Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

Oleg N. GERMAN

Klein polyhedra and lattices with positive norm minima

Tome 19, nº 1 (2007), p. 175-190.

<http://jtnb.cedram.org/item?id=JTNB_2007__19_1_175_0>

Journal de Théorie des Nombres de Bordeaux **19** (2007), 175-190

Klein polyhedra and lattices with positive norm minima

par Oleg N. GERMAN

Résumé. Un polyèdre de Klein est défini comme étant l'enveloppe convexe de tous les points non nuls d'un réseau qui se trouvent dans un orthant de l'espace $\mathbb{R}^n$. On généralise à partir de ce concept la notion de la fraction continue. Dans cet article nous considérons les faces et les configurations étoilées, formées des arêtes issues de chaque sommet d'un polyèdre de Klein, comme les analogues multidimensionnels des quotient partiels, et nous définissons des charactéristiques quantitatives de ces "quotients partiels", qui sont appelés "déterminants". Il est démontré que les faces de tous les 2^n polyèdres de Klein engendrés par un réseau Λ ont leurs déterminants uniformément bornés (dans leur ensemble) si, et seulement si, les faces et les étoiles d'arêtes des sommets du polyèdre de Klein, engendrées par Λ et correspondant à l'orthant positif, ont aussi leurs déterminants uniformément bornés.

Abstract. A Klein polyhedron is defined as the convex hull of nonzero lattice points inside an orthant of $\mathbb{R}^n$. It generalizes the concept of continued fraction. In this paper facets and edge stars of vertices of a Klein polyhedron are considered as multidimensional analogs of partial quotients and quantitative characteristics of these "partial quotients", so called determinants, are defined. It is proved that the facets of all the 2^n Klein polyhedra generated by a lattice Λ have uniformly bounded determinants if and only if the facets and the edge stars of the vertices of the Klein polyhedron generated by Λ and related to the positive orthant have uniformly bounded determinants.

1. Introduction

In this paper we give a complete proof of the results announced in [1]. Here we investigate one of the most natural multidimensional geometric generalizations of continued fractions, the so–called *Klein polyhedra*.

Manuscrit reçu le 11 janvier 2006.

This research was supported by RFBR (grant N° 06–01–00518), INTAS (grant N° 03–51–5070) and grant of the President of Russian Federation N° MK–6370.2006.1.

Continued fractions admit a rather elegant geometric interpretation (see [2]), which can be described as follows. Given a number α: $0 < \alpha < 1$, consider a two–dimensional lattice Λ_α with basis vectors $(1, 1-\alpha)$ and $(0,1)$. The convex hull of the nonzero points of the lattice Λ_α with nonnegative coordinates (in the initial unit basis of $\mathbb{R}^2$) is called a *Klein polygon*. The integer lengths of the Klein polygon's bounded edges are equal to the respective partial quotients of the number α with odd indices, and the integer angles between pairs of adjacent edges are equal to the partial quotients with even indices. The *integer length* of a segment with endpoints in Λ_α is defined as the number of lattice points contained in the interior of this segment plus one. And the *integer angle* between two such segments with a common endpoint is defined as the area of the parallelogram spanned by them divided by the product of their integer lengths, or in other words, the index of the sub-lattice spanned by the primitive vectors of Λ_α parallel to these two segments.

If an arbitrary two–dimensional lattice is considered, then there obviously appear two numbers with their partial quotients describing the combinatorial structure of the corresponding Klein polygon.

The multidimensional generalization of this construction was proposed more than a century ago by F. Klein (see [3]). Let $\Lambda \subset \mathbb{R}^n$ be an n–dimensional lattice with determinant 1.

Definition 1. The convex hulls of the nonzero points of the lattice Λ contained in each orthant are called *Klein polyhedra* of the lattice Λ.

In this paper, we consider only irrational lattices Λ, i.e. we assume that the coordinate planes contain no lattice points except the origin $\mathbf{0}$. Then, as shown in [5], a Klein polyhedron K is a generalized polyhedron, which means that its intersection with an arbitrary bounded polyhedron is itself a polyhedron. Hence the boundary of K is in this case an $(n-1)$–dimensional polyhedral surface homeomorphic to $\mathbb{R}^{n-1}$, consisting of convex $(n-1)$–dimensional (generalized) polyhedra, with each point in it belonging only to a finite number of these polyhedra. Some of the faces of K can be unbounded, but only if the lattice, dual to Λ, is not irrational (see [4]).

Definition 2. The boundary Π of a Klein polyhedron K is called a *sail*.

Definition 3. Let F be a face of K of dimension k. We call F
a) a *vertex* of K, if $k = 0$,
b) an *edge* of K, if $k = 1$,
c) a *facet* of K, if $k = n - 1$.

A few years ago Vladimir Arnold posed a question (see [6], [7]) which local affine invariants of a sail are sufficient to reconstruct the lattice. This question in its initial formulation remains unanswered. However in the current paper we establish a connection between some local invariants of a sail and the property of a lattice to have positive norm minimum.

In the two–dimensional case two neighboring Klein polygons have very much in common, for the integer lengths of edges of one of them equal the integer angles between the correspondent edges of another one (see, for instance, [8]). Due to this fact many statements concerning continued fractions admit "dual" formulations: we can use only integer lengths of edges, and then we will have to consider all the four Klein polygons, or we can use both integer lengths of edges and integer angles between adjacent edges, and then we may content ourselves with only one Klein polygon. The main result of this paper (Theorem 2.1) gives an example of a statement on Klein polyhedra in an arbitrary dimension admitting such a "dual" formulation.

2. Formulation of the main result

In this section the main result is formulated, which is a multidimensional generalization of a well–known statement that a number is badly approximable if and only if its partial quotients are bounded. Recall that a number α is called *badly approximable* if there exists a constant $c > 0$ such that, for all integer p and natural q, the following inequality holds:

$$|q\alpha - p| \geqslant \frac{c}{q}\ .$$

In terms of Klein polygons of the lattice Λ_α from the beginning of the first section, this means exactly that the area $\{\, \mathbf{x} \in \mathbb{R}^2 \mid x_2 > 0 \text{ and } |x_1x_2| < c \,\}$ does not contain any point of Λ_α.

Thus, it is natural to consider the property of a lattice Λ to have a positive norm minimum as a multidimensional generalization of the property of a number to be badly approximable.

Definition 4. *The norm minimum* of a lattice Λ is defined as

$$N(\Lambda) = \inf_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} |\varphi(\mathbf{x})|,$$

where $\varphi(\mathbf{x}) = x_1 \dots x_n$.

We will also need a multidimensional analog of partial quotients. In view of the correspondence between partial quotients and integer lengths and angles mentioned in the previous section, it is rather natural in the n–dimensional case to expect the $(n-1)$–dimensional faces of a sail (we will call them *facets*) and the edge stars of a sail's vertices to play the role of partial quotients. As a numerical characteristic of these multidimensional "partial quotients" we will consider their "determinants".

Definition 5. Let F be an arbitrary facet of a sail Π and let $\mathbf{v}_1,\ldots,\mathbf{v}_m$ be the vertices of F. Then, we define the *determinant of the facet* F as

$$\det F=\sum_{1\leqslant i_1<\ldots<i_n\leqslant m}|\det(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_n})|\,.$$

Definition 6. Suppose a vertex $\mathbf{v}$ of a sail Π is incident to m edges. Let $\mathbf{r}_1,\ldots,\mathbf{r}_m$ denote the primitive vectors of the lattice Λ generating these edges. Then, the *determinant of the edge star* $\mathrm{St}_{\mathbf{v}}$ of the vertex $\mathbf{v}$ is defined as

$$\det\mathrm{St}_{\mathbf{v}}=\sum_{1\leqslant i_1<\ldots<i_n\leqslant m}|\det(\mathbf{r}_{i_1},\ldots,\mathbf{r}_{i_n})|\,.$$

It is clear that when $n=2$, i.e. when the sail is one–dimensional, the determinants of the sail's edges are equal to the integer lengths of these edges, and the determinants of the edge stars of vertices are equal to the integer angles between the correspondent edges.

Note that we can give an equivalent definition of determinants of facets and edge stars in terms of Minkowski sum and mixed volume. Recall (see [9], [10], [11], [12]) that the *Minkowski sum* of segments $[\mathbf{0},\mathbf{x}_1],\ldots,[\mathbf{0},\mathbf{x}_m]$ (we will need only this most simple case) is the set

$$\{\lambda_1\mathbf{x}_1+\cdots+\lambda_m\mathbf{x}_m \mid 0\leqslant\lambda_i\leqslant 1\}$$

and its (Euclidean) volume is called the *mixed volume* of the segments $[\mathbf{0},\mathbf{x}_1],\ldots,[\mathbf{0},\mathbf{x}_m]$. The following simple statement immediately gives us the equivalent way of defining determinants of facets and edge stars:

Statement 2.1. *For any $\mathbf{x}_1,\ldots,\mathbf{x}_m\in\mathbb{R}^n$ the mixed volume of the segments $[\mathbf{0},\mathbf{x}_1],\ldots,[\mathbf{0},\mathbf{x}_m]$ is equal to*

$$\sum_{1\leqslant i_1<\ldots<i_n\leqslant m}|\det(\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_n})|\,.$$

Now that all the needed definitions are given we can formulate the main result of this paper. It is a part of the following

Theorem 2.1. *Given an irrational n–dimensional lattice $\Lambda\subset\mathbb{R}^n$, the following conditions are equivalent:*

(1) $N(\Lambda)>0$.
(2) *The facets of all the 2^n sails generated by Λ have uniformly bounded determinants (i.e. bounded by a constant not depending on a face).*
(3) *The facets and the edge stars of the vertices of the sail generated by Λ and related to the positive orthant have uniformly bounded determinants (i.e. bounded by a constant not depending on a face or an edge star).*

The equivalence of (1) and (2) was established in [4]. In the current paper are proved the two implications (1) & (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2).

Remark 1. Actually, what is proved in this paper is a bit stronger than what is formulated in Theorem 2.1. Namely, it is shown that if $N(\Lambda) = \mu > 0$ then there exists a constant D depending only on n and μ, such that the facets and the edge stars of the vertices of the sail Π have determinants bounded by D. And conversely, if the facets and the edge stars of the vertices of the sail have determinants bounded by a constant D then there exists a constant μ depending only on n and D, such that $N(\Lambda) \geqslant \mu > 0$.

Remark 2. To be precise, the definition of a facet's determinant in [4] is somewhat different from Definition 5. However it is absolutely clear that the uniform boundedness of determinants from [4] is equivalent to the uniform boundedness of determinants from Definition 5.

3. A relation to the Littlewood and Oppenheim conjectures

The following two conjectures are classical:

Littlewood conjecture. *If $\alpha, \beta \in \mathbb{R}$, then $\inf_{m\in\mathbb{N}} m\|m\alpha\|\|m\beta\| = 0$, where $\|\cdot\|$ denotes the distance to the nearest integer.*

Oppenheim conjecture on linear forms. *If $n \geqslant 3$ and $\Lambda \subset \mathbb{R}^n$ is an n–dimensional lattice with $N(\Lambda) > 0$, then Λ is algebraic (i.e. similar modulo the action of the group of diagonal $n \times n$ matrices to the lattice of a complete module of a totally real algebraic field of degree n).*

Note that the converse of the latter statement is an obvious corollary of the Dirichlet theorem on algebraic units (see [13]).

As is well known (see [14]), the three–dimensional Oppenheim conjecture implies the Littlewood conjecture. In [15] and [16] an attempt was made to prove the Oppenheim conjecture, however, there was an essential gap in the proof. Thus, both conjectures remain unproved.

Theorem 2.1 allows to reformulate the Oppenheim conjecture as follows:

Reformulated Oppenheim conjecture. *If $n \geqslant 3$ and a sail Π generated by an n–dimensional lattice $\Lambda \subset \mathbb{R}^n$ is such that all its facets and edge stars of vertices have uniformly bounded determinants, then Λ is algebraic.*

It follows from the Dirichlet theorem on algebraic units that a sail generated by an n–dimensional algebraic lattice has periodic combinatorial structure. The group of "periods" is isomorphic to $\mathbb{Z}^{n-1}$ and the fundamental domain is bounded. Thus the Oppenheim conjecture yields the following corollary: if a sail's facets and edge stars of its vertices have uniformly bounded determinants then this sail has a periodic combinatorial structure.

4. Dual lattices and polar polyhedra

In this section we generalize some of the facts from the theory of polar polyhedra to the case of Klein polyhedra. We also prove some statements connecting Klein polyhedra of dual lattices. It is worth mentioning in this context the book [17], where similar questions are considered.

Definition 7. Let P be an arbitrary (generalized) n–dimensional polyhedron in $\mathbb{R}^n$, $\mathbf{0} \notin P$. Then, the *polar polyhedron* for the P is the set

$$P^\circ = \{\, \mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in P \ \ \langle \mathbf{x}, \mathbf{y} \rangle \geqslant 1 \,\}.$$

The set P° is obviously closed and convex. Hence we can talk about its *faces*, defined as the intersections of P° with its supporting hyperplanes. We will denote by $\mathfrak{B}(P^\circ)$ and $\mathfrak{B}(P)$ the sets of all proper faces of P° and P respectively.

Usually the "inverse" concept of polarity is considered, i.e. for polytopes containing $\mathbf{0}$ in their interior and with the inverse inequality. And it is well known that between the boundary complexes of polar polytopes there exists an inclusion–reversing bijection (see, for instance, [10], [11], [12]). We are going to need a similar statement concerning Klein polyhedra generated by irrational lattices:

Statement 4.1. *Let Λ be an irrational n–dimensional lattice in $\mathbb{R}^n$ and K be the Klein polyhedron generated by Λ and related to the positive orthant. Suppose in addition that all the faces of K are bounded.*
(a) K° is an n–dimensional generalized polyhedron with bounded faces.
(b) If F is a proper face of K then the set F_K° defined as

$$F_K^\circ = K^\circ \cap \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 1 \ \textit{for all} \ \mathbf{y} \in F\}$$

is a face of K° and

$$\dim F_K^\circ = n - 1 - \dim F.$$

(c) The mapping

$$\begin{array}{rl} \beta_K : & \mathfrak{B}(K) \to \mathfrak{B}(K^\circ) \\ \beta_K : & \qquad F \mapsto F_K^\circ \end{array}$$

is an inclusion–reversing bijection.

To prove Statement 4.1 we will need three auxiliary statements. The first one can be proved simply by literal translation of already known arguments for polytopes (see [10], [11], [12]) to our case, so we leave it without proof:

Lemma 4.1. *Let P be an n–dimensional polyhedron in $\mathbb{R}^n$, $\mathbf{0} \notin P$, and let $\lambda P \subset P$ for all $\lambda \geqslant 1$. Suppose that P contains no lines. Let $\mathfrak{B}'(P)$ and $\mathfrak{B}'(P^\circ)$ denote the sets of all proper faces of P and P° respectively, whose affine hulls do not contain $\mathbf{0}$.*
(a) P° is an n–dimensional polyhedron, $\mathbf{0} \notin P$, $\lambda P \subset P$ for all $\lambda \geqslant 1$ and

P contains no lines.
(b) If $F \in \mathfrak{B}'(P)$ then the set F_P° defined as

$$F_P^\circ = P^\circ \cap \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 1 \textit{ for all } \mathbf{y} \in F\}$$

is a face of P° and

$$\dim F_P^\circ = n - 1 - \dim F.$$

(c) The mapping

$$\begin{aligned} \beta_P : \quad & \mathfrak{B}'(P) \to \mathfrak{B}'(P^\circ) \\ \beta_P : \quad & \qquad F \mapsto F_P^\circ \end{aligned}$$

is an inclusion–reversing bijection.

We will also need the following notation: for each $\mathbf{v} \in \mathbb{R}^n$ we will denote by $H_\mathbf{v}^+$ and $H_\mathbf{v}^-$ the half–spaces $\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{v} \rangle \geqslant 1\}$ and $\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{v} \rangle \leqslant 1\}$ respectively.

Lemma 4.2. *If P is an arbitrary generalized polyhedron containing no lines, and all its edges are bounded, then it coincides with the convex hull of its vertices.*

Proof. It is enough to show that P is contained in the convex hull of its vertices. Since P contains no lines, there exists $\mathbf{u} \in \mathbb{R}^n$ such that the set $P_\mathbf{u} = P \cap H_\mathbf{u}^-$ is nonempty and compact. Since P is a generalized polyhedron, $P_\mathbf{u}$ is a bounded polyhedron and, hence, coincides with the convex hull of its vertices. But all the vertices of $P_\mathbf{u}$ are either vertices of P or lie on edges of P, which are bounded. Therefore, $P_\mathbf{u}$ is contained in the convex hull of vertices of P. Hence, so is P. ☐

Lemma 4.3. *Let Λ be an irrational n–dimensional lattice in $\mathbb{R}^n$ and K be the Klein polyhedron generated by Λ and related to the positive orthant. Then, $K^\circ = K'$, where*

$$K' = \bigcap_{\mathbf{v}} H_\mathbf{v}^+$$

and the intersection is taken over all the vertices of K.

Proof. Since Λ is irrational, every edge of K is bounded. Hence by Lemma 4.2, K coincides with the convex hull of its vertices. The inclusion $K' \subseteq K^\circ$ easily follows from this fact and the Carathéodory theorem (see [10], [18]). The inclusion $K^\circ \subseteq K'$ is obvious. ☐

Further by $\mathrm{conv}(M)$ we will denote the convex hull of a set M.

Proof of Statement 4.1. For each $\mathbf{u} \in \mathbb{R}^n$ let us denote by $V_\mathbf{u}$ the set of vertices $\mathbf{v}$ of K such that the open interval $(\mathbf{v}, \mathbf{u})$ does not have common points with K. The set $V_\mathbf{0}$ obviously coincides with the set of all vertices of K. On the other hand, the set $V_\mathbf{u}$ is finite whenever all u_i are strictly positive, since we suppose that K does not have unbounded faces.

Let us consider an arbitrary point $\mathbf{u} \notin K$ with strictly positive coordinates and denote

$$K_{\mathbf{u}} = \bigcup_{\lambda \geqslant 1} \lambda \operatorname{conv}(V_{\mathbf{u}}).$$

Since $V_{\mathbf{u}}$ is finite,

$$K_{\mathbf{u}}^{\circ} = \bigcap_{\mathbf{v} \in V_{\mathbf{u}}} H_{\mathbf{v}}^{+}.$$

At the same time for every $\mathbf{w} \in V_{\mathbf{0}} \backslash V_{\mathbf{u}}$ the interval $(\mathbf{w}, \mathbf{u})$ has at least one common point with $\operatorname{conv}(V_{\mathbf{u}})$, hence, there exist $\lambda_{\mathbf{v}} \geqslant 0$ such that

$$\sum_{\mathbf{v} \in V_{\mathbf{u}}} \lambda_{\mathbf{v}} > 1 \quad \text{and} \quad \mathbf{w} = \mathbf{u} + \sum_{\mathbf{v} \in V_{\mathbf{u}}} \lambda_{\mathbf{v}} (\mathbf{v} - \mathbf{u}).$$

Therefore $K_{\mathbf{u}}^{\circ} \cap H_{\mathbf{u}}^{-} \subset H_{\mathbf{w}}^{+}$. Together with Lemma 4.3 this implies that

$$K^{\circ} \cap H_{\mathbf{u}}^{-} = \bigcap_{\mathbf{v} \in V_{\mathbf{0}}} H_{\mathbf{v}}^{+} \cap H_{\mathbf{u}}^{-} = \bigcap_{\mathbf{v} \in V_{\mathbf{0}} \backslash V_{\mathbf{u}}} H_{\mathbf{v}}^{+} \cap K_{\mathbf{u}}^{\circ} \cap H_{\mathbf{u}}^{-} = K_{\mathbf{u}}^{\circ} \cap H_{\mathbf{u}}^{-}.$$

Thus for each $\mathbf{u} \notin K$ with strictly positive coordinates the set $K^{\circ} \cap H_{\mathbf{u}}^{-}$ is a polyhedron. This shows that K° is a generalized polyhedron. Consider now an arbitrary facet F of K° and a point $\mathbf{u} \notin K$ with strictly positive coordinates such that the facet F has a nonempty intersection with the interior of the half–space $H_{\mathbf{u}}^{-}$. As we have just shown, $K^{\circ} \cap H_{\mathbf{u}}^{-} = K_{\mathbf{u}}^{\circ} \cap H_{\mathbf{u}}^{-}$, so the affine hull of F coincides with the affine hull of some facet of $K_{\mathbf{u}}^{\circ}$. But $K_{\mathbf{u}}$ satisfies the conditions of Lemma 4.1, hence, there exists a vertex $\mathbf{v}$ of K such that the affine hull of F is given by the equation $\langle \mathbf{v}, \mathbf{x} \rangle = 1$. Since the lattice Λ is irrational, all the coordinates of $\mathbf{v}$ are strictly positive, and therefore F is compact. This proves (a).

To prove (b) let us consider an arbitrary proper face F of K and a point $\mathbf{u} \notin K$ with strictly positive coordinates such that the set $V_{\mathbf{u}}$ contains all the vertices of F and such that the set F_K° is contained in $H_{\mathbf{u}}^{-}$. Such points exist since F is compact and is contained in the interior of the positive orthant. Then, due to the equality $K^{\circ} \cap H_{\mathbf{u}}^{-} = K_{\mathbf{u}}^{\circ} \cap H_{\mathbf{u}}^{-}$ we have

$$\begin{aligned} F_K^{\circ} = {} & K^{\circ} \cap \{\mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in F \ \ \langle \mathbf{x}, \mathbf{y} \rangle = 1\} = \\ & K_{\mathbf{u}}^{\circ} \cap \{\mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in F \ \ \langle \mathbf{x}, \mathbf{y} \rangle = 1\} = F_{K_{\mathbf{u}}}^{\circ}. \end{aligned}$$

Applying to $K_{\mathbf{u}}^{\circ}$ Lemma 4.1 we get that F_K° is a face of $K_{\mathbf{u}}^{\circ}$ and $\dim F_K^{\circ} = n - 1 - \dim F$. But $K^{\circ} \subset K_{\mathbf{u}}^{\circ}$, so F_K° is also a face of K°, which proves (b).

It remains to show that β_K maps $\mathfrak{B}(K)$ onto $\mathfrak{B}(K^{\circ})$. Consider an arbitrary $F \in \mathfrak{B}(K^{\circ})$ and a point $\mathbf{u} \notin K$ with strictly positive coordinates such that F is contained in the interior of $H_{\mathbf{u}}^{-}$. The existence of such points follows from (a). Then, F is also an element of $\mathfrak{B}'(K_{\mathbf{u}}^{\circ})$ and by Lemma 4.1 coincides with $G_{K_{\mathbf{u}}}^{\circ}$ for some $G \in \mathfrak{B}'(K_{\mathbf{u}})$. But the affine hull of G does not contain $\mathbf{u}$, hence, $G \in \mathfrak{B}(K)$, and the equality $G_K^{\circ} = G_{K_{\mathbf{u}}}^{\circ} = F$ completes the proof. $\square$

From Statement 4.1 and Lemma 4.2 we get the following

Corollary. *K° coincides with the convex hull of its vertices.*

Definition 8. If vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ form a basis of a lattice $\Lambda \subset \mathbb{R}^n$, then the lattice Λ^* with basis $\mathbf{x}_1^*, \dots, \mathbf{x}_n^*$, such that $\langle \mathbf{x}_i, \mathbf{x}_j^* \rangle = \delta_{ij}$, is called *dual* for the lattice Λ.

The lattice Λ^* also generates a Klein polyhedron in the positive orthant. We will denote it by K^*. From the fact that $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{Z}$ for all $\mathbf{x} \in \Lambda$ and $\mathbf{y} \in \Lambda^*$ one can easily deduce the following

Statement 4.2. *If the boundary of the positive orthant contains no nonzero points of lattices Λ and Λ^*, then $K^* \subseteq K^\circ$.*

Note that in case $n = 2$ we can write in Statement 4.2 that $K^* = K^\circ$. The reason why for $n \geqslant 3$ the equality should be substituted by an inclusion is that the integer distances from facets of K to $\mathbf{0}$ can be greater than 1. Here the *integer distance* from a facet F to the origin $\mathbf{0}$ is defined as

$$\min_{\mathbf{x}_1, \dots, \mathbf{x}_n} |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|,$$

where the minimum is taken over all the n–tuples of linearly independent lattice points lying in the affine hull of F. The following lemma is obvious:

Lemma 4.4. *If the integer distance from a facet F of K to the origin $\mathbf{0}$ equals D, then the vertex of K°, corresponding to F, is a point of the lattice $(D^{-1})\Lambda^*$.*

It is also worth mentioning that when $n = 2$ it is actually the equality $K^* = K^\circ$ that implies that the integer lengths of edges of Klein polygons equal the integer angles between the correspondent pairs of adjacent edges. In case $n \geqslant 3$ there is no such good relation between edge stars of K and facets of K^* (or facets of K and edge stars of K^*). The reason is that K^* and K° do not generally coincide. But even if we consider a vertex $\mathbf{v}$ of K and the facet $F_\mathbf{v}$ of K° corresponding to $\mathbf{v}$, it is not clear yet how to connect $\det \mathrm{St}_\mathbf{v}$ and $\det F_\mathbf{v}$. However we will not need an explicit formula connecting these two values, the inequality yielded by the following lemma will satisfy our needs.

Lemma 4.5. *Let $F_\mathbf{v}$ be a facet of the polyhedron K° corresponding to a vertex $\mathbf{v}$ of the polyhedron K. Then,* $\det F_\mathbf{v} \leqslant (\det \mathrm{St}_\mathbf{v})^{n-1}$.

Before proving Lemma 4.5 let us first prove two auxiliary statements.

Lemma 4.6. *Let $\mathbf{r}_1, \dots, \mathbf{r}_n$ form a basis of $\mathbb{R}^n$ and let $\mathbf{v} \in \mathbb{R}^n$ have positive coordinates in this basis. For each $i = 1, \dots, n$ let F_i denote the simplex* $\mathrm{conv}\big(\{\mathbf{v}, \mathbf{v}+\mathbf{r}_1, \dots, \mathbf{v}+\mathbf{r}_n\} \setminus \{\mathbf{v}+\mathbf{r}_i\}\big)$ *and let $\mathbf{w}_i$ be the vector such that $\langle \mathbf{w}_i, \mathbf{x} \rangle = 1$ for all $\mathbf{x} \in F_i$.*

Then,

$$|\det(\mathbf{w}_1,\dots,\mathbf{w}_n)| = \frac{|\det(\mathbf{r}_1,\dots,\mathbf{r}_n)|^{n-1}}{\det F_1 \dots \det F_n}\,.$$

Proof. Let $\mathbf{r}_1^*,\dots,\mathbf{r}_n^*$ denote the basis of R^n, dual to the basis $\mathbf{r}_1,\dots,\mathbf{r}_n$, i.e. the vectors such that $\langle \mathbf{r}_i, \mathbf{r}_j^*\rangle = \delta_{ij}$. Then, $|\mathbf{w}_i|\det F_i = |\mathbf{r}_i^*||\det(\mathbf{r}_1,\dots,\mathbf{r}_n)|$, which implies that

$$|\det(\mathbf{w}_1,\dots,\mathbf{w}_n)| = |\det(\mathbf{r}_1^*,\dots,\mathbf{r}_n^*)|\frac{|\mathbf{w}_1|}{|\mathbf{r}_1^*|}\cdots\frac{|\mathbf{w}_n|}{|\mathbf{r}_n^*|} = \frac{|\det(\mathbf{r}_1,\dots,\mathbf{r}_n)|^{n-1}}{\det F_1 \dots \det F_n}\,.$$

□

We will denote by $\operatorname{int} P$ and $\operatorname{ext} P$ the relative interior and the vertex set of a polyhedron P. If $M \subset \mathbb{R}^n$ is a finite set and to each point $\mathbf{x} \in M$ a positive mass $\nu_{\mathbf{x}}$ is assigned, then for each subset $M' \subseteq M$ of cardinality $\sharp(M')$ we will denote by $c(M')$ the point $(\sum_{\mathbf{x}\in M'} \nu_{\mathbf{x}}\mathbf{x})/\sharp(M')$, i.e. the center of mass of the set M'.

Lemma 4.7. *Let P be a convex $(n-1)$–dimensional polyhedron with arbitrary positive masses assigned to its vertices. Let $\mathfrak{T}$ be an arbitrary partition of the relative boundary of P into (closed) simplexes with vertices from* $\operatorname{ext} P$. *Then,*

$$\operatorname{int} P = \bigcup_{\Delta\in\mathfrak{T}} \operatorname{int}\left(\operatorname{conv}(\Delta \cup \{c(\operatorname{ext} P\backslash \operatorname{ext}\Delta)\})\right).$$

Proof. Let $\mathbf{x} \in \operatorname{int} P$. Then there obviously exists a simplex $\Delta \in \mathfrak{T}$, such that $\mathbf{x} \in \operatorname{conv}(\Delta \cup \{c(P)\})$. It remains to notice that the set $\operatorname{conv}(\Delta \cup \{c(P)\}) \cap \operatorname{int} P$ is contained in $\operatorname{int}\left(\operatorname{conv}(\Delta \cup \{c(\operatorname{ext} P\backslash \operatorname{ext}\Delta)\})\right)$. □

Proof of Lemma 4.5. The action of the group of diagonal $n \times n$ matrices with determinant 1 obviously preserves the combinatorial structure of sails equipped with determinants of facets and edge stars. Hence we may consider the point $\mathbf{v}$ to have equal coordinates $v_1 = \ldots = v_n$. Suppose $\mathbf{v}$ is incident to m edges of the sail Π. Let $\mathbf{r}_1,\dots,\mathbf{r}_m$ be the primitive vectors of the lattice Λ generating these edges. Let us consider arbitrary positive numbers $k_1,\dots,k_m$ such that the vectors $\mathbf{r}_i' = k_i\mathbf{r}_i$ belong to a same hyperplane and denote $P = \operatorname{conv}(\mathbf{r}_1',\dots,\mathbf{r}_m')$. Consider also an arbitrary point $\lambda\mathbf{v} \in \operatorname{int} P$.

Assign masses k_i^{-1} to the points r_i'. Then, by Lemma 4.7, we can renumber the vectors $\mathbf{r}_1,\dots,\mathbf{r}_m$ (renumbering accordingly the numbers $k_1,\dots,k_m$ and the vectors $\mathbf{r}_1',\dots,\mathbf{r}_m'$) so that $\lambda\mathbf{v} = \lambda_0'\mathbf{r}_0' + \cdots + \lambda_{n-1}'\mathbf{r}_{n-1}'$, where the λ_i' are strictly positive and $\mathbf{r}_0' = (\mathbf{r}_n + \cdots + \mathbf{r}_m)/(m-n+1)$. We set $\mathbf{r}_0 = \mathbf{r}_0'(m-n+1)$, $\lambda_0 = \lambda_0'/(m-n+1)$ and $\lambda_i = k_i\lambda_i'$ for $i = 1,\dots,n-1$ and we get that $\lambda\mathbf{v} = \lambda_0\mathbf{r}_0 + \cdots + \lambda_{n-1}\mathbf{r}_{n-1}$ with strictly positive λ_i.

Clearly, $F_{\mathbf{v}}$ is contained in the $(n-1)$–dimensional polyhedron $F_{\mathbf{r}}$ defined as

$$F_{\mathbf{r}} = \Big\{\mathbf{x} \in \mathbb{R}^n \;\Big|\; \langle \mathbf{x}, \mathbf{v}\rangle = 1 \text{ and } \langle \mathbf{x}, \mathbf{v} + \sum_{i=0}^{n-1} \varkappa_i \mathbf{r}_i\rangle \geqslant 1 \\ \text{for all } \varkappa_0 \geqslant 0, \ldots, \varkappa_{n-1} \geqslant 0\Big\}.$$

Since the vectors $\mathbf{r}_0, \ldots, \mathbf{r}_{n-1}$ are linearly independent and all of the coefficients λ_i are positive, by Lemma 4.6,

$$\det F_{\mathbf{r}} = \frac{|\det(\mathbf{r}_0, \ldots, \mathbf{r}_{n-1})|^{n-1}}{\prod_{i=0}^{n-1} |\det(\mathbf{v}, \{\mathbf{r}_j\}_{j=0}^{n-1} \backslash \{\mathbf{r}_i\})|}\,.$$

All the factors in the denominator are nonzero integers, so, applying the inclusion $F_{\mathbf{v}} \subseteq F_{\mathbf{r}}$, we obtain the required estimate. □

5. Uniform boundedness of determinants of a sail's facets

In this section are given some facts concerning the sails that enjoy the property that the determinants of their facets are uniformly bounded (by a constant depending only on sail). We will make use of them in the subsequent sections.

As before, we denote $\varphi(\mathbf{x}) = x_1 \ldots x_n$. We also denote by $S(F)$ the intersection of the affine hull of a sail's facet F and the positive orthant. We will need a value characterizing the volume of the convex hull of $S(F)$ and the origin $\mathbf{0}$. It is convenient for this purpose to consider the natural extension of Definition 5 (given only for facets of a sail) to the case of arbitrary convex $(n-1)$–dimensional polyhedra and consider the value $\det S(F)$, which in this case is obviously equal to the volume of $\operatorname{conv}(S(F) \cup \{\mathbf{0}\})$ multiplied by $n!$.

In [4] the following is proved:

Theorem 5.1. *Suppose that the boundary of the positive orthant contains no points of a lattice Λ except the origin $\mathbf{0}$. Suppose also that the determinants of all the facets of the sail Π generated by Λ and related to the positive orthant are bounded by a constant D. Then there exists a constant D' depending only on n and D such that*
(a) $\det S(F) \leqslant D'$ *for each facet F of the sail Π;*
(b) $\varphi(\mathbf{v}) \geqslant (D')^{-1}$ *for each vertex $\mathbf{v}$ of K^*.*

Lemma 5.1. *If the determinants of all the facets of a sail Π are bounded by a constant D, then there exists a constant D' depending only on n and D, such that $\varphi(\mathbf{x}) < D'$ for each point $\mathbf{x} \in \Pi$.*

Proof. It is enough to consider a facet F of the sail Π containing a point $\mathbf{x} \in \Pi$, note that $\varphi(\mathbf{x}) < \det S(F)$ and apply Theorem 5.1. □

Lemma 5.2. *Let the determinants of the facets of a sail* Π *be bounded by a constant* D. *Let* $\mathbf{v}$ *be a vertex of* Π *with* $v_1 = \ldots = v_n$ *and let* $\varphi(\mathbf{v}) \geqslant \mu > 0$. *Then, the (Euclidean) lengths of all the edges incident to the vertex* $\mathbf{v}$ *are bounded by a constant* D_{vert} *depending only on* n, D *and* μ.

Proof. Due to Theorem 5.1, there exists a constant D' depending only on n and D, such that $\det S(F) \leqslant D'$ for each facet F of the sail Π.

On the other hand, $v_1 = \ldots = v_n \geqslant \mu^{1/n}$. Hence there exists a constant $D_{\mathrm{vert}} = D_{\mathrm{vert}}(n, D', \mu)$ such that if an edge incident to the vertex $\mathbf{v}$ has length larger than D_{vert}, then for each facet F incident to this edge $\det S(F) > D'$.

Therefore the lengths of all the edges incident to $\mathbf{v}$ should not exceed D_{vert}. □

The following lemma is an obvious corollary of Statement 4.1 and Definitions 5 and 6.

Lemma 5.3. *If the facets and the edge stars of the vertices of a sail* Π *have determinants bounded by a constant* D, *then there exists a constant* D' *depending only on* n *and* D *such that*
(a) each face of K° *has not more than* D' *vertices;*
(b) the number of facets of K° *incident to a vertex of* K° *is bounded by* D'.

6. Boundedness away from zero of the form $\varphi(\mathbf{x})$ in the positive orthant

As before, we suppose that the lattice Λ is irrational.

Lemma 6.1. *If the facets and the edge stars of the vertices of a sail* Π *have determinants bounded by a constant* D, *then there exists a constant* $\mu > 0$ *depending only on* n *and* D *for which*

$$\inf_{\mathbf{v}}(\varphi(\mathbf{v})) \geqslant \mu,$$

where the infimum is taken over all vertices of the sail Π.

Proof. It is easy to show that, if the boundary of the positive orthant contains nonzero points of the lattice Λ^*, then the sail Π has an unbounded facet (see, e.g., [4]). But all facets of Π have bounded determinants, hence, there are no such points. Thus, by Statement 4.2, $K^* \subseteq K^\circ$. On the other hand, the integer distances from facets of K to $\mathbf{0}$ do not exceed D, hence, by Lemma 4.4, all vertices of K° lie in the lattice $(D!)^{-1}\Lambda^*$. Applying the Corollary of Statement 4.1, we get that $D! \cdot K^\circ \subseteq K^* \subseteq K^\circ$.

In virtue of Lemma 4.5, the determinants of facets of K° are bounded by D^{n-1} and, thus, the determinants of facets of $D! \cdot K^\circ$ are bounded by $D^{n-1}(D!)^n$. Let us prove the existence of a constant D' depending only on n and D that bounds the determinants of facets of K^*. Due to the inclusion $D!\cdot K^\circ\subseteq K^*$, it suffices to show that the number of the facets of K° cut off by an arbitrary supporting hyperplane of $D!\cdot K^\circ$ (including those that have nonempty intersection with this hyperplane) is bounded by a constant, which depends only on n and D. Moreover, due to Lemma 5.3, it suffices to consider only hyperplanes that are the affine hulls of facets of $D!\cdot K^\circ$. Let F be a facet of $D!\cdot K^\circ$ and let $\mathrm{aff}(F)$ denote its affine hull. Obviously, the plane $\mathrm{aff}(F)$ contains an $(n-1)$–dimensional sublattice of Λ^*, hence, the lattice Λ^*, as well as the lattice $(D!)^{-1}\Lambda^*$, can be split into $(n-1)$–dimensional layers parallel to $\mathrm{aff}(F)$. Consider now a facet G of K° such that the combinatorial distance between $(D!)^{-1}F$ and G equals k (here we call two different facets neighboring and we define the combinatorial distance between them to equal 1, if they have at least one common point). It follows from the fact that all vertices of K° belong to $(D!)^{-1}\Lambda^*$ that there are at least $k-2$ layers of the lattice $(D!)^{-1}\Lambda^*$ parallel to $\mathrm{aff}(F)$ such that their affine hulls strictly separate the facet G from the facet $(D!)^{-1}F$. But since $\det F\leqslant D^{n-1}(D!)^n$ and $\det\Lambda^*=1$, the number of layers of the lattice $(D!)^{-1}\Lambda^*$ between $(D!)^{-1}\mathrm{aff}(F)$ and $\mathrm{aff}(F)$ is less than $D^{n-1}(D!)^{n+1}$. Therefore, applying Lemma 5.3, we get that the number of facets of K° cut off by $\mathrm{aff}(F)$ is indeed bounded by a constant, which depends only on n and D.

Thus, all the facets of K^* have determinants bounded by a constant D' depending only on n and D. Hence, by Theorem 5.1, there exists a constant D'', which also depends only on n and D, such that $\varphi(\mathbf{v})\geqslant(D'')^{-1}$ for each vertex $\mathbf{v}$ of $(K^*)^*=K$. It remains to set $\mu=(D'')^{-1}$. □

7. The logarithmic plane

Let us denote the positive orthant by $\mathcal{O}_+$. Consider the two mappings:

$$\begin{aligned}&\pi_1:\mathcal{O}_+\to\{\mathbf{x}\in\mathcal{O}_+\mid\varphi(\mathbf{x})=1\},\\&\pi_1(\mathbf{x})=\mathbf{x}\cdot(\varphi(\mathbf{x}))^{-1/n}\end{aligned}$$

and

$$\begin{aligned}&\pi_2:\{\mathbf{x}\in\mathcal{O}_+\mid\varphi(\mathbf{x})=1\}\to\mathbb{R}^{n-1},\\&\pi_2(\mathbf{x})=(\ln(x_1),\dots,\ln(x_{n-1}))\end{aligned}$$

and their composition

$$\pi_{\log}=\pi_2\circ\pi_1.$$

The image of a sail Π under the mapping $\pi_{\log}$ generates a partitioning of $\mathbb{R}^{n-1}$ into cells being curvilinear polyhedra. Each cell is the image of some facet of the sail. Accordingly, each vertex of the partitioning is the image of some vertex of the sail.

We denote this partitioning by $\mathfrak{P}$.

Lemma 7.1. *Suppose that the determinants of the facets of a sail* Π *are bounded by* D *and there exists a constant* $\mu > 0$ *such that, for each vertex* $\mathbf{v}$ *of the sail* Π,

$$\varphi(\mathbf{v}) \geqslant \mu.$$

Then there exists a constant D' *depending only on* n, D *and* μ, *such that each ball* $B \subset \pi_{\log}(\Pi)$ *of radius* D' *contains a cell of the partitioning* $\mathfrak{P}$.

Proof. Consider an arbitrary vertex $\mathbf{v}$ of the sail Π. Applying an appropriate hyperbolic rotation we can assume without loss of generality that $v_1 = \ldots = v_n \geqslant \mu^{1/n}$. Then by Lemma 5.2, the lengths of all the edges incident to $\mathbf{v}$ are bounded by some constant $D_{\mathrm{vert}} = D_{\mathrm{vert}}(n, D, \mu)$. Besides each facet of the sail has not more than $D_1 = D_1(n, D)$ vertices. Therefore there exists such a constant $D_2 = D_2(n, D, \mu)$, that all the facets incident to the vertex $\mathbf{v}$ are contained in a cube of sidelength D_2. But the values of the form φ in all the points of the sail are not less than μ. Hence, there exists a constant $D_3 = D_3(n, D, \mu)$ such that the images under the mapping $\pi_{\log}$ of all the facets incident to the vertex $\mathbf{v}$ are contained in a ball of radius D_3. Thus, for each facet F of the sail Π the cell $\pi_{\log}(F)$ is contained in a ball of radius D_3.

If we now consider an arbitrary ball $B \subset \pi_{\log}(\Pi)$ of radius $D' = 2D_3$, then the cell containing the center of B is contained in a ball of radius D_3, which, in its turn, is contained in B. $\square$

8. Proof of Theorem 2.1

As before, we denote by $S(F)$ the intersection of the affine hull of a sail facet F and the positive orthant.

1. The implication (1) & (2) $\Longrightarrow$ (3) has a rather simple proof. Consider an arbitrary vertex $\mathbf{v}$ of the sail Π. Applying an appropriate hyperbolic rotation we can assume without loss of generality that $v_1 = \ldots = v_n \geqslant \mu^{1/n}$. Then, by Lemma 5.2, the lengths of all edges incident to $\mathbf{v}$ are bounded by a constant $D_{\mathrm{vert}} = D_{\mathrm{vert}}(n, \mu)$. This implies that the number of edges incident to $\mathbf{v}$ does not exceed some constant $m_{\mathrm{vert}} = m_{\mathrm{vert}}(n, \mu)$, because otherwise a ball of radius D_{vert} contains too many lattice points. Now, it is obvious that $\det \mathrm{St}_{\mathbf{v}} \leqslant D' = D'(D_{\mathrm{vert}}, m_{\mathrm{vert}}) = D'(n, \mu)$.

2. The proof of the implication (3) $\Longrightarrow$ (2) is a bit more difficult. We assume that the facets and the edge stars of the vertices of the sail Π have determinants bounded by a constant D. By Lemma 6.1, there exists a constant $\mu = \mu(n, D) > 0$ such that, for each vertex $\mathbf{v}$ of the sail Π, we have $\varphi(\mathbf{v}) \geqslant \mu$, i.e. the conditions of Lemma 7.1 are satisfied.

Consider an arbitrary orthant $\mathcal{O}$ different from $\mathcal{O}_+$ and $-\mathcal{O}_+$ and an arbitrary facet F of the sail corresponding to this orthant. Applying an appropriate hyperbolic rotation, we can assume that the facet F is orthogonal to the bisector line of the orthant $\mathcal{O}$. We set

$$Q(T) = \{\mathbf{x} \in \mathbb{R}^n \mid \max(|x_1|, \dots, |x_n|) < T\},$$

$$Q_+(T) = \{\mathbf{x} \in Q(T) \mid x_i > 0,\ i = 1, \dots, n\}$$

and

$$T_0 = n^{-1/2} (\det F)^{1/n}.$$

Clearly, $Q(T_0) \cap \mathcal{O} \cap \Lambda = \{\mathbf{0}\}$.

By virtue of Lemmas 5.1 and 7.1, there exists a constant $T_1 = T_1(n, D)$ such that the set $\pi_{\log}(\Pi \cap Q_+(\sqrt{T_1}\,))$ contains a cell of the partitioning $\mathfrak{P}$ and, hence, a vertex of this partitioning. This means that $Q_+(\sqrt{T}\,)$ contains a vertex $\mathbf{v}$ of the sail Π for any $T \geqslant T_1$. Consider the parallelepiped

$$P(\mathbf{v}, T) = Q_+(T) \cap (\mathbf{v} + \mathcal{O})$$

for $T \geqslant T_1$. Lemmas 5.1, and 7.1 imply the existence of a constant $T_2 \geqslant T_1$, which also depends only on n and D, such that the set $\pi_{\log}(\Pi \cap P(\mathbf{v}, T_2))$ contains a cell and, hence, a vertex of the partitioning $\mathfrak{P}$. This means that $P(\mathbf{v}, T)$ contains a vertex of the sail Π different from $\mathbf{v}$ for any $T \geqslant T_2$.

But the parallelepiped $P(\mathbf{v}, T_0) - \mathbf{v}$ is contained in the parallelepiped $Q(T_0) \cap \mathcal{O}$ and $Q(T_0) \cap \mathcal{O} \cap \Lambda = \{\mathbf{0}\}$. Hence, $T_0 < T_2$, which means that $\det F$ is bounded by a constant depending only on n and D. $\square$

References

[1] O. N. German, *Klein polyhedra and norm minima of lattices*. Doklady Mathematics **406**:3 (2006), 38–41.

[2] P. Erdös, P. Gruber, J. Hammer, *Lattice Points*. Pitman Monographs and Surveys in Pure and Applied Mathematics **39**. Longman Scientific & Technical, Harlow (1989).

[3] F. Klein, *Uber eine geometrische Auffassung der gewohnlichen Kettenbruchentwichlung*. Nachr. Ges. Wiss. Gottingen **3** (1895), 357–359.

[4] O. N. German, *Sails and norm minima of lattices*. Mat. Sb. **196**:3 (2005), 31–60; English transl., Russian Acad. Sci. Sb. Math. **196**:3 (2005), 337–367.

[5] J.–O. Moussafir, *Convex hulls of integral points*. Zapiski nauch. sem. POMI **256** (2000).

[6] V. I. Arnold, *Continued fractions*. Moscow: Moscow Center of Continuous Mathematical Education (2002).

[7] V. I. Arnold, *Preface*. Amer. Math. Soc. Transl. **197**:2 (1999), ix–xii.

[8] E. I. Korkina, *Two–dimensional continued fractions. The simplest examples*. Proc. Steklov Math. Inst. RAS **209** (1995), 143–166.

[9] T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper*. Berlin: Springer (1934).

[10] B. Grünbaum, *Convex polytopes*. London, New York, Sydney: Interscience Publ. (1967).

[11] P. McMullen, G. C. Shephard, *Convex polytopes and the upper bound conjecture*. Cambridge (GB): Cambridge University Press (1971).

[12] G. Ewald, *Combinatorial convexity and algebraic geometry*. Sringer–Verlag New York, Inc. (1996).

[13] Z. I. Borevich, I. R. Shafarevich, *Number theory*. NY Academic Press (1966).

[14] J. W. S. Cassels, H. P. F. Swinnerton–Dyer, *On the product of three homogeneous linear forms and indefinite ternary quadratic forms*. Phil. Trans. Royal Soc. London **A 248** (1955), 73–96.

[15] B. F. Skubenko, *Minima of a decomposable cubic form of three variables.* Zapiski nauch. sem. LOMI **168** (1988).
[16] B. F. Skubenko, *Minima of decomposable forms of degree n of n variables for $n \geqslant 3$.* Zapiski nauch. sem. LOMI **183** (1990).
[17] G. Lachaud, *Voiles et Polyèdres de Klein.* Act. Sci. Ind., Hermann (2002).
[18] L. Danzer, B. Grünbaum, V. Klee, *Helly's Theorem and its relatives.* in Convexity (Proc. Symp. Pure Math. 7) 101–180, AMS, Providence, Rhode Island, 1963.

Oleg N. German
Moscow State University
Vorobiovy Gory, GSP–2
119992 Moscow, RUSSIA
E-mail: german@mech.math.msu.su