} \to \mathbf{C}$ be the arithmetical function defined by

$$u(n) = \begin{cases} 0, & \text{if } n = 0, \\ (-1)^{n-1}, & \text{if } n \in \mathbf{N}, \end{cases}$$

and $\xi : [0,\infty) \to \mathbf{C}$ be its summatory function:

$$\xi(x) = \sum_{0\le n\le x} u(n),$$

the value is, for $n \in \mathbf{N} \cup \{0\}$,

$$\xi(x) = \begin{cases} 0, & \text{if } x \in [2n, 2n+1), \\ 1, & \text{if } x \in [2n+1, 2n+2). \end{cases}$$

We notice that there exists a strong relation between $u(n)$ and $S_2(n)$. In fact, since the 2-adic coefficients satisfy the relation $a_k(n) = \xi(n/2^k)$, we have

$$S_2(n) = \sum_{k=0}^{\infty} \xi\Big(\frac{n}{2^k}\Big). \tag{1.1}$$

From the expression

$$\frac{1}{N}\sum_{n=0}^{N-1} S_2(n) = \frac{1}{N}\sum_{k=0}^{\infty}\sum_{n=0}^{N-1} \xi\Big(\frac{n}{2^k}\Big),$$

we can derive Delange's (I) directly, not easily but straightforward. Moreover, the Dirichlet series whose coefficients are $u(n)$ has the expression

$$\sum_{n=1}^{\infty} \frac{u(n)}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re s > 1,$$

then Perron's method and the residue analysis give Delange's (II) directly (see [8]). It seems that the function $u(n)$ plays a crucial role in the study of $S_2(n)$. In order to clarify and generalize the relation between $u(n)$ and $S_2(n)$, we present here a much more general result, which is our first main result.

Let $\mathcal{A}$ be the set of all arithmetical functions $f : \mathbf{N} \cup \{0\} \to \mathbf{C}$ with $f(0) = 0$. For $f \in \mathcal{A}$, define the map $\Phi : \mathcal{A} \to \mathcal{A}$ by

$$(\Phi(f))(n) = \sum_{k=0}^{\infty} \sum_{0\le a\le \frac{n}{2^k}} f(a). \tag{1.2}$$

And for $S \in \mathcal{A}$, define the map $\Psi : \mathcal{A} \to \mathcal{A}$ by

$$(\Psi(S))(n) = \begin{cases} 0, & \text{if } n = 0, \\ S(n) - S(n-1) - \Big(S(\frac{n}{2}) - S(\frac{n}{2}-1)\Big), & \text{if } n \ge 2 \text{ is even}, \\ S(n) - S(n-1), & \text{if } n \text{ is odd}. \end{cases} \tag{1.3}$$

Then we have

Theorem 1.1. *The map Φ is bijective with the inverse map $\Phi^{-1} = \Psi$.*

Theorem 1.1 guarantees the existence of the bijective relation between simple arithmetical functions and sum of digits type functions. According to this notation, the relation (1.1) is expressed as $S_2 = \Phi(u)$. We will prove Theorem 1.1 in Section 2.

We mention here another example. Besides the usual binary code (the ordered sequence of all 2-adic expansions), we have another binary code – the *reflected binary code* (RBC),

$$\mathrm{RBC} = \{0, 1, 11, 10, 110, 111, 101, 100, \ldots\},$$

whose definition will be given in Section 3. For the reflected binary code, its sum of digits function $S_{\mathrm{RBC}}(n)$ is naturally defined. According to the notation of Theorem 1.1, we have

$$\Psi(S_{\mathrm{RBC}}) = \chi_4, \quad \Phi(\chi_4) = S_{\mathrm{RBC}}, \tag{1.4}$$

where χ_4 is the Dirichlet character mod 4 defined by

$$\chi_4(n) = \begin{cases} 0, & \text{if } n \equiv 0 \mod 4, \\ 1, & \text{if } n \equiv 1 \mod 4, \\ 0, & \text{if } n \equiv 2 \mod 4, \\ -1, & \text{if } n \equiv 3 \mod 4, \end{cases}$$

see Table 1 in Section 3.

Flajolet and Ramshaw [5] studied the average of $S_{\mathrm{RBC}}(n)$, and they obtained Delange's (II)-type result for it. Flajolet et al. [6] obtained Delange's (II)-type result for the average of various interesting arithmetical functions, their main tools were Mellin transform and Perron's formula. Dumont and Thomas [4] studied the average of arithmetical functions coming from substitutions, by making use of the theory of substitution matrices.

The reflected binary code, which was used in the patent of F. Gray [7], is the simplest example of the *Gray codes*. The Gray codes have an outstanding property – successive words differ by a single bit. Taking into account the fact (1.4), now let us introduce a family of infinite Gray codes $\mathcal{G}$, define the sum of digits functions $S_{\mathcal{G}}$ for $\mathcal{G}$, and study the behaviour of $S_{\mathcal{G}}$ and $\Psi(S_{\mathcal{G}})$.

Definition 1.1. (finite Gray code) Let L be a positive integer, and

$$\mathcal{G}_0 = \{\mathcal{G}_0(0), \mathcal{G}_0(1), \ldots, \mathcal{G}_0(2^L - 1)\}$$

be a list of all binary L-bit words, where $\mathcal{G}_0(0)$ consists of all bits 0, i.e., $\mathcal{G}_0(0) = \underbrace{00\cdots00}_{L}$. If successive L-bit words in $\mathcal{G}_0$ differ by a single bit, then $\mathcal{G}_0$ is called an *L-bit Gray code*. If the first and last words in a Gray code $\mathcal{G}_0$ also differ by a single bit, then $\mathcal{G}_0$ is called a *cyclic Gray code*.

Starting from an L-bit Gray code $\mathcal{G}_0$, we construct the infinite Gray code $\mathcal{G}$ induced by $\mathcal{G}_0$. We use the conventions:

1. For any list of words $M = \{m_1, m_2, \ldots, m_l\}$,
$$\widetilde{M} = \{m_l, m_{l-1}, \ldots, m_1\}.$$
2. For any list M of binary words with maximam length s, M' denotes the list built from M by replacing each words m in M by adding enough 0's on the left, giving $0^{s-|m|}m$, where $|m|$ denotes the length of the word m.
3. For any list of words $M = \{m_1, m_2, \ldots, m_l\}$,
$$1 \cdot M = \{1m_1, 1m_2, \ldots, 1m_l\}.$$

Definition 1.2. (infinite Gray code) For an L-bit Gray code $\mathcal{G}_0 = \{\mathcal{G}_0(0), \mathcal{G}_0(1), \ldots, \mathcal{G}_0(2^L - 1)\}$, the list $\mathcal{G}_1 = \{\mathcal{G}_0, 1 \cdot \widetilde{\mathcal{G}_0}'\}$ is the $(L+1)$-bit Gray code. For the $(L+1)$-bit Gray code $\mathcal{G}_1$, the list $\mathcal{G}_2 = \{\mathcal{G}_1, 1 \cdot \widetilde{\mathcal{G}_1}'\}$ is the $(L+2)$-bit Gray code, and inductively the Gray codes $\mathcal{G}_3, \mathcal{G}_4, \ldots$ are defined. Then *the infinite Gray code $\mathcal{G}$ induced by $\mathcal{G}_0$* is defined by
$$\mathcal{G} = \lim_{j \to \infty} \mathcal{G}_j.$$

Definition 1.3. (sum of digits function for Gray code) Let $\mathcal{G}$ be an infinite Gray code, and put $\mathcal{G} = \{\mathcal{G}(0), \mathcal{G}(1), \ldots, \mathcal{G}(n), \ldots\}$. Then the *sum of digits function $S_{\mathcal{G}}$ for $\mathcal{G}$* is defined by
$$S_{\mathcal{G}}(n) = \text{the sum of digits of } \mathcal{G}(n).$$

In Section 3, we will give some examples of infinite Gray codes and sum of digits functions.

When we take a sum of digits function $S_{\mathcal{G}}$ for an infinite Gray code $\mathcal{G}$, then $S_{\mathcal{G}} \in \mathcal{A}$. Theorem 1.1 shows the existence of the arithmetical function $\Psi(S_{\mathcal{G}}) \in \mathcal{A}$. The next theorem states that the arithmetical functions $\Psi(S_{\mathcal{G}})$ have some common properties. We will prove this result in Section 5.

Theorem 1.2. *Let $\mathcal{G}$ be the infinite Gray code induced by an L-bit Gray code $\mathcal{G}_0$, $S_{\mathcal{G}}$ be the sum of digits function for $\mathcal{G}$, and $f_{\mathcal{G}} = \Psi(S_{\mathcal{G}})$. Then the following properties hold*:

(i) [values of $f_{\mathcal{G}}$]
$$f_{\mathcal{G}}(n) = \begin{cases} \pm 1, & \text{if } n \text{ is odd}, \\ 0, \pm 2, & \text{if } n \text{ is even}. \end{cases}$$

(ii) [periodicity]
$$f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2}), \quad n \geq 2^{L+2}.$$

(iii) [point-symmetry]

$$f_{\mathcal{G}}(n) = -f_{\mathcal{G}}(2^{L+2} - n), \quad 0 < n < 2^{L+2}.$$

(iv) [zero-sum property]

$$\sum_{n=0}^{2^{L+2}-1} f_{\mathcal{G}}(n) = 0. \tag{1.5}$$

Moreover, if the Gray code $\mathcal{G}_0$ is cyclic, then

$$\sum_{n=0}^{2^{L+1}-1} f_{\mathcal{G}}(n) = 0. \tag{1.6}$$

The sum of digits function $S_{\mathcal{G}}$ for an infinite Gray code $\mathcal{G}$ has a connection with automaton. More precisely, the difference sequence of $S_{\mathcal{G}}(n)$ is generated by an automaton.

Theorem 1.3. *Let $\mathcal{G}$ be the infinite Gray code induced by an L-bit Gray code $\mathcal{G}_0$, and $S_{\mathcal{G}}$ be the sum of digits function for $\mathcal{G}$. Let $H_{\mathcal{G}}$ be the sequence defined by*

$$H_{\mathcal{G}}(n) = \begin{cases} 1, & \text{if } n = 0, \\ S_{\mathcal{G}}(n) - S_{\mathcal{G}}(n-1), & \text{if } n \in \mathbf{N}. \end{cases} \tag{1.7}$$

Then $H_{\mathcal{G}}$ is a 2-automatic sequence.

We notice that H_{RBC} coincides with the *regular paperfolding sequence*, see Table 1 in Section 3. For the regular paperfolding sequence and automatic sequences, see Allouche and Shallit [1] Chapter 5.

Now let us study an analytic aspect of the arithmetical function $\Psi(S_{\mathcal{G}})$ for an infinite Gray code $\mathcal{G}$. Since Theorem 1.2 shows that those arithmetical functions $f_{\mathcal{G}} = \Psi(S_{\mathcal{G}})$ are periodic and satisfy the zero-sum property, we consider a little more general situation.

Let $f \in \mathcal{A}$, and $p \geq 2$ be an integer. We assume two properties on f:

[Periodicity]: f is a periodic function with period p,

[Zero-sum]: $\sum_{n=0}^{p-1} f(n) = 0$.

Let us introduce the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \Re s > 1. \tag{1.8}$$

From [Periodicity] and [Zero-sum] this Dirichlet series can be analytically extended to the whole complex s-plane, and this extension is also denoted by $L(s, f)$. For the average of $S = \Phi(f)$, we can prove the following analogous result to Delange's Theorem.

Theorem 1.4. *Let $p \geq 2$ be an integer. Assume that $f \in \mathcal{A}$ satisfies* [Periodicity] *with period p and* [Zero-sum]*. Let $S = \Phi(f)$. Let $\xi : [0, \infty) \to \mathbf{C}$ be the function defined by $\xi(x) = \sum_{0 \leq n \leq x} f(n)$. Then, for any positive integer N,*

$$\frac{1}{N}\sum_{n=0}^{N-1} S(n) = \frac{\log N}{\log 2} L(0, f) + F\Big(\frac{\log N}{\log 2}\Big) - \frac{1}{N} G(N), \tag{1.9}$$

where the function $F(x)$ is defined by either of the following two ways (I) *and* (II)*, and $F(x)$ is periodic with period* 1 *and continuous*:

(I) $F(x) = (1 + \lfloor x \rfloor - x) L(0, f) + 2^{\lfloor x \rfloor - x} \sum_{r=0}^{\infty} \int_0^{2^{x - \lfloor x \rfloor}} (\xi(2^r t) - L(0, f)) dt,$

(II) $F(x) = \sum_{k \in \mathbf{Z}} D_k e^{2\pi i k x}$

whose Fourier coefficients are given by

$$\begin{cases} D_0 = \Big(\dfrac{1}{2} - \dfrac{1}{\log 2}\Big) L(0, f) + \dfrac{L'(0, f)}{\log 2}, & \\ D_k = \dfrac{L\big(\frac{2\pi i k}{\log 2}, f\big)}{2\pi i k\big(\frac{2\pi i k}{\log 2} + 1\big)}, & k \neq 0, \end{cases}$$

the function $G(N)$ is periodic with period $p/2$, i.e., $G(N + p/2) = G(N)$, and defined by

$$G(N) = \sum_{r=1}^{\infty} \frac{1}{2^r} \int_0^{2^r N} (\xi(x) - L(0, f)) dx.$$

We will prove this in Section 7.

In Section 8, we specialize to $f = f_{\mathcal{G}} = \Psi(S_{\mathcal{G}})$ with an infinite Gray code $\mathcal{G}$. Then we can prove $L(0, f_{\mathcal{G}}) = 1/2$, which gives the following two results.

Corollary 1.1. *Let $\mathcal{G}$ be the infinite Gray code induced by an L-bit Gray code $\mathcal{G}_0$, and $S_{\mathcal{G}}$ be the sum of digits function for $\mathcal{G}$. Then, for any positive integer N,*

$$\frac{1}{N}\sum_{n=0}^{N-1} S_{\mathcal{G}}(n) = \frac{\log N}{2 \log 2} + F\Big(\frac{\log N}{\log 2}\Big) - \frac{1}{N} G(N),$$

where $F(x)$ and $G(N)$ are defined in Theorem 1.4 *with $f = \Psi(S_{\mathcal{G}})$.*

Theorem 1.5. *Under the same notation as in Theorem* 1.2*,*

$$\sum_{m=0}^{2^{L+2}} m f_{\mathcal{G}}(m) = -2^{L+1}. \tag{1.10}$$

If the L-bit Gray code $\mathcal{G}_0$ is cyclic, then

$$\sum_{m=0}^{2^{L+1}} m f_{\mathcal{G}}(m) = -2^L. \tag{1.11}$$

2. A bijection between arithmetical functions

Let $f(n)$ be an arithmetical function $f : \mathbf{N} \cup \{0\} \to \mathbf{C}$ with $f(0) = 0$, i.e., $f \in \mathcal{A}$. Define the function $\xi : [0, \infty) \to \mathbf{C}$ by

$$\xi(x) = \sum_{0 \le n \le x} f(n), \tag{2.1}$$

and the arithmetical function $S : \mathbf{N} \cup \{0\} \to \mathbf{C}$ by

$$S(n) = \sum_{k=0}^{\infty} \xi\Big(\frac{n}{2^k}\Big). \tag{2.2}$$

Since $\xi(\frac{n}{2^k}) = 0$ for k's with $2^k > n$, $S(n)$ is well-defined for any $n \in \mathbf{N} \cup \{0\}$. Obviously, $S(0) = 0$. Hence $S \in \mathcal{A}$, and $S = \Phi(f)$ (cf. (1.2)).

Lemma 2.1. *We have*

$$\xi(n) = \begin{cases} S(n) - S(\frac{n}{2}), & \text{if } n \text{ is even,} \\ S(n) - S(\frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then by (2.2),

$$S(n) = \xi(n) + S\Big(\frac{n}{2}\Big).$$

If n is odd, then $\lfloor \frac{n-1}{2^k} \rfloor = \lfloor \frac{n}{2^k} \rfloor$ for any $k \ge 1$. Since $\xi(x)$ is constant on the interval $[m, m+1)$ with $m \in \mathbf{N} \cup \{0\}$, it follows that

$$\begin{aligned} S(n) &= \xi(n) + \sum_{k=1}^{\infty} \xi\Big(\frac{n-1}{2^k}\Big) \\ &= \xi(n) + S\Big(\frac{n-1}{2}\Big). \end{aligned}$$

□

Proposition 2.1. *For $f \in \mathcal{A}$, let $S = \Phi(f)$. Then*

$$f(n) = \begin{cases} S(n) - S(n-1) - \Big(S(\frac{n}{2}) - S(\frac{n}{2} - 1)\Big), & \text{if } n \ge 2 \text{ is even,} \\ S(n) - S(n-1), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From (2.1) we have $f(n) = \xi(n) - \xi(n-1)$, then combining with Lemma 2.1, we obtain the result. □

Proposition 2.2. *For $n \in \mathbf{N}$, put $n = 2^j m$ with $j \in \mathbf{N} \cup \{0\}$ and $2 \nmid m$. Then*

$$S(n) - S(n-1) = \sum_{k=0}^{j} f(2^k m).$$

Proof. It follows from Proposition 2.1 that

$$\begin{aligned}
&\sum_{k=0}^{j} f(2^k m) \\
&= \sum_{k=1}^{j} \left(S(2^k m) - S(2^k m - 1) - \left(S\Big(\frac{2^k m}{2}\Big) - S\Big(\frac{2^k m}{2} - 1\Big) \right) \right) + f(m) \\
&= S(2^j m) - S(2^j m - 1) - \Big(S(m) - S(m-1) \Big) + f(m) \\
&= S(n) - S(n-1).
\end{aligned}$$

□

Proof of Theorem 1.1. Proposition 2.1 shows that Φ is injective, and that $\Psi(\Phi(f)) = f$ (cf. (1.3)).

Take an arbitrary $S \in \mathcal{A}$ and put $\Psi(S) = f$. We prove $\Phi(\Psi(S))(n) = S(n)$ by induction on n. Obviously, $\Phi(\Psi(S))(0) = S(0)$. Assume that $\Phi(\Psi(S))(n-1) = S(n-1)$ for an $n \in \mathbf{N}$, i.e.,

$$\sum_{k=0}^{\infty} \sum_{0 \le a \le \frac{n-1}{2^k}} f(a) = S(n-1).$$

Let $n = 2^j m$ with $j \in \mathbf{N} \cup \{0\}$ and $2 \nmid m$. Then, by Proposition 2.2,

$$\begin{aligned}
\Phi(\Psi(S))(n) &= \sum_{k=0}^{\infty} \sum_{0 \le a \le \frac{n-1}{2^k}} f(a) + \sum_{k=0}^{\infty} \sum_{\frac{n-1}{2^k} < a \le \frac{n}{2^k}} f(a) \\
&= S(n-1) + \sum_{k=0}^{j} f\Big(\frac{n}{2^k}\Big) \\
&= S(n-1) + \sum_{k=0}^{j} f(2^k m) = S(n).
\end{aligned}$$

This proves $\Phi(\Psi(S)) = S$ and therefore Φ is surjective. □

Remark. Theorem 1.1 is easily generalized to the q-adic case. Let $q \ge 2$ be an integer. For $f \in \mathcal{A}$, define the map $\Phi_q : \mathcal{A} \to \mathcal{A}$ by

$$(\Phi_q(f))(n) = \sum_{k=0}^{\infty} \sum_{0 \le a \le \frac{n}{q^k}} f(a).$$

And for $S \in \mathcal{A}$, define the map $\Psi_q : \mathcal{A} \to \mathcal{A}$ by

$$(\Psi_q(S))(n)$$
$$= \begin{cases} 0, & \text{if } n = 0, \\ S(n) - S(n-1) - \Big(S(\frac{n}{q}) - S(\frac{n}{q} - 1)\Big), & \text{if } n \geq q \text{ and } n \equiv 0 \mod q, \\ S(n) - S(n-1), & \text{if } n \geq 1 \text{ and } n \not\equiv 0 \mod q. \end{cases}$$

Then *the map* Φ_q *is bijective with the inverse map* $\Phi_q^{-1} = \Psi_q$.

In order to study sum of digits functions for infinite Gray codes, the map Φ_2 is most appropriate. By making use of Φ_q, we can prove Delange's Theorem in a different way.

3. Examples of infinite Gray codes

In this section we present some infinite Gray codes and related functions.

The reflected binary code (RBC). This is the simplest infinite Gray code. $\mathcal{G}_0 = \{0, 1\}$ is the 1-bit cyclic Gray code. According to the procedure described in Definition 1.2, we can construct the infinite Gray code induced by this $\mathcal{G}_0$:

$$\begin{aligned} \mathrm{RBC} &= \{0, 1, 11, 10, 110, 111, 101, 100, 1100, 1101, 1111, 1110, \ldots\} \\ &= \{\mathrm{RBC}(0), \mathrm{RBC}(1), \mathrm{RBC}(2), \mathrm{RBC}(3), \ldots, \mathrm{RBC}(n), \ldots\}, \quad \text{say.} \end{aligned}$$

RBC is a permutation of the usual binary code, and it has the property of Gray codes: $\mathrm{RBC}(n)$ and $\mathrm{RBC}(n+1)$ differ by a single bit.

We define the sum of digits function $S_{\mathrm{RBC}}(n)$ for RBC by

$$S_{\mathrm{RBC}}(n) = \text{the sum of digits of } \mathrm{RBC}(n),$$

see Table 1.

The infinite Gray code AG3. We have the 3-bit cyclic Gray code $\mathcal{G}_0 = \{000, 001, 011, 111, 101, 100, 110, 010\}$, which is an example of *antipodal Gray codes* (see Killian and Savage [9]). The infinite Gray code induced by this $\mathcal{G}_0$ is

$$\begin{aligned} \mathrm{AG3} = \{&000, 001, 011, 111, 101, 100, 110, 010, \\ &1010, 1110, 1100, 1101, 1111, 1011, 1001, 1000, 11000, \ldots\} \\ = \{&\mathrm{AG3}(0), \mathrm{AG3}(1), \mathrm{AG3}(2), \mathrm{AG3}(3), \ldots, \mathrm{AG3}(n), \ldots\}, \quad \text{say.} \end{aligned}$$

We define the sum of digits function $S_{\mathrm{AG3}}(n)$ for AG3 by

$$S_{\mathrm{AG3}}(n) = \text{the sum of digits of } \mathrm{AG3}(n),$$

see Table 2 in the last section (p. 332–333).

Table 1. RBC and its sum of digits function.

n	RBC	S_{RBC}	H_{RBC}	$H_{\mathrm{RBC}}(\cdot/2)$	f_{RBC}
0	0	0			
1	1	1	1		1
2	11	2	1	1	0
3	10	1	−1		−1
4	110	2	1	1	0
5	111	3	1		1
6	101	2	−1	−1	0
7	100	1	−1		−1
8	1100	2	1	1	0
9	1101	3	1		1
10	1111	4	1	1	0
11	1110	3	−1		−1
12	1010	2	−1	−1	0
13	1011	3	1		1
14	1001	2	−1	−1	0
15	1000	1	−1		−1
16	11000	2	1	1	0
17	11001	3	1		1
18	11011	4	1	1	0
19	11010	3	−1		−1
20	11110	4	1	1	0
21	11111	5	1		1
22	11101	4	−1	−1	0
23	11100	3	−1		−1
24	10100	2	−1	−1	0
25	10101	3	1		1
26	10111	4	1	1	0
27	10110	3	−1		−1
28	10010	2	−1	−1	0
29	10011	3	1		1
30	10001	2	−1	−1	0
31	10000	1	−1		−1

The infinite Gray code AG4. We start from the 4-bit antipodal Gray code

$$\begin{aligned}\mathcal{G}_0 = \{&0000, 0001, 0011, 0111, 1111, 1110, 1100, 1000,\\ &1010, 1011, 1001, 1101, 0101, 0100, 0110, 0010\}\end{aligned}$$

(see [9]). Then the infinite Gray code induced by this $\mathcal{G}_0$ is

$$\begin{aligned}\mathrm{AG4} = \{&0000, 0001, 0011, 0111, 1111, 1110, 1100, 1000,\\ &1010, 1011, 1001, 1101, 0101, 0100, 0110, 0010, 10010, \ldots\}\\ = \{&\mathrm{AG4}(0), \mathrm{AG4}(1), \mathrm{AG4}(2), \mathrm{AG4}(3), \ldots, \mathrm{AG4}(n), \ldots\}, \quad \text{say.}\end{aligned}$$

We define the sum of digits function $S_{\mathrm{AG4}}(n)$ similarly (see Table 3, p. 335–336).

4. Properties of $S_{\mathcal{G}}$ and $H_{\mathcal{G}}$

In this section we derive the properties of $S_{\mathcal{G}}$ and $H_{\mathcal{G}}$.

Lemma 4.1. *For $n \in \mathbf{N} \cup \{0\}$, put*

$$n = 2^{L+1}q + n_0, \quad q \in \mathbf{N} \cup \{0\}, \quad 0 \le n_0 < 2^{L+1}.$$

Then

$$S_{\mathcal{G}}(n) = \begin{cases} S_{\mathrm{RBC}}(2q) + S_{\mathcal{G}}(n_0), & \text{if } 0 \le n_0 < 2^L, \\ S_{\mathrm{RBC}}(2q+1) + S_{\mathcal{G}}(2^{L+1} - 1 - n_0), & \text{if } 2^L \le n_0 < 2^{L+1}, \end{cases}$$

where S_{RBC} is defined in Section 3.

Proof. Let $\mathcal{G}_0 = \{\mathcal{G}_0(0), \mathcal{G}_0(1), \ldots, \mathcal{G}_0(2^L - 1)\}$ be an L-bit Gray code, and let $\mathcal{G}$ be the infinite Gray code induced by $\mathcal{G}_0$ (cf. Definition 1.2). We notice that, when

$$n = 2^{L+1}q + n_0, \quad 0 \le n_0 < 2^L,$$

then

$$\mathcal{G}(n) = \mathrm{RBC}(2q) \cdot \mathcal{G}_0(n_0),$$

where we use $\cdot$ by the meaning of Convention 3 in Section 1, and when

$$\begin{aligned} n &= 2^{L+1}q + n_0 \\ &= 2^L(2q+1) + n_0 - 2^L, \quad 0 \le n_0 - 2^L < 2^L, \end{aligned}$$

then

$$\begin{aligned} \mathcal{G}(n) &= \mathrm{RBC}(2q+1) \cdot \mathcal{G}_0(2^L - 1 - (n_0 - 2^L)) \\ &= \mathrm{RBC}(2q+1) \cdot \mathcal{G}_0(2^{L+1} - 1 - n_0). \end{aligned}$$

From here we get this lemma directly. □

Proposition 4.1. *For $n \ge 2^{L+1}$, put*

$$n = 2^{L+1}q + n_0, \quad q \in \mathbf{N}, \quad 0 \le n_0 < 2^{L+1}.$$

Let $H_{\mathcal{G}}$ be the sequence defined by (1.7)*.*

(i) *If n is odd, then*

$$H_{\mathcal{G}}(n) = H_{\mathcal{G}}(n - 2^{L+1}).$$

(ii) *If n is even and n_0 is neither $n_0 = 0$ nor $n_0 = 2^L$, then*

$$H_{\mathcal{G}}(n) = H_{\mathcal{G}}(n - 2^{L+1}).$$

(iii) *If $n_0 = 2^L$, then*

$$H_{\mathcal{G}}(n) = (-1)^q.$$

(iv) *If $n_0 = 0$, we define $m \in \mathbf{N}$ as $q = 2^j m$, $j \in \mathbf{N} \cup \{0\}$, and $2 \nmid m$. Then*

$$H_{\mathcal{G}}(n) = (-1)^{\frac{m-1}{2}}.$$

Proof. (i) When n_0 is in the range $0 < n_0 < 2^L$, it follows from Lemma 4.1 that

$$\begin{aligned} H_{\mathcal{G}}(n) &= S_{\mathrm{RBC}}(2q) + S_{\mathcal{G}}(n_0) - S_{\mathrm{RBC}}(2q) - S_{\mathcal{G}}(n_0 - 1) \\ &= H_{\mathcal{G}}(n_0) \end{aligned}$$

and

$$\begin{aligned} H_{\mathcal{G}}(n - 2^{L+1}) &= S_{\mathrm{RBC}}(2(q-1)) + S_{\mathcal{G}}(n_0) \\ &\quad - S_{\mathrm{RBC}}(2(q-1)) - S_{\mathcal{G}}(n_0 - 1) \\ &= H_{\mathcal{G}}(n_0). \end{aligned}$$

When n_0 is in the range $2^L < n_0 < 2^{L+1}$, it follows from Lemma 4.1 that

$$\begin{aligned} H_{\mathcal{G}}(n) &= S_{\mathrm{RBC}}(2q+1) + S_{\mathcal{G}}(2^{L+1} - 1 - n_0) \\ &\quad - S_{\mathrm{RBC}}(2q+1) - S_{\mathcal{G}}(2^{L+1} - n_0) \\ &= -H_{\mathcal{G}}(2^{L+1} - n_0) \end{aligned}$$

and

$$\begin{aligned} H_{\mathcal{G}}(n - 2^{L+1}) &= S_{\mathrm{RBC}}(2(q-1)+1) + S_{\mathcal{G}}(2^{L+1} - 1 - n_0) \\ &\quad - S_{\mathrm{RBC}}(2(q-1)+1) - S_{\mathcal{G}}(2^{L+1} - n_0) \\ &= -H_{\mathcal{G}}(2^{L+1} - n_0). \end{aligned}$$

Hence in both cases we obtain (i).

We can prove (ii) similarly to (i).

(iii) In this case, $n = 2^{L+1}q + 2^L$ and $n - 1 = 2^{L+1}q + 2^L - 1$. Then, by Lemma 4.1,

$$\begin{aligned} H_{\mathcal{G}}(n) &= S_{\mathrm{RBC}}(2q+1) + S_{\mathcal{G}}(2^{L+1} - 1 - 2^L) \\ &\quad - S_{\mathrm{RBC}}(2q) - S_{\mathcal{G}}(2^L - 1) \\ &= H_{\mathrm{RBC}}(2q+1). \end{aligned}$$

Here we apply Lemma 4.1 to the case $\mathcal{G} = \mathrm{RBC}$. For $l \in \mathbf{N} \cup \{0\}$, put

$$l = 2^2 q' + l_0, \quad q' \in \mathbf{N} \cup \{0\}, \quad 0 \le l_0 < 2^2.$$

Then

$$(4.1)\quad S_{\mathrm{RBC}}(l)=\begin{cases}S_{\mathrm{RBC}}(2q')+S_{\mathrm{RBC}}(l_0), & \text{if } 0\le l_0<2,\\ S_{\mathrm{RBC}}(2q'+1)+S_{\mathrm{RBC}}(2^2-1-l_0), & \text{if } 2\le l_0<2^2.\end{cases}$$

When $q=2r$, then $2q+1=2^2r+1$. By (4.1),

$$S_{\mathrm{RBC}}(2q+1)=S_{\mathrm{RBC}}(2r)+S_{\mathrm{RBC}}(1)=S_{\mathrm{RBC}}(2r)+1$$

and

$$S_{\mathrm{RBC}}(2q)=S_{\mathrm{RBC}}(2r)+S_{\mathrm{RBC}}(0)=S_{\mathrm{RBC}}(2r).$$

Hence

$$(4.2)\qquad H_{\mathrm{RBC}}(2q+1)=1=(-1)^q.$$

When $q=2r+1$, then $2q+1=2^2r+3$. By (4.1),

$$S_{\mathrm{RBC}}(2q+1)=S_{\mathrm{RBC}}(2r+1)+S_{\mathrm{RBC}}(2^2-1-3)=S_{\mathrm{RBC}}(2r+1),$$

and

$$S_{\mathrm{RBC}}(2q)=S_{\mathrm{RBC}}(2r+1)+S_{\mathrm{RBC}}(2^2-1-2)=S_{\mathrm{RBC}}(2r+1)+1.$$

Hence

$$(4.3)\qquad H_{\mathrm{RBC}}(2q+1)=-1=(-1)^q.$$

Hence in both cases we obtain (iii).

(iv) In this case, $n=2^{L+1}q$ and $n-1=2^{L+1}(q-1)+2^{L+1}-1$. Then, by Lemma 4.1,

$$(4.4)\qquad \begin{aligned}H_{\mathcal{G}}(n)&=S_{\mathrm{RBC}}(2q)+S_{\mathcal{G}}(0)\\ &\quad -S_{\mathrm{RBC}}(2(q-1)+1)-S_{\mathcal{G}}(2^{L+1}-1-(2^{L+1}-1))\\ &=H_{\mathrm{RBC}}(2q).\end{aligned}$$

When $q=2^jm=2^j(2r+1)$, $j,r\in\mathbf{N}\cup\{0\}$, then

$$2q=\begin{cases}2^2(2^jr+2^{j-1}), & \text{if } j\in\mathbf{N},\\ 2^2r+2, & \text{if } j=0.\end{cases}$$

By (4.1),

$$\begin{aligned}S_{\mathrm{RBC}}(2q)&=\begin{cases}S_{\mathrm{RBC}}(2^{j+1}r+2^j)+S_{\mathrm{RBC}}(0), & \text{if } j\in\mathbf{N},\\ S_{\mathrm{RBC}}(2r+1)+S_{\mathrm{RBC}}(2^2-1-2), & \text{if } j=0,\end{cases}\\ &=\begin{cases}S_{\mathrm{RBC}}(q), & \text{if } j\in\mathbf{N},\\ S_{\mathrm{RBC}}(q)+1, & \text{if } j=0,\end{cases}\end{aligned}$$

and

$$S_{\mathrm{RBC}}(2q-1) = \begin{cases} S_{\mathrm{RBC}}(2^{j+1}r + 2^j - 1) + S_{\mathrm{RBC}}(2^2 - 1 - 3), & \text{if } j \in \mathbf{N}, \\ S_{\mathrm{RBC}}(2r) + S_{\mathrm{RBC}}(1), & \text{if } j = 0, \end{cases}$$

$$= \begin{cases} S_{\mathrm{RBC}}(q-1), & \text{if } j \in \mathbf{N}, \\ S_{\mathrm{RBC}}(q-1) + 1, & \text{if } j = 0. \end{cases}$$

Hence

$$H_{\mathrm{RBC}}(2q) = H_{\mathrm{RBC}}(q) = \cdots = H_{\mathrm{RBC}}(m).$$

(4.2) and (4.3) give $H_{\mathrm{RBC}}(m) = (-1)^{\frac{m-1}{2}}$, and hence

$$H_{\mathrm{RBC}}(2q) = (-1)^{\frac{m-1}{2}}.$$

Substituting this into (4.4), we obtain (iv). □

5. Proof of Theorem 1.2

Proof of Theorem 1.2 (i). Since $\mathcal{G}$ is an infinite Gray code, $\mathcal{G}(n)$ and $\mathcal{G}(n-1)$ differ by a single bit. This means $|S_{\mathcal{G}}(n) - S_{\mathcal{G}}(n-1)| = 1$ for any $n \in \mathbf{N}$, then Proposition 2.1 gives our assertion directly. □

First we assume (ii) and (iii), and prove (iv).

Proof of Theorem 1.2 (iv). (1.5) is a direct consequence of (iii).

If the Gray code $\mathcal{G}_0$ is cyclic, then, by Proposition 2.1,

$$\begin{aligned} \sum_{n=0}^{2^{L+1}-1} f_{\mathcal{G}}(n) &= \sum_{\substack{n=1 \\ n:\text{odd}}}^{2^{L+1}-1} f_{\mathcal{G}}(n) + \sum_{\substack{n=2 \\ n:\text{even}}}^{2^{L+1}-1} f_{\mathcal{G}}(n) \\ &= \sum_{n=1}^{2^{L+1}-1} \Big(S_{\mathcal{G}}(n) - S_{\mathcal{G}}(n-1) \Big) \\ &\quad - \sum_{\substack{n=2 \\ n:\text{even}}}^{2^{L+1}-1} \Big(S_{\mathcal{G}}\Big(\frac{n}{2}\Big) - S_{\mathcal{G}}\Big(\frac{n}{2} - 1\Big) \Big) \\ &= S_{\mathcal{G}}(2^{L+1} - 1) - S_{\mathcal{G}}(2^L - 1). \end{aligned}$$

From the construction of the infinite Gray code $\mathcal{G}$, we know

$$\mathcal{G}(2^{L+1} - 1) = 1\underbrace{00\cdots00}_{L},$$

hence $S_{\mathcal{G}}(2^{L+1} - 1) = 1$. Since $\mathcal{G}_0$ is cyclic, $\mathcal{G}(2^L - 1)$ and $\mathcal{G}(0)$ differ by a single bit, and hence $S_{\mathcal{G}}(2^L - 1) = 1$. These give (1.6). □

Proof of Theorem 1.2 (ii). By (1.3) and (1.7),

$$f_{\mathcal{G}}(n) = \begin{cases} H_{\mathcal{G}}(n) - H_{\mathcal{G}}\left(\frac{n}{2}\right), & \text{if } n \text{ is even}, \\ H_{\mathcal{G}}(n), & \text{if } n \text{ is odd}. \end{cases} \tag{5.1}$$

When n is odd, it follows from Proposition 4.1 (i) and (5.1) that $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2})$.

Now we consider the case n is even. We put

$$n = 2^{L+2}q_1 + n_1, \quad q_1 \in \mathbf{N}, \quad 0 \le n_1 < 2^{L+2}, \quad (n_1\text{: even}),$$

and divide this case into some subcases according to the size of n_1. In each case, taking into account of (5.1), we calculate $H_{\mathcal{G}}(n)$ and $H_{\mathcal{G}}(n/2)$, one by one.

(Case 1): $0 < n_1 < 2^{L+1}$ and $n_1 \neq 2^L$. Since $n = 2^{L+1}(2q_1) + n_1$ and $n - 2^{L+1} = 2^{L+1}(2q_1 - 1) + n_1$, Proposition 4.1 (ii) gives $H_{\mathcal{G}}(n) = H_{\mathcal{G}}(n - 2^{L+2})$. Since

$$\frac{n}{2} = 2^{L+1}q_1 + \frac{n_1}{2}, \quad 0 < \frac{n_1}{2} < 2^L,$$

Proposition 4.1 (i) and (ii) give

$$H_{\mathcal{G}}\left(\frac{n}{2}\right) = H_{\mathcal{G}}\left(\frac{n}{2} - 2^{L+1}\right) = H_{\mathcal{G}}\left(\frac{n - 2^{L+2}}{2}\right).$$

Thus $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2})$.

(Case 2): $2^{L+1} < n_1 < 2^{L+2}$ and $n_1 \neq 2^{L+1} + 2^L$. In this case we rewrite n as $n = 2^{L+1}(2q_1 + 1) + n_1 - 2^{L+1}$, where $0 < n_1 - 2^{L+1} < 2^{L+1}$ and $n_1 - 2^{L+1} \neq 2^L$. Then Proposition 4.1 (ii) gives $H_{\mathcal{G}}(n) = H_{\mathcal{G}}(n - 2^{L+2})$. Since

$$\frac{n}{2} = 2^{L+1}q_1 + \frac{n_1}{2}, \quad 2^L < \frac{n_1}{2} < 2^{L+1},$$

Proposition 4.1 (i) and (ii) give

$$H_{\mathcal{G}}\left(\frac{n}{2}\right) = H_{\mathcal{G}}\left(\frac{n}{2} - 2^{L+1}\right) = H_{\mathcal{G}}\left(\frac{n - 2^{L+2}}{2}\right).$$

Thus $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2})$.

(Case 3): $n_1 = 2^L$, i.e., $n = 2^{L+1}(2q_1) + 2^L$. Proposition 4.1 (iii) gives $H_{\mathcal{G}}(n) = (-1)^{2q_1} = 1$. Since $n - 2^{L+2} = 2^{L+1}(2q_1 - 2) + 2^L$, $H_{\mathcal{G}}(n - 2^{L+2}) = (-1)^{2q_1 - 2} = 1$. Hence

$$H_{\mathcal{G}}(n) = H_{\mathcal{G}}(n - 2^{L+2}).$$

Since $n/2 = 2^{L+1}q_1 + 2^{L-1}$, Proposition 4.1 (i) and (ii) give

$$H_{\mathcal{G}}\left(\frac{n}{2}\right) = H_{\mathcal{G}}\left(\frac{n}{2} - 2^{L+1}\right) = H_{\mathcal{G}}\left(\frac{n - 2^{L+2}}{2}\right).$$

Thus $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2})$.

(Case 4): $n_1 = 2^{L+1} + 2^L$, i.e., $n = 2^{L+1}(2q_1 + 1) + 2^L$. In this case, we can prove $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2})$ similarly to Case 3.

(Case 5): When $n_1 = 0$, define m as $q_1 = 2^j m$ and $2 \nmid m$, i.e., $n = 2^{L+1}(2^{j+1}m)$. Then Proposition 4.1 (iv) gives

$$H_{\mathcal{G}}(n) = (-1)^{\frac{m-1}{2}}. \tag{5.2}$$

And $n - 2^{L+2} = 2^{L+1}(2(2^j m - 1))$. If $j \geq 1$, then Proposition 4.1 (iv) gives

$$H_{\mathcal{G}}(n - 2^{L+2}) = (-1)^{\frac{2^j m-2}{2}}. \tag{5.3}$$

If $j = 0$, define m' as $m - 1 = 2^{j'}m'$ and $2 \nmid m'$, i.e., $n - 2^{L+2} = 2^{L+1}(2^{j'+1}m')$. Then Proposition 4.1 (iv) gives

$$H_{\mathcal{G}}(n - 2^{L+2}) = (-1)^{\frac{m'-1}{2}}. \tag{5.4}$$

Now $n/2 = 2^{L+1}(2^j m)$. Then Proposition 4.1 (iv) gives

$$H_{\mathcal{G}}\Big(\frac{n}{2}\Big) = (-1)^{\frac{m-1}{2}}. \tag{5.5}$$

And $n/2 - 2^{L+1} = 2^{L+1}(2^j m - 1)$. By Proposition 4.1 (iv), if $j \geq 1$, then

$$H_{\mathcal{G}}\Big(\frac{n}{2} - 2^{L+1}\Big) = (-1)^{\frac{2^j m-2}{2}}, \tag{5.6}$$

and, if $j = 0$, then

$$H_{\mathcal{G}}\Big(\frac{n}{2} - 2^{L+1}\Big) = (-1)^{\frac{m'-1}{2}}. \tag{5.7}$$

Combining (5.2) – (5.7) with (5.1), we get $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2}) = 0$.

(Case 6): $n_1 = 2^{L+1}$, i.e., $n = 2^{L+1}(2q_1 + 1)$. Proposition 4.1 (iv) gives

$$H_{\mathcal{G}}(n) = (-1)^{q_1}. \tag{5.8}$$

Since $n - 2^{L+2} = 2^{L+1}(2q_1 - 1)$,

$$H_{\mathcal{G}}(n - 2^{L+2}) = (-1)^{q_1-1}. \tag{5.9}$$

Now

$$\frac{n}{2} = 2^{L+1}q_1 + 2^L \quad \text{and} \quad \frac{n}{2} - 2^{L+1} = 2^{L+1}(q_1 - 1) + 2^L.$$

Then Proposition 4.1 (iii) gives

$$H_{\mathcal{G}}\Big(\frac{n}{2}\Big) = (-1)^{q_1} \tag{5.10}$$

and

$$H_{\mathcal{G}}\Big(\frac{n}{2} - 2^{L+1}\Big) = (-1)^{q_1-1}. \tag{5.11}$$

Combining (5.8) – (5.11) with (5.1), we get $f_{\mathcal{G}}(n) = f_{\mathcal{G}}(n - 2^{L+2}) = 0$.

This completes the proof of Theorem 1.2 (ii). □

Proof of Theorem 1.2 (iii). By the procedure to get $\mathcal{G}_2$ from $\mathcal{G}_1$ (cf. Definition 1.2),

$$S_{\mathcal{G}}(n) = S_{\mathcal{G}}(2^{L+2} - 1 - n) + 1, \quad 2^{L+1} \leq n < 2^{L+2}, \tag{5.12}$$

and

$$S_{\mathcal{G}}(n-1) = S_{\mathcal{G}}(2^{L+2}-n)+1, \quad 2^{L+1} < n \le 2^{L+2}. \tag{5.13}$$

From (5.12) and (5.13), it follows that

$$H_{\mathcal{G}}(n) = -H_{\mathcal{G}}(2^{L+2}-n), \quad 0 < n < 2^{L+2},\ n \ne 2^{L+1}. \tag{5.14}$$

If n is odd, then (5.1) and (5.14) give the desired formula.

If n is even with $0 < n < 2^{L+2}$ and $n \ne 2^{L+1}$, then (5.1) and (5.14) give

$$f_{\mathcal{G}}(n) = -H_{\mathcal{G}}(2^{L+2}-n) + H_{\mathcal{G}}\Big(2^{L+2}-\frac{n}{2}\Big). \tag{5.15}$$

Since

$$2^{L+2}-\frac{n}{2} = 2^{L+1} + \Big(2^{L+1}-\frac{n}{2}\Big), \quad 0 < 2^{L+1}-\frac{n}{2} < 2^{L+1},\ 2^{L+1}-\frac{n}{2} \ne 2^{L},$$

Proposition 4.1 (i) and (ii) give

$$H_{\mathcal{G}}\Big(2^{L+2}-\frac{n}{2}\Big) = H_{\mathcal{G}}\Big(2^{L+1}-\frac{n}{2}\Big) = H_{\mathcal{G}}\Big(\frac{2^{L+2}-n}{2}\Big).$$

Substituting this into (5.15), we obtain the desired formula.

Finally, Theorem 1.2 (ii), Proposition 4.1 (iv), and Proposition 4.1 (iii) give $f_{\mathcal{G}}(2^{L+1}) = f_{\mathcal{G}}(2^{L+2}+2^{L+1})$, $H_{\mathcal{G}}(2^{L+2}+2^{L+1}) = -1$, and $H_{\mathcal{G}}(2^{L+1}+2^{L}) = -1$, respectively. Hence $f_{\mathcal{G}}(2^{L+1}) = 0$. □

6. Proof of Theorem 1.3

Let $K_2(H_{\mathcal{G}})$ be the 2-kernel of $H_{\mathcal{G}}$, i.e., the set of subsequences of $H_{\mathcal{G}}$ of the form $H_{\mathcal{G}}(2^a n+b)$, where $a \in \mathbf{N}\cup\{0\}$ and $0 \le b < 2^a$. In order to prove Theorem 1.3, it is sufficient to prove the finiteness of the 2-kernel $K_2(H_{\mathcal{G}})$, see Theorem 6.6.2 of Allouche and Shallit [1] p.185.

If $a \le L$, then the number of sequences of the form $H_{\mathcal{G}}(2^a n+b)$, $0 \le a \le L$, $0 \le b < 2^L$, is at most $(L+1)2^L$. We divide the case $a \ge L+1$ into some subcases according to the property of b.

(Case 1): b is odd. Let b_0 be a non-negative integer with $b \equiv b_0 \pmod{2^{L+1}}$, $0 \le b_0 < 2^{L+1}$. Then b_0 is odd and $2^a n + b = 2^{L+1}m + b_0$ for some $m \in \mathbf{N}\cup\{0\}$. Proposition 4.1 (i) gives

$$H_{\mathcal{G}}(2^a n+b) = H_{\mathcal{G}}(2^{L+1}m + b_0 - 2^{L+1}) = \cdots = H_{\mathcal{G}}(b_0).$$

Since the values of $H_{\mathcal{G}}$ are 1 or -1, the number of sequences of the form $H_{\mathcal{G}}(2^a n+b)$ is at most 2.

(Case 2): b is even, $b \not\equiv 0 \pmod{2^{L+1}}$, and $b \not\equiv 2^L \pmod{2^{L+1}}$. Let b_0 be a non-negative integer chosen similarly as in Case 1. Then b_0 is even, $b_0 \ne 0$, and $b_0 \ne 2^L$. By Proposition 4.1 (ii) and the same argument as above, we see that the number of sequences of the form $H_{\mathcal{G}}(2^a n+b)$ is at most 2.

(Case 3): $b \equiv 2^L \pmod{2^{L+1}}$. Let $b = 2^{L+1}r + 2^L$, $r \in \mathbf{N} \cup \{0\}$. Then $2^a n + b = 2^{L+1}(2^{a-L-1}n + r) + 2^L$. Proposition 4.1 (iii) gives

$$\begin{aligned} H_{\mathcal{G}}(2^a n + b) &= (-1)^{2^{a-L-1}n+r} \\ &= \begin{cases} (-1)^n(-1)^r, & \text{if } a = L+1, \\ (-1)^r, & \text{if } a > L+1. \end{cases} \end{aligned}$$

Hence the number of sequences of the form $H_{\mathcal{G}}(2^a n + b)$ is at most 4.

(Case 4): $b \equiv 0 \pmod{2^{L+1}}$ and $b \neq 0$. Let $b = 2^j r$, $j \in \mathbf{N} \cup \{0\}$, $2 \nmid r$. Obviously, $L + 1 \leq j < a$. Then $2^a n + b = 2^{L+1}2^{j-L-1}(2^{a-j}n + r)$, and $2^{a-j}n + r$ is odd. Proposition 4.1 (iv) gives

$$\begin{aligned} H_{\mathcal{G}}(2^a n + b) &= (-1)^{\frac{2^{a-j}n+r-1}{2}} \\ &= \begin{cases} (-1)^n(-1)^{\frac{r-1}{2}}, & \text{if } a = j+1, \\ (-1)^{\frac{r-1}{2}}, & \text{if } a > j+1. \end{cases} \end{aligned}$$

Hence the number of sequences of the form $H_{\mathcal{G}}(2^a n + b)$ is at most 4.

(Case 5): $b = 0$. Proposition 2.2 gives

$$H_{\mathcal{G}}(2^a n) = f_{\mathcal{G}}(2^a n) + f_{\mathcal{G}}(2^{a-1}n) + \cdots + f_{\mathcal{G}}(2n) + H_{\mathcal{G}}(n).$$

By Theorem 1.2 (ii), $f_{\mathcal{G}}(2^a n) = f_{\mathcal{G}}(0) = 0$ for $a \geq L + 2$. Hence

$$H_{\mathcal{G}}(2^a n) = f_{\mathcal{G}}(2^{L+1}n) + f_{\mathcal{G}}(2^L n) + \cdots + f_{\mathcal{G}}(2n) + H_{\mathcal{G}}(n),$$

and the number of sequences of the form $H_{\mathcal{G}}(2^a n)$ is 1.

Thus the 2-kernel $K_2(H_{\mathcal{G}})$ is a finite set. □

Corollary 6.1. *Let $\mathcal{G}$ be the infinite Gray code induced by an L-bit Gray code $\mathcal{G}_0$, and $S_{\mathcal{G}}$ be the sum of digits function for $\mathcal{G}$. Then $S_{\mathcal{G}}$ is 2-regular.*

Proof. Since $H_{\mathcal{G}}$ is a 2-automatic sequence and $S_{\mathcal{G}}(n) = \sum_{m=1}^{n} H_{\mathcal{G}}(m)$, Theorems 16.1.5 and 16.4.1 of Allouche and Shallit [1] give that $S_{\mathcal{G}}$ is 2-regular. □

7. Proof of Theorem 1.4

Let $p \geq 2$ be an integer. Let $f : \mathbf{N} \cup \{0\} \to \mathbf{C}$ be the arithmetical function with $f(0) = 0$, $\xi : [0, \infty) \to \mathbf{C}$ be the function defined by (2.1), and $S = \Phi(f)$. Throughout this section, we assume that

[Periodicity]: f is a periodic function with period p,
[Zero-sum]: $\sum_{n=0}^{p-1} f(n) = 0$.

It is obvious that

[C1]: $\xi(x)$ is constant on the interval $[m, m+1)$ with $m \in \mathbf{N} \cup \{0\}$,
[C2]: $\xi(x) = 0$ for $x \in [0, 1)$.

Moreover, [Periodicity] and [Zero-sum] give that

[C3]: ξ is a periodic function with period p.

Here we prepare some lemmas.

Lemma 7.1. *We have*

$$\frac{1}{N}\sum_{n=0}^{N-1} S(n) = \frac{1}{N}\sum_{k=0}^{\infty} 2^k \int_0^{\frac{N}{2^k}} \xi(x)dx.$$

Proof. By (2.2),

$$\sum_{n=0}^{N-1} S(n) = \sum_{k=0}^{\infty}\sum_{n=0}^{N-1} \xi\Big(\frac{n}{2^k}\Big). \tag{7.1}$$

Since $0 \le \{\frac{x}{2^k}\} - \frac{\{x\}}{2^k} < 1$ and [C1], the inner sum of (7.1) is expressed as

$$\begin{aligned}\sum_{n=0}^{N-1} \xi\Big(\frac{n}{2^k}\Big) &= \int_0^N \xi\Big(\frac{\lfloor x\rfloor}{2^k}\Big)dx = \int_0^N \xi\Big(\Big\lfloor\frac{x}{2^k}\Big\rfloor + \Big\{\frac{x}{2^k}\Big\} - \frac{\{x\}}{2^k}\Big)dx \\ &= \int_0^N \xi\Big(\Big\lfloor\frac{x}{2^k}\Big\rfloor\Big)dx = \int_0^N \xi\Big(\frac{x}{2^k}\Big)dx.\end{aligned}$$

Substituting this into (7.1), we obtain the desired formula. □

Lemma 7.2. *We have*

$$\frac{L(s,f)}{s} = \int_0^{\infty} \xi(x)\frac{1}{x^{s+1}}dx, \quad \Re s > 0. \tag{7.2}$$

Proof. The Dirichlet series (1.8) is analytic for $\Re s > 1$ because of the boundedness of f. By [Periodicity],

$$L(s,f) = \sum_{j=0}^{\infty}\sum_{m=1}^{p} \frac{f(m)}{(m+jp)^s}, \quad \Re s > 1. \tag{7.3}$$

By integration by parts, [Zero-sum], [C2], and [C3], the inner sum is expressed as

$$\begin{aligned}\sum_{m=1}^{p} \frac{f(m)}{(m+jp)^s} &= \sum_{m=1}^{p} f(m)\frac{1}{(p+jp)^s} + s\int_1^p \xi(x)\frac{1}{(x+jp)^{s+1}}dx \\ &= s\int_{jp}^{(j+1)p} \xi(x)\frac{1}{x^{s+1}}dx.\end{aligned}$$

Substituting this into (7.3), we obtain (7.2) for $\Re s > 1$. (7.2) is also valid for $\Re s > 0$ because of the boundedness of ξ and [C2]. □

Lemma 7.3. *We have*

$$\frac{1}{N}\sum_{n=0}^{N-1} S(n) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{2^s}{2^s-1}\cdot\frac{L(s,f)N^s}{s(s+1)}ds, \quad \alpha > 1. \tag{7.4}$$

Proof. Let

$$\Xi(y) = \int_0^y \xi(x)dx.$$

Then the boundedness of ξ gives $|\Xi(y)| \le C|y|$ for some positive constant C. Hence, applying integration by parts to (7.2), we have

$$\frac{L(s,f)}{s(s+1)} = \int_0^\infty \Xi(y)\frac{1}{y^{s+2}}dy, \quad \Re s > 0.$$

Applying the Mellin inversion formula to the above, we have

$$\frac{1}{y}\int_0^y \xi(x)dx = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(s,f)y^s}{s(s+1)}ds, \quad \alpha > 1. \tag{7.5}$$

Lemma 7.1 and (7.5) give (7.4). □

Remark. By comparing the Dirichlet coefficients, (1.3) is equivalent to

$$\sum_{n=1}^\infty \frac{S(n)-S(n-1)}{n^s} = \frac{2^s}{2^s-1}\cdot L(s,f), \quad \Re s > 1. \tag{7.6}$$

Lemma 7.3 is directly derived from (7.6). In fact, by the Perron type formula with (7.6),

$$\frac{1}{N}\sum_{n=1}^N (S(n)-S(n-1))(N-n) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{2^s}{2^s-1}\cdot\frac{L(s,f)N^s}{s(s+1)}ds,$$

and the left-hand side is equal to $\frac{1}{N}\sum_{n=0}^{N-1} S(n)$.

Lemma 7.4. *For $L(s,f)$ the following properties hold*:

(i) *The Dirichlet series* (1.8) *can be analytically extended to the whole complex s-plane.*

(ii)
$$L(s,f) \ll (1+|t|)^{\frac{1}{2}-\sigma}, \quad s = \sigma + it,$$
where the implied constant is uniform for s in a vertical strip $\sigma_1 \le \sigma \le \sigma_2 < 0$.

Proof. (i) From (7.3) it follows that

$$L(s,f) = \frac{1}{p^s}\sum_{m=1}^p f(m)\zeta\Big(s,\frac{m}{p}\Big), \quad \Re s > 1, \tag{7.7}$$

where $\zeta(s,a)$ is the Hurwitz zeta-function defined by

$$\zeta(s,a) = \sum_{n=0}^\infty \frac{1}{(n+a)^s}, \quad 0 < a \le 1,\ \Re s > 1.$$

The function $\zeta(s,a)$ is analytically extended to the whole complex s-plane with a simple pole at $s=1$, then $L(s,f)$ has the same property. (7.2) shows that $L(s,f)$ is also analytic at $s=1$.

(ii) Substituting Hurwitz's formula (see [2] p.257) into (7.7), we get

$$L(s,f)=\frac{1}{p^s}\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Bigg(e^{-\frac{\pi i(1-s)}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\sum_{m=1}^{p}f(m)e^{2\pi i n\frac{m}{p}} + e^{\frac{\pi i(1-s)}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\sum_{m=1}^{p}f(m)e^{-2\pi i n\frac{m}{p}}\Bigg).$$

Using this expression and the estimate

$$\Gamma(s)\ll e^{-\frac{\pi}{2}|t|}(1+|t|)^{\sigma-\frac{1}{2}},\quad C_1\le\sigma\le C_2,\ |t|\ge 1,$$

we have the desired estimate. □

Lemma 7.5. *Let* $\eta:[0,\infty)\to\mathbf{C}$ *be the function defined by*

$$\eta(y)=\int_0^y(\xi(x)-L(0,f))dx.$$

Then $\eta(p)=0$ *and* η *is a periodic function with period* p.

Proof. By (7.7), $\zeta(0,a)=\frac{1}{2}-a$, and [Zero-sum],

$$L(0,f)=-\frac{1}{p}\sum_{m=0}^{p}mf(m). \tag{7.8}$$

Then, by integration by parts and [Zero-sum],

$$L(0,f)=\frac{1}{p}\int_0^p\xi(y)dy.$$

Hence

$$\eta(p)=\frac{1}{p}\int_0^p\int_0^p(\xi(x)-\xi(y))dydx=0.$$

By $\eta(p)=0$ and [C3],

$$\begin{aligned}\eta(y+p)&=\int_0^p(\xi(x)-L(0,f))dx+\int_p^{y+p}(\xi(x)-L(0,f))dx\\&=\int_0^y(\xi(x+p)-L(0,f))dx=\eta(y).\end{aligned}$$

□

Now let us prove Theorem 1.4 (II). By Lemma 7.4 and the Phragmén-Lindelöf convexity principle, we can shift the contour of integration of (7.4) to the vertical line $\Re s = \beta$ with $-\frac{1}{2} < \beta < 0$. Then

$$\begin{aligned}(7.9)\qquad \frac{1}{N}\sum_{n=0}^{N-1} S(n) &= \operatorname*{Res}_{s=0} \frac{1}{2^s-1}\cdot\frac{L(s,f)(2N)^s}{s(s+1)} \\ &\quad + \sum_{\substack{k\in\mathbf{Z}\\ k\neq 0}} \operatorname*{Res}_{s=\frac{2\pi ik}{\log 2}} \frac{1}{2^s-1}\cdot\frac{L(s,f)(2N)^s}{s(s+1)} \\ &\quad - \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{2^s}{1-2^s}\cdot\frac{L(s,f)N^s}{s(s+1)}ds \\ &= R_1 + \sum_{\substack{k\in\mathbf{Z}\\ k\neq 0}} R_2 - I(N), \quad \text{say.}\end{aligned}$$

Firstly, let us consider $I(N)$. We have

$$I(N) = \sum_{r=1}^{\infty}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{L(s,f)(2^rN)^s}{s(s+1)}ds.$$

After shifting the contour of integration of (7.5) to the vertical line $\Re s = \beta$,

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{L(s,f)y^s}{s(s+1)}ds = \frac{1}{y}\int_0^y(\xi(x)-L(0,f))dx,$$

and hence

$$(7.10)\qquad I(N) = \sum_{r=1}^{\infty}\frac{1}{2^rN}\int_0^{2^rN}(\xi(x)-L(0,f))dx,$$

where the series converges absolutely by Lemma 7.5. $G(N)$ in Theorem 1.4 is equal to $N\cdot I(N)$, and it is a periodic function with period $p/2$. In fact, by Lemma 7.5,

$$G(N+p/2) = \sum_{r=1}^{\infty}\frac{1}{2^r}\eta(2^rN+p2^{r-1}) = \sum_{r=1}^{\infty}\frac{1}{2^r}\eta(2^rN) = G(N).$$

Secondly, let us consider R_1. We get, near the point $s=0$,

$$\frac{1}{2^s-1} = \frac{1}{s\log 2}\sum_{n=0}^{\infty}\frac{B_n}{n!}(s\log 2)^n = \frac{1}{s\log 2} - \frac{1}{2} + O(|s|),$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ... are the Bernoulli numbers,

$$\frac{1}{s+1} = 1 - s + O(|s|^2),$$

$$(2N)^s = 1 + s\log(2N) + O(|s|^2),$$

and

$$L(s,f) = L(0,f) + L'(0,f)s + O(|s|^2).$$

Using these estimates, we get

$$R_1 = \Big(\frac{1}{2} + \frac{\log N - 1}{\log 2}\Big) L(0,f) + \frac{L'(0,f)}{\log 2}. \tag{7.11}$$

Finally, let us consider R_2. We get, near the point $s = \frac{2\pi ik}{\log 2}$,

$$\begin{aligned}\frac{1}{2^s - 1} &= \frac{1}{s\log 2 - 2\pi ik} \sum_{n=0}^{\infty} \frac{B_n}{n!} (s\log 2 - 2\pi ik)^n \\ &= \frac{1}{s\log 2 - 2\pi ik} + O(1),\end{aligned}$$

and hence

$$R_2 = \frac{L(\frac{2\pi ik}{\log 2}, f) N^{\frac{2\pi ik}{\log 2}}}{2\pi ik(\frac{2\pi ik}{\log 2} + 1)}. \tag{7.12}$$

Substituting (7.10), (7.11), and (7.12) into (7.9), we obtain Theorem 1.4 (II). □

Next, let us prove Theorem 1.4 (I). For any $N \in \mathbf{N}$, put $M,\ R \in \mathbf{N} \cup \{0\}$ with $N = 2^M + R$, $0 \le R < 2^M$, i.e., $M = \lfloor \frac{\log N}{\log 2} \rfloor$. By Lemma 7.1,

$$\frac{1}{N} \sum_{n=0}^{N-1} S(n) = \frac{1}{N} \sum_{k=0}^{M} \int_0^N \xi\Big(\frac{x}{2^k}\Big) dx + \frac{1}{N} \sum_{k=M+1}^{\infty} \int_0^N \xi\Big(\frac{x}{2^k}\Big) dx,$$

and the second term on the right-hand side is 0 by [C2]. Hence

$$\begin{aligned}\frac{1}{N} \sum_{n=0}^{N-1} S(n) &= (M+1) L(0,f) + \frac{1}{N} \sum_{k=0}^{M} \int_0^N \Big(\xi\Big(\frac{x}{2^k}\Big) - L(0,f)\Big) dx \\ &= (M+1) L(0,f) + \frac{1}{N} J, \quad \text{say.}\end{aligned} \tag{7.13}$$

We have

$$\begin{aligned}J &= \sum_{r=0}^{M} \int_0^N \Big(\xi\Big(2^r \cdot \frac{x}{2^M}\Big) - L(0,f)\Big) dx \\ &= 2^M \sum_{r=0}^{M} \int_0^{\frac{N}{2^M}} (\xi(2^r x) - L(0,f)) dx.\end{aligned}$$

Here

$$\begin{aligned}2^M &\sum_{r=M+1}^{\infty} \int_0^{\frac{N}{2^M}} (\xi(2^r x) - L(0,f))dx \\ &= 2^M \sum_{r=1}^{\infty} \int_0^{\frac{N}{2^M}} (\xi(2^r \cdot 2^M x) - L(0,f))dx \\ &= \sum_{r=1}^{\infty} \int_0^{N} (\xi(2^r x) - L(0,f))dx = G(N).\end{aligned}$$

Hence

$$J = 2^M \sum_{r=0}^{\infty} \int_0^{\frac{N}{2^M}} (\xi(2^r x) - L(0,f))dx - G(N). \tag{7.14}$$

Substituting (7.14) into (7.13), we obtain

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} S(n) &= (M+1)L(0,f) \\ &\quad + \frac{2^M}{N}\sum_{r=0}^{\infty} \int_0^{\frac{N}{2^M}} (\xi(2^r x) - L(0,f))dx - \frac{1}{N}G(N),\end{aligned}$$

which is identical with Theorem 1.4 (I). □

8. Proof of Corollary 1.1 and Theorem 1.5

Proof of Corollary 1.1. By Theorem 1.2, the formula (1.9) is valid in the setting of $p = 2^{L+2}$, $f = f_{\mathcal{G}}$, $\xi = \xi_{\mathcal{G}}$, and $S = S_{\mathcal{G}}$. An L-bit Gray code $\mathcal{G}_0$, which contains 2^L words, is a permutation of the usual binary code. Hence, for all $N = 2^{L+l}$, $l \geq 0$,

$$\frac{1}{N}\sum_{n=0}^{N-1} S_{\mathcal{G}}(n) = \frac{1}{N}\sum_{n=0}^{N-1} S_2(n).$$

Thus the main term of (1.9) and that of Delange's (2-adic case) are identical. This concludes $L(0, f_{\mathcal{G}}) = 1/2$. □

Proof of Theorem 1.5. (7.8) with $f = f_{\mathcal{G}}$ and $L(0, f_{\mathcal{G}}) = 1/2$ give (1.10). By $f_{\mathcal{G}}(2^{L+1}) = 0$, Theorem 1.2 (iii), and (1.6), we have

$$\begin{aligned}\sum_{m=0}^{2^{L+2}} m f_{\mathcal{G}}(m) &= \sum_{m=0}^{2^{L+1}-1} m f_{\mathcal{G}}(m) + \sum_{m=0}^{2^{L+1}-1} (2^{L+2}-m) f_{\mathcal{G}}(2^{L+2}-m) \\ &= 2\sum_{m=0}^{2^{L+1}-1} m f_{\mathcal{G}}(m).\end{aligned}$$

This and (1.10) conclude (1.11). □

Table 2. AG3 and its sum of digits function.

n	AG3	S_{AG3}	H_{AG3}	$H_{\mathrm{AG3}}(\cdot/2)$	f_{AG3}
0	000	0			
1	001	1	1		1
2	011	2	1	1	0
3	111	3	1		1
4	101	2	−1	1	−2
5	100	1	−1		−1
6	110	2	1	1	0
7	010	1	−1		−1
8	1010	2	1	−1	2
9	1110	3	1		1
10	1100	2	−1	−1	0
11	1101	3	1		1
12	1111	4	1	1	0
13	1011	3	−1		−1
14	1001	2	−1	−1	0
15	1000	1	−1		−1
16	11000	2	1	1	0
17	11001	3	1		1
18	11011	4	1	1	0
19	11111	5	1		1
20	11101	4	−1	−1	0
21	11100	3	−1		−1
22	11110	4	1	1	0
23	11010	3	−1		−1
24	10010	2	−1	1	−2
25	10110	3	1		1
26	10100	2	−1	−1	0
27	10101	3	1		1
28	10111	4	1	−1	2
29	10011	3	−1		−1
30	10001	2	−1	−1	0
31	10000	1	−1		−1

9. Examples and Tables

The infinite Gray code AG3 (cf. Section 3) is induced by the 3-bit cyclic Gray code. Then Theorem 1.2 (ii) shows that $f_{\mathrm{AG3}} = \Psi(S_{\mathrm{AG3}})$ is a periodic function with period $32 (= 2^{3+2})$. Table 2 gives the list of the values of $f_{\mathrm{AG3}}(n)$.

n	AG3	S_{AG3}	H_{AG3}	$H_{\mathrm{AG3}}(\cdot/2)$	f_{AG3}
32	110000	2	1	1	0
33	110001	3	1		1
34	110011	4	1	1	0
35	110111	5	1		1
36	110101	4	−1	1	−2
37	110100	3	−1		−1
38	110110	4	1	1	0
39	110010	3	−1		−1
40	111010	4	1	−1	2
41	111110	5	1		1
42	111100	4	−1	−1	0
43	111101	5	1		1
44	111111	6	1	1	0
45	111011	5	−1		−1
46	111001	4	−1	−1	0
47	111000	3	−1		−1
48	101000	2	−1	−1	0
49	101001	3	1		1
50	101011	4	1	1	0
51	101111	5	1		1
52	101101	4	−1	−1	0
53	101100	3	−1		−1
54	101110	4	1	1	0
55	101010	3	−1		−1
56	100010	2	−1	1	−2
57	100110	3	1		1
58	100100	2	−1	−1	0
59	100101	3	1		1
60	100111	4	1	−1	2
61	100011	3	−1		−1
62	100001	2	−1	−1	0
63	100000	1	−1		−1

The periodicity of f_{AG3} and Table 2 give

Proposition 9.1. *Let* $f_{\mathrm{AG3}} = \Psi(S_{\mathrm{AG3}})$. *Then*

$$f_{\mathrm{AG3}}(n) = \chi_8(n) + \varepsilon_{32}(n),$$

where χ_8, ε_{32} *are periodic functions with periods* 8, 32, *respectively, defined by*

$$\chi_8(n) = \begin{cases} 0, & \textit{if } n \equiv 0 \bmod 8, \\ 1, & \textit{if } n \equiv 1 \bmod 8, \\ 0, & \textit{if } n \equiv 2 \bmod 8, \\ 1, & \textit{if } n \equiv 3 \bmod 8, \\ 0, & \textit{if } n \equiv 4 \bmod 8, \\ -1, & \textit{if } n \equiv 5 \bmod 8, \\ 0, & \textit{if } n \equiv 6 \bmod 8, \\ -1, & \textit{if } n \equiv 7 \bmod 8, \end{cases} \qquad \varepsilon_{32}(n) = \begin{cases} -2, & \textit{if } n \equiv 4 \bmod 32, \\ 2, & \textit{if } n \equiv 8 \bmod 32, \\ -2, & \textit{if } n \equiv 24 \bmod 32, \\ 2, & \textit{if } n \equiv 28 \bmod 32, \\ 0, & \textit{otherwise.} \end{cases}$$

Similarly, $f_{\mathrm{AG4}} = \Psi(S_{\mathrm{AG4}})$ is a periodic function with period 64 $(= 2^{4+2})$, see Table 3.

Proposition 9.2. *Let* $f_{\mathrm{AG4}} = \Psi(S_{\mathrm{AG4}})$. *Then*

$$f_{\mathrm{AG4}}(n) = \chi_8(n) + \varepsilon_{64}(n),$$

where χ_8 *is the same one as in Proposition* 9.1, *and* ε_{64} *is a periodic function with period* 64 *defined by*

$$\varepsilon_{64}(n) = \begin{cases} -2, & \textit{if } n \equiv 6, 18, 28, 38, 44, 50 \bmod 64, \\ 2, & \textit{if } n \equiv 14, 20, 26, 36, 46, 58 \bmod 64, \\ 0, & \textit{otherwise.} \end{cases}$$

f_{AG3} (resp. f_{AG4}) is expressed as the Dirichlet character χ_8 plus the additional arithmetical function ε_{32} (resp. ε_{64}). This situation is different from $f_{\mathrm{RBC}} = \chi_4$. We will try to understand the meanings of ε_{32} and ε_{64} in a future study.

Acknowledgement

The authors would like to express their sincere gratitude to the referee for his valuable comments.

Table 3. AG4 and its sum of digits function.

n	AG4	S_{AG4}	H_{AG4}	$H_{\mathrm{AG4}}(\cdot/2)$	f_{AG4}
0	0000	0			
1	0001	1	1		1
2	0011	2	1	1	0
3	0111	3	1		1
4	1111	4	1	1	0
5	1110	3	−1		−1
6	1100	2	−1	1	−2
7	1000	1	−1		−1
8	1010	2	1	1	0
9	1011	3	1		1
10	1001	2	−1	−1	0
11	1101	3	1		1
12	0101	2	−1	−1	0
13	0100	1	−1		−1
14	0110	2	1	−1	2
15	0010	1	−1		−1
16	10010	2	1	1	0
17	10110	3	1		1
18	10100	2	−1	1	−2
19	10101	3	1		1
20	11101	4	1	−1	2
21	11001	3	−1		−1
22	11011	4	1	1	0
23	11010	3	−1		−1
24	11000	2	−1	−1	0
25	11100	3	1		1
26	11110	4	1	−1	2
27	11111	5	1		1
28	10111	4	−1	1	−2
29	10011	3	−1		−1
30	10001	2	−1	−1	0
31	10000	1	−1		−1

n	AG4	S_{AG4}	H_{AG4}	$H_{\mathrm{AG4}}(\cdot/2)$	f_{AG4}
32	110000	2	1	1	0
33	110001	3	1		1
34	110011	4	1	1	0
35	110111	5	1		1
36	111111	6	1	−1	2
37	111110	5	−1		−1
38	111100	4	−1	1	−2
39	111000	3	−1		−1
40	111010	4	1	1	0
41	111011	5	1		1
42	111001	4	−1	−1	0
43	111101	5	1		1
44	110101	4	−1	1	−2
45	110100	3	−1		−1
46	110110	4	1	−1	2
47	110010	3	−1		−1
48	100010	2	−1	−1	0
49	100110	3	1		1
50	100100	2	−1	1	−2
51	100101	3	1		1
52	101101	4	1	1	0
53	101001	3	−1		−1
54	101011	4	1	1	0
55	101010	3	−1		−1
56	101000	2	−1	−1	0
57	101100	3	1		1
58	101110	4	1	−1	2
59	101111	5	1		1
60	100111	4	−1	−1	0
61	100011	3	−1		−1
62	100001	2	−1	−1	0
63	100000	1	−1		−1

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