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Abstract. We prove density modulo 1 of the sets of the form
\[ \{\mu^m \lambda^n \xi + r_m : n, m \in \mathbb{N}\}, \]
where \(\lambda, \mu \in \mathbb{R}\) is a pair of rationally independent algebraic integers of degree \(d \geq 2\), satisfying some additional assumptions, \(\xi \neq 0\), and \(r_m\) is any sequence of real numbers.

1. Introduction

It is a very well known result in the theory of distribution modulo 1 that for every irrational \(\xi\) the sequence \(\{n\xi : n \in \mathbb{N}\}\) is dense modulo 1 (and even uniformly distributed modulo 1) [11].

In 1967, in his seminal paper [4], Furstenberg proved the following

Theorem 1.1 (Furstenberg, [4, Theorem IV.1]). If \(p, q > 1\) are rationally independent integers (i.e., they are not both integer powers of the same integer) then for every irrational \(\xi\) the set
\[ \{p^n q^m \xi : n, m \in \mathbb{N}\} \]
is dense modulo 1.
One possible direction of generalizations is to consider \( p \) and \( q \) in Theorem 1.1 not necessarily integer. This was done by Berend in [3].

According to [10], Furstenberg conjectured that under the assumptions of Theorem 1.1, the set \( \{ (p^n + q^m)\xi : n, m \in \mathbb{N} \} \) is dense modulo 1. As far as we know, this conjecture is still open. However, there are some results concerning related questions. For example, B. Kra in [9], proved the following

**Theorem 1.3** (Kra, [9, Theorem 1.2 and Corollary 2.2]). For \( i = 1, 2 \), let \( 1 < p_i < q_i \) be two rationally independent integers. Assume that \( p_1 \neq p_2 \) or \( q_1 \neq q_2 \). Then, for every \( \xi_1, \xi_2 \in \mathbb{R} \) with at least one \( \xi_i \notin \mathbb{Q} \), the set

\[
\{ p_1^n q_1^m \xi_1 + p_2^n q_2^m \xi_2 : n, m \in \mathbb{N} \}
\]

is dense modulo 1.

Furthermore, let \( r_m \) be any sequence of real numbers and \( \xi \notin \mathbb{Q} \). Then, the set

\[
\{ p_1^n q_1^m \xi + r_m : n, m \in \mathbb{N} \}
\]

is dense modulo 1.

Inspired by Berend’s result [3], we prove some kind of a generalization of the second part of Theorem 1.3 (some kind of an extension of the first part is given in [15]). Namely, we allow algebraic integers, satisfying some additional assumption, to appear in (1.4) instead of integers, and we prove the following

**Theorem 1.5.** Let \( \lambda, \mu \) be a pair of rationally independent real algebraic integers of degree \( d \geq 2 \), with absolute values greater than 1. Let \( \lambda_2, \ldots, \lambda_d \) denote the conjugates of \( \lambda = \lambda_1 \). Assume that either \( \lambda \) or \( \mu \) has the property that for every \( n \in \mathbb{N} \), its \( n \)-th power is of degree \( d \), and that \( \mu \) may be expressed in the form \( g(\lambda) \), where \( g \) is a polynomial with integer coefficients, i.e.,

\[
\mu = g(\lambda), \textrm{ for some } g \in \mathbb{Z}[x]. \tag{1.6}
\]

Assume further that

\[
\text{for each } i = 2, \ldots, d, \text{ either } |\lambda_i| > 1 \text{ or } |g(\lambda_i)| > 1, \tag{1.7}
\]

and

\[
\text{for each } i = 2, \ldots, d, \ |\lambda_i| \neq 1. \tag{1.8}
\]

Then for any non-zero \( \xi \), and any sequence of real numbers \( r_m \), the set

\[
\{ \mu^n \lambda^n \xi + r_m : n, m \in \mathbb{N} \}
\]

is dense modulo 1.
As an example illustrating Theorem 1.5 we can consider the following expressions

$$(\sqrt{23} + 1)^n(\sqrt{23} + 2)^m + 2^m \beta \text{ or } (3 + \sqrt{3})^n(\sqrt{3})^m 5 + 7^m \beta, \beta \in \mathbb{R}.$$  

**Remark.** We believe that assumption (1.6) is not necessary to conclude density modulo 1 of the sets of the form (1.9).

Another kind of a generalization of Furstenberg’s Theorem 1.1, which we are going to use in the proof of our result, is to consider higher-dimensional analogues. A generalization to a commutative semigroup of non-singular $d \times d$-matrices with integer coefficients acting by endomorphisms on the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and to the commutative semigroups of continuous endomorphisms of other compact abelian groups was given by Berend in [1] and [2], respectively (see Sect. 2.3). Recently some results for non-commutative semigroups of endomorphisms of $\mathbb{T}^d$ have been obtained in [5, 6, 13].

The structure of the paper is as follows. In Sect. 2 we recall some notions and facts from ergodic theory and topological dynamics. Following Berend [1, 2], we recall the definition of an ID-semigroup of endomorphisms of the $d$-dimensional torus $\mathbb{T}^d$. Then we state Berend’s theorem, [1], which gives conditions that guarantee that a given semigroup of endomorphisms of $\mathbb{T}^d$ is an ID-semigroup. This theorem is crucial for the proof of our main result. Finally in Sect. 3, using some ideas from [9, 3] we prove Theorem 1.5.

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## 2. Preliminaries

### 2.1. Algebraic numbers.**

We say that $P \in \mathbb{Z}[x]$ is *monic* if the leading coefficient of $P$ is one, and *reduced* if its coefficients are relatively prime. A *real algebraic integer* is any real root of a monic polynomial $P \in \mathbb{Z}[x]$, whereas an *algebraic number* is any root (real or complex) of a (not necessarily monic) non-constant polynomial $P \in \mathbb{Z}[x]$. The *minimal polynomial* of an algebraic number $\theta$ is the reduced element $Q$ of $\mathbb{Z}[x]$ of the least degree such that $Q(\theta) = 0$. If $\theta$ is an algebraic number, the roots of its minimal polynomial are simple. The *degree* of an algebraic number is the degree of its minimal polynomial.

Let $\theta$ be an algebraic integer of degree $n$ and let $P \in \mathbb{Z}[x]$ be the minimal polynomial of $\theta$. The $n - 1$ other distinct (real or complex) roots $\theta_2, \ldots, \theta_n$ of $P$ are called *conjugates* of $\theta$.

### 2.2. Topological transitivity, ergodicity and hyperbolic toral endomorphisms.**

We start with some basic notions, [12, 7]. We consider a *discrete topological dynamical system* $(X, f)$ given by a compact metric
space $X$ and a continuous map $f : X \to X$. We say that a topological dynamical system $(X, f)$ (or simply that a map $f$) is topologically transitive if for any two nonempty open sets $U, V \subset X$ there exists $n = n(U, V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. One can show that $f$ is topologically transitive if for every nonempty open set $U$ in $X$, $\bigcup_{n \geq 0} f^{-n}(U)$ is dense in $X$ (see [8] for other equivalent definitions). If there exists a point $x \in X$ such that its orbit $\{ f^n(x) : n \in \mathbb{N} \}$ is dense in $X$, then we say that $x$ is a transitive point. Under some additional assumptions on $X$, the map $f$ is topologically transitive if and only if there is a transitive point $x \in X$. Namely, we have the following

**Proposition 2.1** ([14]). If $X$ has no isolated point and $f$ has a transitive point then $f$ is topologically transitive. If $X$ is separable, second category and $f$ is topologically transitive then $f$ has a transitive point.

Consider a probability space $(X, \mathcal{F}, \mu)$ and a continuous transformation $f : X \to X$. We say that the map $f$ is measure preserving, and that $\mu$ is $f$-invariant, if for every $A \in \mathcal{F}$ we have $\mu(f^{-1}(A)) = \mu(A)$. Recall that $f$ is said to be ergodic if every set $A$ such that $f^{-1}(A) = A$ has measure $0$ or $1$.

Let $L$ be a hyperbolic matrix, that is a $d \times d$-matrix with integer entries, with non-zero determinant, and without eigenvalues of absolute value $1$. Then $L\mathbb{Z}^d \subset \mathbb{Z}^d$, so $L$ determines a map of the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. Such a map is called a hyperbolic toral endomorphism. It is known (see e.g. [12]) that the Haar measure $m$ of $\mathbb{T}^d$ is invariant under surjective continuous homomorphisms. In particular, it is $L$-invariant. We state two propositions about toral endomorphisms. Their proofs can be found in [12].

**Proposition 2.2.** Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a hyperbolic toral endomorphism. Then $L$ is ergodic.

The next proposition gives an elementary and useful relation between ergodicity and topological transitivity.

**Proposition 2.3.** Let $L$ be a continuous endomorphism of $\mathbb{T}^d$ which preserves the Haar measure $m$. If $L$ is ergodic then it is topologically transitive. In particular, if $L$ is a hyperbolic toral endomorphism then $L$ has a transitive point $t \in \mathbb{T}^d$, i.e., $\{ L^n t : n \in \mathbb{N} \}$ is dense in $\mathbb{T}^d$.

We will also need the following lemma about finite invariant sets of ergodic endomorphisms. For the proof see [1, Lemma 5.2].

**Lemma 2.4.** Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an ergodic endomorphism. A finite $L$-invariant set is necessarily composed of torsion elements only.

### 2.3. ID semigroups of endomorphisms acting on $\mathbb{T}^d$.

Following [1, 2], we say that the semigroup $\Sigma$ of endomorphisms of a compact group
\( G \) has the **ID-property** (or simply that \( \Sigma \) is an **ID-semigroup**) if the only infinite closed \( \Sigma \)-invariant subset of \( G \) is \( G \) itself. (ID-property stands for \( \text{infinite invariant is dense} \).) A subset \( A \subset G \) is said to be \( \Sigma \)-invariant if \( \Sigma A \subset A \).

We say, exactly like in the case of real numbers, that two endomorphisms \( \sigma \) and \( \tau \) are **rationally dependent** if there are integers \( m \) and \( n \), not both of which are 0, such that \( \sigma^m = \tau^n \), and **rationally independent** otherwise.

Berend in \([1]\) gave necessary and sufficient conditions in arithmetical terms for a commutative semigroup \( \Sigma \) of endomorphisms of the \( d \)-dimensional torus \( T^d = \mathbb{R}^d / \mathbb{Z}^d \) to have the ID-property. Namely, he proved the following.

**Theorem 2.5** (Berend, \([1\), Theorem 2.1]). A commutative semigroup \( \Sigma \) of continuous endomorphisms of \( T^d \) has the ID-property if and only if the following hold:

(i) There exists an endomorphism \( \sigma \in \Sigma \) such that the characteristic polynomial \( f_{\sigma^n} \) of \( \sigma^n \) is irreducible over \( \mathbb{Z} \) for every positive integer \( n \).

(ii) For every common eigenvector \( v \) of \( \Sigma \) there exists an endomorphism \( \sigma_v \in \Sigma \) whose eigenvalue in the direction of \( v \) is of norm greater than 1.

(iii) \( \Sigma \) contains a pair of rationally independent endomorphisms.

**Remark.** Let \( \Sigma \) be a commutative ID-semigroup of endomorphisms of \( T^d \). Then the \( \Sigma \)-orbit of the point \( x \in T^d \) is finite if and only if \( x \) is a rational element, i.e., \( x = r/q, r \in \mathbb{Z}^d, q \in \mathbb{N} \) (see \([1]\)).

**3. Proof of Theorem 1.5**

Let \( \lambda > 1 \) be a real algebraic integer of degree \( d \) with minimal (monic) polynomial \( Q_\lambda \in \mathbb{Z}[x] \),

\[
Q_\lambda(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_1x + c_0.
\]

We associate with \( \lambda \) the following **companion matrix** of \( Q_\lambda \),

\[
\sigma_\lambda = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-c_0 & -c_1 & -c_2 & \ldots & -c_{d-1}
\end{pmatrix}.
\]

**Remark.** We can think of \( \sigma_\lambda \) as a matrix of multiplication by \( \lambda \) in the algebraic number field \( \mathbb{Q}(\lambda) \). Namely, if \( x \in \mathbb{Q}(\lambda) \) has coordinates \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}) \) in the basis consisting of \( 1, \lambda, \ldots, \lambda^{d-1} \), then \( \lambda x \) has coordinates \( \alpha \sigma_\lambda \).
Let $\mu = g(\lambda)$, where $g \in \mathbb{Z}[x]$ is a polynomial with integer coefficients, and define the matrix $\sigma_{\mu} = g(\sigma_{\lambda})$.

Denote by $\Sigma$ the semigroup of endomorphisms of $\mathbb{T}^d$ generated by $\sigma_{\lambda}$ and $\sigma_{\mu}$. The vector $v = (1, \lambda, \lambda^2, \ldots, \lambda^{d-1})^t$ is an eigenvector of the matrix $\sigma_{\lambda}$ with an eigenvalue $\lambda$, that is $\sigma_{\lambda}v = \lambda v$. Since $\Sigma$ is a commutative semigroup, it follows that $v$ is a common eigenvector of $\Sigma$, in particular $\sigma_{\mu}v = g(\sigma_{\lambda})v = g(\lambda)v = \mu v$.

Clearly, under the assumptions on $\lambda$ and $\mu$, the operators $\sigma_{\lambda}$ and $\sigma_{\mu}$ are rationally independent endomorphisms of $\mathbb{T}^d$ and the characteristic polynomial either of $\sigma_{\lambda}^n$ or $\sigma_{\mu}^n$ is irreducible over $\mathbb{Z}$ for every $n \in \mathbb{N}$. Furthermore, it follows from (1.7) that the condition (ii) of Theorem 2.5 is also satisfied. Thus we have proved the following

**Lemma 3.1.** Let $\lambda$ and $\mu$ be as in Theorem 1.5. Let $\Sigma$ be the semigroup of endomorphisms of $\mathbb{T}^d$ generated by $\sigma_{\lambda}$ and $\sigma_{\mu}$. Then $\Sigma$ is the ID-semigroup.

The next lemma is a generalization of [9, Lemma 2.1] to the higher-dimensional case. Let $X$ be a compact metric space with a distance $d$. Consider the space $C_X$ of all closed subsets of $X$. The Hausdorff metric $d_H$ on the space $C_X$ is defined as

$$d_H(A, B) = \max\{\max_{x \in A} d(x, B), \max_{x \in B} d(x, A)\},$$

where $d(x, B) = \min_{y \in B} d(x, y)$ is the distance of $x$ from the set $B$. It is known that if $X$ is a compact metric space then $C_X$ is also compact.

**Lemma 3.2.** Let $\sigma, \tau$ be a pair of rationally independent and commuting endomorphisms of $\mathbb{T}^d$. Assume that the semigroup $\Sigma = \langle \sigma, \tau \rangle$ generated by $\sigma$ and $\tau$ satisfies the conditions of Theorem 2.5, and $\sigma$ is a hyperbolic toral endomorphism of $\mathbb{T}^d$. Let $A$ be an infinite $\sigma$-invariant subset of $\mathbb{T}^d$. Then for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that the set $\tau^mA$ is $\varepsilon$-dense.

**Proof.** It is clear that, taking the closure of $A$ if necessary, we can assume that $A$ is closed. We consider the space $C_{\mathbb{T}^d}$ of all closed subsets of $\mathbb{T}^d$ with the Hausdorff metric $d_H$. Let

$$\mathcal{F} := \{\tau^nA : n \in \mathbb{N}\} \subset C_{\mathbb{T}^d}.$$

Since the set $A$ is $\sigma$-invariant, it follows that every element (set) $F \in \mathcal{F}$ is also $\sigma$-invariant. Define,

$$T = \bigcup_{F \in \mathcal{F}} F \subset \mathbb{T}^d.$$

Since $A$ is an infinite set and $A \subset T$, it follows that $T$ is infinite. Notice that $T$ is closed in $\mathbb{T}^d$, since $\mathcal{F}$ is closed in $C_{\mathbb{T}^d}$. Moreover, $T$ is $\sigma$- and $\tau$-invariant. Hence, by Theorem 2.5, we get

$$T = \mathbb{T}^d.$$
Since $\sigma$ is a hyperbolic toral endomorphism, it follows by Proposition 2.3, that there exists $t \in T$ such that the orbit $\{\sigma^n t : n \in \mathbb{N}\}$ is dense in $\mathbb{T}^d$, i.e.,
\begin{equation}
\{\sigma^n t : n \in \mathbb{N}\} = \mathbb{T}^d
\end{equation}

Clearly, $t \in F$ for some $F \in \mathcal{F}$. By definition of $\mathcal{F}$, there is a sequence $\{n_k\} \subset \mathbb{N}$ such that $F = \lim_k \tau^{n_k} A$, and the limit is taken in the Hausdorff metric $d_H$. Since $t \in F$ and $F$ is $\sigma$-invariant, we get $F \supset \{\sigma^n t : n \in \mathbb{N}\} = \mathbb{T}^d$ (see (3.3)). Hence, $F = \mathbb{T}^d$. Therefore, for sufficiently large $k$, $\tau^{n_k} A$ is $\epsilon$-dense. \hfill $\square$

Now we are ready to give

**Proof of Theorem 1.5.** Let $\alpha = \xi(1, \lambda, \lambda^2, \ldots, \lambda^{d-1})^t \in \mathbb{R}^d$ be a common eigenvector of the semigroup $\Sigma$. Consider

$$A = \{\sigma^\alpha \pi(\alpha) : n \in \mathbb{N}\} = \{\pi(\lambda^n \xi, \lambda^{n+1} \xi, \ldots, \lambda^{n+d-1} \xi)^t : n \in \mathbb{N}\},$$

where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection. By (1.8), $\sigma_\lambda$ is a hyperbolic toral endomorphism. In particular, by Proposition 2.2, $\sigma_\lambda$ is ergodic. Since $\pi(\alpha)$ is not a torsion element, it follows from Lemma 2.4 that $A$ is infinite. By Lemma 3.1, $\Sigma = \langle \sigma_\lambda, \sigma_\mu \rangle$ is the ID-semigroup of $\mathbb{T}^d$. Thus, by Lemma 3.2 applied to $\sigma_\lambda$ and $\sigma_\mu$, there exists $m \in \mathbb{N}$ such that $\sigma_\mu^m A$ is $\epsilon$-dense. Let $v_m = \pi(r_m, 0, \ldots, 0)^t$. Since

$$\sigma_\mu^m A + v_m = \{\pi(\mu^m \lambda^n \xi + r_m, \mu^m \lambda^{n+1} \xi, \ldots, \mu^m \lambda^{n+d-1} \xi)^t : n \in \mathbb{N}\}$$

is a translate of an $\epsilon$-dense set, it is also $\epsilon$-dense. Now, taking the projection of the set $\sigma_\mu^m A + v_m$ on the first coordinate we get the result. \hfill $\square$

**References**


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