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par Florian LUCA et Paul POLLACK

Résumé. Soit ϕ la fonction indicatrice d'Euler. Une conjecture de Carmichael qui a 100 ans affirme que pour chaque n, l'équation $\phi(n) = \phi(m)$ a au moins une solution $m \neq n$. Ceci suggère que l'on définisse $F(n)$ comme le nombre de solutions m de l'équation $\phi(n) = \phi(m)$. (Donc, la conjecture de Carmichael est équivalente à l'inégalité $F(n) \geq 2$ pour tout n.) Les résultats sur F sont répandus dans la littérature. Par exemple, Sierpiński a conjecturé, et Ford a démontré, que l'image de F contient tous les nombres $k \geq 2$. Aussi, l'ordre maximal de F a été recherché par Erdős et Pomerance. Dans notre article, nous étudions l'ordre normal de F. Soit

$$K(x) := (\log x)^{(\log\log x)(\log\log\log x)}.$$

On démontre que pour chaque $\varepsilon > 0$, l'inégalité

$$K(n)^{1/2-\varepsilon} < F(n) < K(n)^{3/2+\varepsilon}$$

est vraie pour presque tous les entiers positifs n. Comme application, on montre que $\phi(n) + 1$ est sans facteur carré pour presque tous les n. On conclut avec quelques remarques sur les valeurs de n telles que $F(n)$ est proche de sa valeur maximale conjecturée.

Abstract. Let ϕ denote Euler's totient function. A century-old conjecture of Carmichael asserts that for every n, the equation $\phi(n) = \phi(m)$ has a solution $m \neq n$. This suggests defining $F(n)$ as the number of solutions m to the equation $\phi(n) = \phi(m)$. (So Carmichael's conjecture asserts that $F(n) \geq 2$ always.) Results on F are scattered throughout the literature. For example, Sierpiński conjectured, and Ford proved, that the range of F contains every natural number $k \geq 2$. Also, the maximal order of F has been investigated by Erdős and Pomerance. In this paper we study the normal behavior of F. Let

$$K(x) := (\log x)^{(\log\log x)(\log\log\log x)}.$$

We prove that for every fixed $\epsilon > 0$,

$$K(n)^{1/2-\epsilon} < F(n) < K(n)^{3/2+\epsilon}$$

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for almost all natural numbers n. As an application, we show that $\phi(n)+1$ is squarefree for almost all n. We conclude with some remarks concerning values of n for which $F(n)$ is close to the conjectured maximum size.

1. Introduction

Let ϕ denote Euler's totient function, so that $\phi(n) = \#(\mathbf{Z}/n\mathbf{Z})^\times$. In 1907, R. D. Carmichael [3] claimed to have shown for every n, the equation $\phi(n) = \phi(m)$ has a solution m with $m \neq n$. This claim, which for a time appeared as an exercise in Carmichael's introductory number theory text, was eventually retracted [4] and is now known as *Carmichael's conjecture.*

Define $F(n)$ as the number of solutions m to the equation $\phi(m) = \phi(n)$. While Carmichael's conjecture remains elusive, other aspects of F have succumbed to detailed study. For example, we know from the work of Ford [13] that every natural number > 1 belongs to the range of F. (This had been conjectured by Sierpiński in the 1950s.) In 1935, Erdős [7] showed the existence of a positive constant c for which $F(n) > n^c$ infinitely often. Work of Baker and Harman [1] implies that we may take $c = 0.7038$, and it is conjectured that any $c < 1$ is permissible. Erdős's investigations were extended by Pomerance [20] (see also [22, §4], and cf. [21, p. 591–592]), who showed that

$$\max_{n\le x} F(n) \le x/L(x)^{1+o(1)}, \quad \text{where} \quad L(x) := x^{\frac{\log\log\log x}{\log\log x}}, \tag{1.1}$$

and that equality holds in this estimate under a plausible (but seemingly difficult) hypothesis on the distribution of smooth shifted primes.

We study the normal behavior of F, i.e., how many solutions m there typically are to an equation of the form $\phi(m) = \phi(n)$. Our main result is as follows:

Theorem 1.1. *Fix $\epsilon > 0$. For almost all natural numbers n (i.e., all n outside of a set of asymptotic density zero), we have*

$$K(n)^{1/2-\epsilon} < F(n) < K(n)^{3/2+\epsilon},$$
$$\textit{where} \quad K(x) := (\log x)^{(\log\log x)(\log\log\log x)}.$$

Let $\mathcal{V}$ denote the set of elements in the range of ϕ (so-called *totients*), and let $V(x) = \#\mathcal{V}\cap[1,x]$ be the corresponding counting function. Theorem 1.1 contrasts with Ford's result (see [12, Theorem 3]) that if $k = k(x)$ tends to infinity (however slowly), then only $o(V(x))$ totients $v \le x$ are such that the equation $\phi(m) = v$ has more than k solutions m. This is one manifestation of the seeming paradox that $\phi(n)$, for a typical n, displays very different properties from a typical $v \in \mathcal{V}$. Another example of this phenomenon is given by the theorem of Erdős and Pomerance mentioned in §3 below.

Theorem 1.1 can also be compared with [2, Theorem 4(i)]. In that paper, the analogue of F is defined with respect to the Carmichael λ-function. (Recall that $\lambda(n)$ denotes the exponent of the group $(\mathbf{Z}/n\mathbf{Z})^{\times}$.) It is shown there that $\lambda(m) = \lambda(n)$ almost always has more than $\exp((\log\log n)^{10/3})$ solutions m.

It seems that the method used to establish Theorem 1.1 may provide insight into other aspects of the Euler function. We offer the following application to the study of shifted totients:

Theorem 1.2. *For $x \geq 20$, we have*

$$\sum_{n\leq x}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2 \ll x\frac{(\log\log\log x)^5}{\log\log x}. \tag{1.2}$$

An immediate consequence of Theorem 1.2 is that the number $\phi(n)+1$ is squarefree for almost all n. One might expect that this last result could be established by a routine argument, but it seems surprisingly difficult to estimate the frequency with which $\phi(n)+1$ is divisible by the square of a large prime. We work around this by observing that when this occurs, there are not many possibilities for $\phi(n)$; the proof of Theorem 1.1 then shows that there must be few corresponding values of n.

Our proof of Theorem 1.2 can be adapted to yield a similar estimate for the κth moment, for any $\kappa > 0$. (In the general case, the exponent of $\log\log\log x$ on the right-hand side of (1.2) may depend on κ.) We choose to deal only with the case $\kappa = 2$, which already involves all of the central ideas.

In addition to the normal order, is sensible to ask also about the average order of F. If equality holds in (1.1), as we expect, then a superficial argument shows that

$$x/L(x)^{2+o(1)} \leq \frac{1}{x}\sum_{n\leq x}F(n) \leq x/L(x)^{1+o(1)}, \tag{1.3}$$

as $x \to \infty$. Of course the upper bound here is immediate from (1.1), but the lower bound is almost as trivial: Put $x' = x/(2\log\log x)$, and pick an $m \leq x'$ for which $F(m)$ is as large as possible. Assuming equality in (1.1),

$$F(m) \geq x'/L(x')^{1+o(1)} = x/L(x)^{1+o(1)}.$$

Now $F(n) = F(m)$ for any solution n to $\phi(n) = \phi(m)$, and there are precisely $F(m)$ such values of n. Moreover, all such n belong to $[1, x]$ if x is large, by a well-known result on the minimal order of the Euler function (see the start of the proof of Lemma 2.1 below). So the contribution of these n to the sum is at least $F(m)^2 = x^2/L(x)^{2+o(1)}$, which gives the left-half of (1.3). We do not know which side of (1.3), if either, gives the truth about the average size of F.

A natural way to attack this problem is to study the structure of those n for which equality is attained in (1.1). The examples of n of this type (conditionally) obtained by Pomerance are products of primes p for which $p-1$ is $(\log x)$-*smooth* (i.e., $p-1$ has no prime factors $> \log x$), where the number of distinct primes dividing n is about $\log x/(\log\log x)^2$. The next theorem shows that whenever equality holds in (1.1), $v := \phi(n)$ has at least $(\log x)^{1-o(1)}$ prime factors below $(\log x)^{1+o(1)}$, and almost all of the $F(n)$ elements of $\phi^{-1}(v)$ have at least $\log x/(\log\log x)^{2+o(1)}$ distinct prime factors:

Theorem 1.3. *Fix δ with $0 < \delta < 1$.*

(i) *If $v \le x$ has fewer than $(\log x)^{1-\delta}$ distinct prime factors from the interval $[1, (\log x)^{1+\delta}]$, then $\#\phi^{-1}(v) \le x/L(x)^{1+\delta+o(1)}$.*

(ii) *For any $v \le x$, the number of preimages m of v with $\omega(m) \le \log x/(\log\log x)^{2+\delta}$ is bounded by $x/L(x)^{1+\delta+o(1)}$.*

In both parts, the $o(1)$ is as $x \to \infty$ and is uniform in v.

Notation. Most of our notation is standard in analytic number theory. We use the arithmetic functions ω and Ω with their usual meanings, so that $\omega(n) := \sum_{p|n} 1$ and $\Omega(n) := \sum_{p^\ell|n} 1$. We also use the function

$$\Omega(n,z) := \sum_{p\le z,\ p^\ell|n} 1,$$

which counts with multiplicity the number of primes p dividing n not exceeding z. We write $\mathrm{rad}(n) := \prod_{p|n} p$ for the *radical* of n, i.e., the largest squarefree divisor of n. We let $P(n)$ denote the largest prime factor of n (with $P(1) = 1$) and put

$$\Psi(x,y) := \#\{n \le x : P(n) \le y\}.$$

It is convenient to introduce abbreviated notation for iterated logarithms: For $x > 0$, we put $\log_1 x := \max\{\log x, 1\}$ and define $\log_k$ as the kth iterate of $\log_1$. We use $C_1, C_2, \dots$ for absolute positive constants, which are numbered consecutively in order of appearance.

2. The key lemma

Define $S(x;d)$ as the number of n for which $\phi(n)$ is a multiple of d belonging to $[1,x]$, i.e.,

$$S(x;d) = \sum_{\substack{v\le x\\ d|v}} \#\phi^{-1}(v).$$

In the case $d = 1$, the study of $S(x;d)$ goes back to Erdős and Turán [8], who showed that $\frac{1}{x}S(x;1)$ tends to a nonzero limit as $x \to \infty$; later Dressler [6] computed the limit as $\frac{\zeta(2)\zeta(3)}{\zeta(6)}$.

Our principal tool is a uniform upper bound for $S(x;d)$. In what follows, we let B_k (the kth *Bell number*) denote the number of set partitions of a k-element set.

Lemma 2.1 (cf. [19, Lemma 2.4])**.** *Let x be sufficiently large. Then for each squarefree $d \le x$, we have*

$$S(x;d) \le B_{\omega(d)}(C_1 \log_2 x)^{\omega(d)} \frac{x(\log_2 x)^2}{d},$$

where C_1 is an absolute positive constant.

Proof. When $d = 1$, we have already remarked that a stronger estimate holds, so we assume that $d > 1$. Since $\phi(n) \ge (e^{-\gamma} + o(1))n/\log_2 n$ as $n \to \infty$ (see [16, Theorem 328]), we have that for large x, the relation $\phi(n) \le x$ implies that $n \le 2x \log_2 x$. Thus, $S(x;d)$ is bounded above by the number of $n \le 2x\log_2 x$ for which $d \mid \phi(n)$. For each such n, write its unique factorization in the form $\prod_i p_i^{e_i}$. Since $d \mid \phi(n)$, it follows that there is a factorization of d of the form $d = d_1 d_2 \cdots$ for which each $d_i \mid \phi(p_i^{e_i})$. By discarding those $d_i = 1$ and reordering, we can assume $d = d_1 \cdots d_l$, where each $d_i > 1$. Note that $l \le \omega(d)$.

Now consider the factorization '$d = d_1 \dots d_l$' as fixed and count the number of corresponding n. Clearly this number is bounded by

$$2x \log_2 x \prod_{i=1}^{l} \left(\sum_{\substack{p^e \le x \\ d_i \mid \phi(p^e)}} \frac{1}{p^e} \right).$$

The terms of the inner sum corresponding to primes p for which $d_i \mid p-1$ contribute

$$\le \sum_{p \equiv 1 \pmod{d_i}} \left(\frac{1}{p} + \frac{1}{p^2} + \cdots \right) \ll \sum_{p \equiv 1 \pmod{d_i}} \frac{1}{p} \ll \frac{\log_2 x}{\phi(d_i)},$$

by the Brun–Titchmarsh inequality and partial summation (see [9, eq. (3.1)]). So, suppose that $d_i \nmid p-1$ but that $d_i \mid \phi(p^e) = (p-1)p^{e-1}$ for some $e \ge 2$. Then $p \mid d_i$. Moreover, since d_i is squarefree, we have that d_i/p is a divisor of $p-1$ in this case, so that $d_i \mid p(p-1) = \phi(p^2)$. In particular, $p \ge \sqrt{d_i/2}$. Since $p \mid d_i$, the number of possibilities for p is $O(1)$, uniformly in d_i, and the powers of each such p contribute

$$\le \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{1}{p(p-1)} \le \frac{1}{d_i}.$$

Hence, the inner sum is

$$\ll \frac{\log_2 x}{\phi(d_i)} + \frac{1}{d_i} \ll \frac{\log_2 x}{\phi(d_i)},$$

uniformly in $1 \le i \le l$. Substituting this above, we obtain a bound for the number of such n of the form

$$2x \log_2 x \prod_{i=1}^{l} \left(\frac{C_1 \log_2 x}{\phi(d_i)} \right) \le \frac{2x \log_2 x}{\phi(d)} (C_1 \log_2 x)^{\omega(d)} \le \frac{4x (\log_2 x)^2}{d} (C_1 \log_2 x)^{\omega(d)}.$$

Now we sum over the $B_{\omega(d)}$ (unordered) factorizations of the squarefree number d. This gives the estimate of the lemma, apart from an additional factor of 4, which can be absorbed into the constant C_1. □

Lemma 2.1 gives strong results when $\omega(d)$ is fairly small, which is all that is needed for our proof of Theorem 1.1. Nevertheless, it seems also worth highlighting what the method yields in more extremal cases. To state our results, let us define the *roundness* $R(n)$ of the natural number n by the ratio

$$R(n) := \frac{\omega(n)}{\log n / \log \log n}.$$

It is worthwhile to recall that $\limsup R(n) = 1$ (see, e.g., [15, p. 263]); in fact, it follows from the prime number theorem with error term that

$$R(n) \le 1 + O((\log_2 n)^{-1}). \tag{2.1}$$

(See, e.g., [23], where versions of this result are established with explicit constants.) One should also keep in mind that very round numbers are quite rare: A theorem of Erdős and Nicolas [10, Theorem 2] asserts that for each fixed $\eta \in (0, 1)$, the number of $n \le x$ with $R(n) \ge 1 - \eta$ is $x^{\eta + o(1)}$.

Theorem 2.1. *Suppose that $d \le x$ is squarefree and that*

$$d \ge \exp((\log x)^{1/\log_3 x}).$$

(i) *Fix $\eta \in (0, 1)$. If $R(d) \le 1 - \eta$, then $S(x; d) \le x/d^{\eta + o(1)}$, as $x \to \infty$.*
(ii) *$S(x; d) \le x/L(d)^{1+o(1)}$, as $x \to \infty$, uniformly in d. Here $L(\cdot)$ is as in* (1.1).

Proof. We start with (i). Put $Z(d) := \lfloor (1 - \eta) \log d / \log \log d \rfloor$. To simplify notation, we write Z instead of $Z(d)$ when d is understood. Hence, $\omega(d) \le Z$. Since the Bell numbers satisfy the crude upper bound $B_k \le k^k$ (by a straightforward combinatorial argument), we have

$$S(x; d) \le (C_1 Z \log_2 x)^Z \frac{x (\log_2 x)^2}{d},$$

and so

$$\log \left(\frac{S(x; d)}{x/d} \right) \le Z(\log Z + \log_3 x + O(1)) + O(\log_3 x).$$

Our hypothesized lower bound on d asserts that

$$\log_3 d \geq \log_3 x - \log_4 x.$$

In particular, $\log_3 x \sim \log_3 d$ as $x \to \infty$; also, for large x,

$$\begin{aligned} \log Z + \log_3 x &\leq \log_2 d - \log_3 d + \log_3 x \\ &\leq \log_2 d + \log_4 x < \log_2 d + 2\log_4 d. \end{aligned} \tag{2.2}$$

Inserting this bound and the definition of $Z := Z(d)$ into our earlier estimate, we deduce that

$$\begin{aligned} \log\left(\frac{S(x;d)}{x/d}\right) &\leq (1-\eta)\frac{\log d}{\log_2 d}\left(\log_2 d + O(\log_4 d)\right) + O(\log_3 x) \\ &= (1-\eta)(1 + O(\log_4 d/\log_2 d))\log d + O(\log_3 d) \\ &= (1 - \eta + o(1))\log d, \end{aligned}$$

as $x \to \infty$, and now (i) follows upon exponentiating.

The proof of (ii) is similar, except that in place of the trivial upper bound $B_Z \leq Z^Z$, we use the bound $B_Z \leq Z^Z/(\log Z)^{Z(1+o(1))}$, as $Z \to \infty$ (see [5, p. 108]). Let C_2 be a sufficiently large constant. We put

$$Z(d) := \lfloor \log d/\log\log d + C_2 \log d/(\log\log d)^2 \rfloor.$$

Keeping with our previous convention of writing Z instead of $Z(d)$, we have $\omega(d) \leq Z$ uniformly in d, by (2.1). By (2.2),

$$\log Z + \log_3 x \leq \left(\log\left(\frac{\log d}{\log_2 d}\right) + o(1)\right) + \log_3 x < \log_2 d + 3\log_4 d.$$

Hence,

$$\begin{aligned} \log\left(\frac{S(x;d)}{x/d}\right) &\leq Z(\log Z + \log_3 x + O(1)) + O(\log_3 x) - (1+o(1))Z\log_2 Z \\ &\leq Z(\log_2 d + O(\log_4 d)) + O(\log_3 d) - (1+o(1))Z\log_2 Z \\ &= \log d + O\left(\frac{\log d \log_4 d}{\log_2 d}\right) - (1+o(1))\frac{\log d \log_3 d}{\log_2 d} \\ &= \log d - (1+o(1))\frac{\log d\log_3 d}{\log_2 d} \qquad (\text{as } x \to \infty). \end{aligned}$$

Once again, exponentiating gives the result. □

3. Proof of Theorem 1.1

Because the logarithm function grows so slowly, it is enough to show that all but $o(x)$ of the natural numbers $n \leq x$ satisfy

$$K(x)^{1/2-\epsilon} < F(n) < K(x)^{3/2+\epsilon}. \tag{3.1}$$

Our proof makes heavy use of the determination by Erdős and Pomerance [11] of the normal number of prime factors of $\phi(n)$: For each fixed $\delta > 0$,

$$\left(\frac{1}{2} - \delta\right) (\log_2 x)^2 \le \omega(\phi(n)) \le \left(\frac{1}{2} + \delta\right) (\log_2 x)^2$$

for all but $o(x)$ natural numbers $n \le x$ (as $x \to \infty$). Moreover, the same holds with Ω in place of ω. (By contrast, it is known that a typical totient $v \in \mathcal{V} \cap [1, x]$ has about $2.186 \log_2 x$ prime divisors, counted with or without multiplicity; see [12, Theorem 10].)

Lower bound: Set

$$\mathcal{V}_1 := \left\{\phi(n) : n \le x, F(n) \le K(x)^{1/2-\epsilon}, \omega(\phi(n)) \ge \frac{1-\epsilon}{2}(\log\log x)^2\right\}.$$

By the Erdős–Pomerance result with $\delta = \epsilon/2$,

$$\#\{n \le x : F(n) \le K(x)^{1/2-\epsilon}\} \le o(x) + \#\phi^{-1}(\mathcal{V}_1) \le o(x) + K(x)^{1/2-\epsilon}\#\mathcal{V}_1.$$

But by a well-known result of Hardy and Ramanujan [15, Lemma B],

$$\#\mathcal{V}_1 \le \frac{x}{\log x} \sum_{l \ge Z} \frac{1}{(l-1)!}(\log_2 x + C_3)^{l-1}, \text{ where } Z := \left\lceil \frac{1-\epsilon}{2}(\log\log x)^2 \right\rceil.$$

The sum is dominated by its first term, and using the elementary inequality $(Z-1)! \ge ((Z-1)/e)^{Z-1}$, we see that $\#\mathcal{V}_1 \le x/K(x)^{(1-\epsilon)/2+o(1)}$. Substituting above shows that the lower bound in (3.1) holds for all but $o(x)$ values of $n \le x$.

Upper bound: We prove what at first glance appears to be a stronger result. Observe that with $\mathcal{V}_2 := \{\phi(n) : n \le x, F(n) > K(x)^{3/2+\epsilon}\}$, we have

$$K(x)^{3/2+\epsilon}\#\mathcal{V}_2 \le \#\phi^{-1}(\mathcal{V}_2) \le 2x \log_2 x$$

for large x, so that

$$\#\mathcal{V}_2 \le \frac{2x \log_2 x}{K(x)^{3/2+\epsilon}}. \tag{3.2}$$

Clearly

$$\#\{n \le x : F(n) > K(x)^{3/2+\epsilon}\} \le \#\{n \le x : \phi(n) \in \mathcal{V}_2\}. \tag{3.3}$$

The upper bound aspect of (3.1) asserts that the left-hand side of (3.3) is $o(x)$, as $x \to \infty$. We now show that if $\mathcal{V}_2$ is *any* subset of $[1, x]$ satisfying the upper bound (3.2), then $\#\{n \le x : \phi(n) \in \mathcal{V}_2\} = o(x)$, as $x \to \infty$, uniformly in $\mathcal{V}_2$.

We may suppose each $v \in \mathcal{V}_2$ satisfies the following four conditions:

(i) $v \ge x/(\log_2 x \log_3 x)$,
(ii) $\Omega(v) \le (1/2 + \epsilon/6)(\log_2 v)^2$,
(iii) if p is a prime for which $p^2 \mid v$, then $p \le (\log_2 x)^2$,
(iv) $\Omega(v, (\log_2 x)^2) \le 2 \log_2 x \log_4 x$.

Indeed, any n for which $\phi(n) < x/\log_2 x \log_3 x$ satisfies $n \ll x/\log_3 x$, and the number of such n is clearly $o(x)$. The result of Erdős–Pomerance (with $\delta = \epsilon/6$) implies that we may assume (ii). In their paper [11], it is also shown explicitly that the number of $n \le x$ for which $\phi(n)$ is divisible by p^2 for some $p > (\log_2 x)^2$ is $o(x)$ (see [11, p. 350, middle]) and that the number of $n \le x$ with $\Omega(\phi(n), (\log_2 x)^2) > 2\log_2 x \log_4 x$ is $o(x)$ (see [11, p. 350, top]). So, we can assume (iii) and (iv).

Now fix $v \in \mathcal{V}_2$. Clearly, the number of $n \le x$ with $\phi(n) = v$ is bounded by $S(x; \mathrm{rad}(v))$. We now apply Lemma 2.1. Notice that by (iii), (iv), and (i),

$$\mathrm{rad}(v) \ge v / \prod_{\substack{p^e \| v \\ p \le (\log_2 x)^2}} p^e \ge v/((\log_2 x)^2)^{\Omega(v,(\log_2 x)^2)}$$
$$\ge v((\log_2 x)^{4\log_2 x \log_4 x})^{-1} \ge x/K(x)^{o(1)},$$

and that by (ii),

$$\omega(\mathrm{rad}(v)) \le \Omega(v) \le Z, \quad \text{where now} \quad Z := \left\lfloor \left(\frac{1}{2} + \frac{\epsilon}{6}\right)(\log_2 x)^2 \right\rfloor.$$

Using again the inequality $B_k \le k^k$, Lemma 2.1 shows that

$$S(x; \mathrm{rad}(v)) \le (C_1 Z \log_2 x)^Z K(x)^{o(1)} = K(x)^{3/2+\epsilon/2+o(1)}.$$

Since this holds uniformly for $v \in \mathcal{V}_2$, the result now follows from (3.2).

4. Proof of Theorem 1.2

We let x be large and we start by eliminating some integers $n \le x$. Let $\mathcal{N}$ be the set of all positive integers $n \le x$ which fulfill the following five conditions:

(i) $P(n) > y$, where $y := L(x)^{1/4}$.
(ii) $P(n)^2 \nmid n$;
(iii) $\Omega(n) < 10\log_2 x$;
(iv) $\Omega(\phi(n)) < 110(\log_2 x)^2$;
(v) $\Omega(\phi(n)+1) < 10(\log_2 x)^2$.

We claim that

$$\#\{n \le x : n \notin \mathcal{N}\} \ll \frac{x}{(\log x)^3}.$$

The number of positive integers $n \le x$ failing (i) is $\Psi(x, y)$. From well-known estimates for smooth numbers (see, e.g., [14], esp. pp. 269–270), we have $\Psi(x, y) = \Psi(x, L(x)^{1/4}) \le x/(\log x)^{4+o(1)}$. Positive integers $n \le x$ passing (i) but failing (ii) have a prime factor $p > y$ with $p^2 \mid n$. The number of such n is at most $x\sum_{p>y} 1/p^2 \ll x/(\log x)^3$. Lemma 13 in [18] shows that uniformly for each positive integer Z we have

$$\sum_{\substack{n\le x\\ \Omega(n)\ge Z}} 1 \ll \frac{Z}{2^Z} x\log x. \tag{4.1}$$

Applying this with $Z := \lfloor 10\log_2 x\rfloor$, we get that the set of positive integers $n \le x$ failing (iii) has cardinality $\ll x/(\log x)^3$. If $n \le x$ has passed (iii) but failed (iv), then, since

$$\Omega(\phi(n)) \le \Omega(n) + \sum_{p\mid n} \Omega(p-1),$$

it follows that $n \le x$ has a prime factor p for which $\Omega(p-1) \ge Z$. The number of such n is at most

$$\sum_{\substack{p\le x\\ \Omega(p-1)\ge Z}} \frac{x}{p} < \sum_{\substack{p\le x\\ \Omega(p-1)\ge Z}} \frac{x}{p-1} \le \sum_{\substack{d\le x\\ \Omega(d)\ge Z}} \frac{x}{d}$$
$$\ll x\frac{Z}{2^Z}\int_2^x \frac{\log t}{t}dt \ll x\frac{Z(\log x)^2}{2^Z} \ll \frac{x}{(\log x)^3},$$

by estimate (4.1) and Abel summation. Assume now that $n \le x$ passes (i)–(iv) but fails (v). Then there are at least $\lfloor 9(\log_2 x)^2\rfloor$ prime factors of $\phi(n)+1$ counting multiplicities not exceeding $x^{1/(\log_2 x)^2}$. Writing d for the product of the first Z such prime factors of $\phi(n)+1$, we get that

$$d < x^{Z/(\log_2 x)^2} < y^{1/2},$$

where the last inequality holds for large values of x. We fix d. Now writing $n = Pm$, where $P = P(n)$, we have that $\phi(n)+1 = (P-1)\phi(m)+1$. We fix m and observe that $P \le x/m$. Since $d \mid \phi(n)+1$, we get that $\phi(m)$ is invertible modulo d and $P \equiv 1-\phi(m)^{-1} \pmod d$. Let $a_{d,m} \in \{1,2,\dots,d\}$ be the first positive integer in the above progression modulo d. By the Brun–Titchmarsh theorem, for fixed d and m, the number of possibilities for P is

$$\pi(x/m; d, a_{d,m}) \ll \frac{x}{m\phi(d)\log(x/(md))} \ll \frac{x\log_2 x}{md\log(y^{1/2})} \ll \frac{x(\log_2 x)^2}{md\log x}, \tag{4.2}$$

where in the above chain of inequalities we use that $x/(md) \ge P/d \ge y^{1/2}$. Summing (4.2) over all $m \le x$ and all $d \le x$ with $\Omega(d) = Z$, we get that the number of $n \le x$ passing (i)–(iv) but failing (v) is

$$\ll \sum_{m\le x} \sum_{\substack{d\le x\\ \Omega(d)=Z}} \frac{x(\log_2 x)^2}{md\log x} \ll \frac{x(\log_2 x)^2}{\log x}\left(\sum_{m\le x}\frac{1}{m}\right)\left(\sum_{\substack{d\le x\\ \Omega(d)=Z}}\frac{1}{d}\right)$$
$$\ll \frac{Z}{2^Z} x(\log x)^2(\log_2 x)^2 \ll \frac{x}{(\log x)^3},$$

which is what we wanted. Since $\Omega(\phi(n)+1) \ll \log x$ for all $n \le x$, we have

$$\sum_{n\le x}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2$$
$$= \sum_{n\in\mathcal{N}}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2 + O\left(\frac{x}{\log x}\right).$$

Next we shrink $\mathcal{N}$ by throwing away those $n \le x$ for which $P(n) \le y_1$, where

$$y_1 := \exp\left(\frac{\log x \log_4 x}{6\log_3 x}\right).$$

We denote the resulting set by $\mathcal{N}_0$. Then

$$\#\left(\mathcal{N}\backslash\mathcal{N}_0\right) \le \Psi(x,y_1) \le x/(\log_2 x)^5$$

for large x. Since $\Omega(\phi(n)+1) \ll (\log_2 x)^2$ holds for all $n \in \mathcal{N}$, it follows that

$$\sum_{n\in\mathcal{N}\backslash\mathcal{N}_0}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2 \ll \frac{x}{\log_2 x},$$

so that

$$\sum_{n\le x}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2$$
$$= \sum_{n\in\mathcal{N}_0}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2 + O\left(\frac{x}{\log_2 x}\right).$$

We now look at the subset of $n \in \mathcal{N}_0$ for which $\phi(n)+1$ is not squarefree. We put

$$z_1 := \frac{C_4\log_2 x}{\log_3 x}, \qquad z_2 := \log x, \qquad z_3 := y_1^{1/5},$$

where $C_4 > 0$ is a constant to be specified later, and we let

$$\mathcal{I}_1 := [2,z_1), \quad \mathcal{I}_2 := [z_1,z_2), \quad \mathcal{I}_3 := [z_2,z_3), \quad \mathcal{I}_4 := [z_3,\sqrt{x}].$$

For $i = 1,\ldots,4$, write

$$\Omega_i(n) := \sum_{\substack{p^{a_p}\|\phi(n)+1\\ p\in\mathcal{I}_i}}(a_p-1), \qquad \text{for} \qquad i=1,\ldots,4,$$

and let $\mathcal{N}_i$ be the subset of $\mathcal{N}_0$ such that $\Omega_i(n) = \max_{1\le j\le 4}\{\Omega_j(n)\}$. Clearly,

$$\sum_{n\le x}(\Omega(\phi(n)+1)-\omega(\phi(n)+1))^2 \ll \sum_{i=1}^{4}\Omega_i^2\#\mathcal{N}_i + \frac{x}{\log_2 x}. \tag{4.3}$$

We now deal with the various $i = 1,\dots,4$.

When $i = 1$, the proof of Lemma 2 in [17] shows that there exists a constant $C_4 > 0$ such that the set of $n \le x$ with the property that $\phi(n)$ is not a multiple of all primes $p \le z_1$ has cardinality $\ll x/(\log_2 x)^{10}$. Let C_4 have the above value. Since $\Omega_1 \ll (\log_2 x)^2$, it follows that

$$\Omega_1^2\#\mathcal{N}_1 \ll \frac{x}{(\log_2 x)^6}. \tag{4.4}$$

Assume now that $i = 2$. Let $n \in \mathcal{N}_2$. Put

$$d := \prod_{\substack{p^{a_p}\,\|\phi(n)+1\\ a_p>1\\ p\in\mathcal{I}_2}} p^{a_p}.$$

Observe that

$$d < z_2^{10(\log_2 x)^2} = \exp(10(\log_2 x)^3) < y_1^{1/2},$$

where the last inequality holds for all sufficiently large x. We now write again $n = Pm$, so $\phi(n)+1 = (P-1)\phi(m)+1$. We fix both m and d. Then, as in the argument used to bound the number of $n \le x$ failing condition (v) from the definition of $\mathcal{N}$, we have that the number of such $n \le x$ when both m and d are fixed is at most

$$\pi(x/m; d, a_{d,m}) \ll \frac{x}{m\phi(d)\log(x/(md))} \ll \frac{x}{m\phi(d)\log(y_1^{1/2})} \ll \frac{x\log_3 x}{m\phi(d)\log x}.$$

Since $\omega(d) \le \Omega(d) \le T := \lfloor 10(\log_2 x)^2\rfloor$, it follows that if we write $2 = p_1 < p_2 < \cdots < p_k < \cdots$ for the increasing sequence of all prime numbers, we have that

$$\frac{\phi(d)}{d} \ge \prod_{i=s}^{s+T}\left(1-\frac{1}{p_i}\right),$$

where $s := \pi(z_1)$. For large x, we have $p_{s+T} < (\log_2 x)^3$, and so

$$\frac{\phi(d)}{d} \ge \prod_{p\le(\log_2 x)^3}\left(1-\frac{1}{p}\right) \gg \frac{1}{\log_3 x},$$

where the last inequality follows from Mertens' formula. Hence, for our values of d and m, we in fact have

$$\pi(x/m; d, a_{d,m}) \ll \frac{x(\log_3 x)^2}{md\log x}.$$

Summing the above bound up over all values of $m \le x$, we get an upper bound of

$$\ll \frac{x(\log_3 x)^2}{d}.$$

Let $\mathcal{D}$ be the set of all allowable values for d. Observe that $\mathcal{D}$ consists of squarefull numbers. (Recall that a number d is *squarefull* if $q^2 \mid d$ whenever q is a prime factor of d.) The above argument shows that

$$\Omega_2^2 \# \mathcal{N}_2 \ll \sum_{d \in \mathcal{D}} \frac{x\Omega(d)^2(\log_3 x)^2}{d} \ll x(\log_3 x)^2 \sum_{\substack{d > z_1^2 \\ d \text{ squarefull}}} \frac{(\log d)^2}{d}. \tag{4.5}$$

The counting function of the set of squarefull positive integers $d \le s$ is $O(s^{1/2})$. By Abel's summation formula, it follows immediately that

$$\sum_{\substack{d > z_1^2 \\ d \text{ squarefull}}} \frac{(\log d)^2}{d} \ll \frac{(\log z_1)^2}{z_1} \ll \frac{(\log_3 x)^3}{\log_2 x},$$

which together with estimate (4.5) implies that

$$\Omega_2^2 \# \mathcal{N}_2 \ll \frac{x(\log_3 x)^5}{\log_2 x}. \tag{4.6}$$

We now deal with the case $i = 3$. In this case, $\Omega_3^2 \ll (\log_2 x)^4$ for all $n \in \mathcal{N}_3$. Furthermore, for each such n, there is a prime $p \in \mathcal{I}_3$ such that $p^2 \mid \phi(n) + 1$. Fix the prime p. Write again n as $n = Pm$, where $P = P(n)$. Fix also m. Then $\phi(n) + 1 = (P - 1)\phi(m) + 1$ is a multiple of p^2. By an argument already used several times, the number of possibilities for P is bounded by

$$\pi(x/m; p^2, a_{p^2,m}) \ll \frac{x}{m\phi(p^2)\log(x/(mp^2))} \ll \frac{x}{mp^2 \log(y_1^{1/2})} \ll \frac{x \log_3 x}{mp^2 \log x}.$$

Here, we use that $x/(mp^2) \ge P/z_3^2 > y_1^{1/2}$. Summing up the above bound over all $m \le x$ and over all $p \in \mathcal{I}_3$, we get that

$$\begin{aligned} \Omega_3^2 \# \mathcal{N}_3 &\ll \frac{x(\log_2 x)^4 \log_3 x}{\log x} \left(\sum_{m \le x} \frac{1}{m} \right) \left(\sum_{z_2 \le p \le z_3} \frac{1}{p^2} \right) \\ &\ll \frac{x(\log_2 x)^4 \log_3 x}{z_2 \log z_2} \ll \frac{x(\log_2 x)^3 (\log_3 x)}{\log x}. \end{aligned} \tag{4.7}$$

Finally, take $i = 4$. Let $\mathcal{N}_4'$ consist of those $n \in \mathcal{N}_4 \cap [x/\log_2 x, x]$ for which $n\phi(n)$ is not divisible by the square of a prime $p > (\log_2 x)^3$. From [18, Proposition 8], we have that

$$\#(\mathcal{N}_4 \setminus \mathcal{N}_4') \ll x/\log_2 x. \tag{4.8}$$

We now turn our attention to $\#\mathcal{N}_4'$. Suppose that $n \in \mathcal{N}_4'$, and say that $p^2 \mid \phi(n)+1$ for some $p \in \mathcal{I}_4$ (so that $p > y_1^{1/5}$). Then the number of possibilities for $m = \phi(n)$ is clearly $\ll x \sum_{p > y_1^{1/5}} p^{-2} \ll x/y_1^{1/5}$. For each fixed value of m, the number of $n \in \mathcal{N}_4'$ with $\phi(n) = m$ is at most $S(x; \mathrm{rad}(m))$. We have

$$\begin{aligned}\mathrm{rad}(m) \geq \prod_{\substack{p \mid m \\ p > (\log_2 x)^3}} p = \phi(n) \prod_{\substack{p^{e_p} \| \phi(n) \\ p \leq (\log_2 x)^3}} p^{-e_p} &\geq \frac{\phi(n)}{((\log_2 x)^3)^{\Omega(\phi(n))}} \\ &\geq \frac{x/(2(\log_2 x)^2)}{\exp(330(\log_2 x)^2 \log_3 x)} > \frac{x}{K(x)^{O(1)}};\end{aligned}$$

here we use use that $n \geq x/\log_2 x$ for $n \in \mathcal{N}_4'$, so that $\phi(n) \geq x/(2(\log_2 x)^2)$ for large x. Moreover, $\omega(\mathrm{rad}(m)) \leq \Omega(\phi(n)) \leq T$. Hence, by Lemma 2.1,

$$\begin{aligned}S(x; \mathrm{rad}(m)) &\leq B_{\omega(m)} (C_1 \log_2 x)^{\omega(m)} K(x)^{O(1)} \\ &\leq T^T (C_1 \log_2 x)^T K(x)^{O(1)} \leq K(x)^{O(1)}.\end{aligned}$$

Summing over the possible values of m, we see that

$$\#\mathcal{N}_4' \ll (x/y_1^{1/5}) K(x)^{O(1)} \ll x/y_1^{1/10}.$$

With (4.8), this shows that

$$\#\mathcal{N}_4 \ll x/y_1^{1/10} + x/\log_2 x \ll x/\log_2 x.$$

Since obviously $\Omega_4 \ll \log_3 x$, it follows from the above analysis that

$$\Omega_4^2 \#\mathcal{N}_4 \ll \frac{x(\log_3 x)^2}{\log_2 x}. \tag{4.9}$$

The estimate (1.2) in the statement of the theorem follows now by inserting estimates (4.4), (4.6), (4.7) and (4.9) into estimate (4.3).

5. Proof of Theorem 1.3

The proof uses the upper-bound technique of [20]. Put $z := 2x \log_2 x$, so that for large x, the set $\phi^{-1}([1, x])$ is a subset of $[1, z]$. If $v \leq x$ and c is a positive real number, we have

$$\#\phi^{-1}(v) \leq \sum_{\substack{m \leq z \\ \phi(m) = v}} (z/m)^c \leq z^c \sum_{m:\, p \mid m \Rightarrow p-1 \mid v} \frac{1}{m^c} = z^c \prod_{p-1 \mid v} (1 - p^{-c})^{-1}, \tag{5.1}$$

where the product is over those primes p for which $p-1$ divides v. Assuming $c \geq 2/3$ (say), we have

$$\prod_{p-1|v} (1-p^{-c})^{-1} \ll \exp\left(\sum_{p-1|v} \frac{1}{p^c}\right)$$

$$\leq \exp\left(\sum_{d|v} \frac{1}{d^c}\right) = \exp\left(\exp\left(O\left(\sum_{p|v} \frac{1}{p^c}\right)\right)\right). \tag{5.2}$$

For the proof of (i), we choose

$$c := 1 - (1+\delta)\log_3 x/\log_2 x.$$

(So certainly $c \geq 2/3$ for large x.) Then $z^c = x/L(x)^{1+\delta+o(1)}$, and it suffices to show that the contribution from the product in (5.1) is $L(x)^{o(1)}$. We will show that

$$\sum_{p|v} \frac{1}{p^c} \ll (\log_2 x)^{1-\delta^2}, \tag{5.3}$$

which by (5.2) is enough. Write this sum in the form $\sum_1 + \sum_2 + \sum_3$, corresponding to splitting p into three ranges:

(1) $p \leq (\log x)^{1-\delta}$,
(2) $(\log x)^{1-\delta} < p \leq (\log x)^{1+\delta}$,
(3) $p > (\log x)^{1+\delta}$.

To estimate $\sum_1$, we use the estimate (see [22, eq. (2.4)])

$$\sum_{p \leq y} \frac{1}{p^c} = \mathrm{li}(y^{1-c})\,(1 + O(1/\log y)) + O(|\log(1-c)|),$$

valid with uniform implied constants whenever $0 < c < 1$ and $y^{1-c} \geq 2$. (Since we require only a crude bound on $\sum_1$, this could be avoided, but this estimate will be needed in the proof of (ii).) Putting $y := (\log x)^{1-\delta}$, we find that

$$y^{1-c} = (\log x)^{(1-\delta)(1+\delta)\log_3 x/\log_2 x} = (\log_2 x)^{1-\delta^2}.$$

Thus, $\sum_1 \ll \mathrm{li}(y^{1-c}) = o((\log_2 x)^{1-\delta^2})$, which fits nicely inside the right-hand side of (5.3). The contribution from $\sum_2$ is more easily estimated; by our hypothesis on the number of prime factors of v in $[1, (\log x)^{1+\delta}]$, we have

$$\begin{aligned}\textstyle\sum_2 &\leq \frac{1}{((\log x)^{1-\delta})^c} \#\{p \mid v : (\log x)^{1-\delta} < p \leq (\log x)^{1+\delta}\} \\ &\leq \frac{1}{((\log x)^{1-\delta})^c} (\log x)^{1-\delta} = y^{1-c} = (\log_2 x)^{1-\delta^2}.\end{aligned}$$

Finally, $\sum_3 = o(1)$, since the number of terms is trivially bounded by $\omega(v) = o(\log x)$, while each term is smaller than $(\log x)^{-1}$ (for large x). Collecting our results gives (5.3) and completes the proof of (i).

To prove (ii), we modify the method slightly, in order to insert the condition $\omega(m) \leq Z := \lfloor \log x/(\log_2 x)^{2+\delta} \rfloor$. By the multinomial theorem,

$$\sum_{\substack{m:\ \phi(m)=v \\ \omega(m)\leq Z}} 1 \leq z^c \sum_{\substack{m:\ p|m \Rightarrow p-1|v \\ \omega(m)\leq Z}} \frac{1}{m^c}$$
$$\leq z^c \sum_{0\leq k\leq Z} \frac{1}{k!} \left(\sum_{p-1|v} \left(\frac{1}{p^c} + \frac{1}{p^{2c}} + \cdots \right) \right)^k .$$

As in (i), we take

$$c = 1 - (1+\delta)\log_3 x/\log_2 x,$$

so that $z^c = x/L(x)^{1+\delta+o(1)}$, and we seek to show that the multiplicative contribution from the sums is $L(x)^{o(1)}$. Since $c > 2/3$, we have

$$\sum_{p-1|v} \left(\frac{1}{p^c} + \frac{1}{p^{2c}} + \cdots \right) \leq \sum_{p-1|v} \frac{1}{p^c} + O(1)$$
$$\leq \sum_{d|v} \frac{1}{d^c} + O(1) \ll \exp\left(\sum_{p|v} \frac{1}{p^c} \right).$$

For a given value of $\omega(v)$, the remaining sum is largest when the primes are smallest. From the prime number theorem,

$$\sum_{p|v} \frac{1}{p^c} \leq \sum_{p\leq 2\log x} \frac{1}{p^c} \ll \mathrm{li}((2\log x)^{1-c}) \ll \frac{(\log_2 x)^{1+\delta}}{\log_3 x}$$

for large x. It follows that for certain absolute constants C_5 and C_6, we have

$$\sum_{0\leq k\leq Z} \frac{1}{k!} \left(\sum_{p-1|v} \left(\frac{1}{p^c} + \frac{1}{p^{2c}} + \cdots \right) \right)^k$$
$$\leq \sum_{0\leq k\leq Z} \frac{1}{k!} \left(C_5 \exp(C_6 (\log_2 x)^{1+\delta}/\log_3 x) \right)^k .$$

This sum is dominated by the term corresponding to $k = Z$, and so it is

$$\ll \frac{1}{Z!} C_5^Z \exp\left(C_6 \log x/(\log_2 x \log_3 x) \right) = L(x)^{o(1)} \qquad (\text{as } x \to \infty),$$

as desired.

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