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A class–field theoretical calculation

par CRISTIAN D. POPESCU

RÉSUMÉ. Dans cet article, nous donnons la caractérisation complète des sous-groupes de p -torsion de certains groupes de classes d'idèles associés à des corps de fonctions de caractéristique p . Nous utilisons ce résultat pour répondre à une question qui a surgi dans le contexte de l'approche employée par Tan [6] pour résoudre un important cas particulier d'une généralisation d'une conjecture de Gross [4] sur des valeurs spéciales des fonctions L .

ABSTRACT. In this paper, we give the complete characterization of the p -torsion subgroups of certain idèle–class groups associated to characteristic p function fields. As an application, we answer a question which arose in the context of Tan's approach [6] to an important particular case of a generalization of a conjecture of Gross [4] on special values of L -functions.

1. Notation and motivation

Let p be a prime number. As usual, the term *characteristic p function field* (equivalently, characteristic p global field) refers to a finite extension of a field $\mathbb{F}_p(T)$, where \mathbb{F}_p is the finite field of p elements and T is a variable. Let K/k be a finite abelian extension of characteristic p function fields, of Galois group $\Gamma := \text{Gal}(K/k)$. Let S be a finite, nonempty set of primes in k , containing all the primes which ramify in K/k . We denote by S_K the set of primes in K dividing primes in S . Let $K_S^{\text{ab},p}$ be the maximal pro- p abelian extension of K , unramified outside S_K . Since S_K is Γ -invariant, $K_S^{\text{ab},p}/k$ is a Galois extension. Let $G := \text{Gal}(K_S^{\text{ab},p}/k)$, and $H := \text{Gal}(K_S^{\text{ab},p}/K)$. As usual, the group Γ acts by lift–and–conjugation on H . More precisely, $\gamma * h := \tilde{\gamma}h\tilde{\gamma}^{-1}$, for all $h \in H$ and $\gamma \in \Gamma$, where $\tilde{\gamma}$ denotes any lift of γ to G with respect to the usual epimorphism $G \rightarrow \Gamma$. This way, since H is an abelian group, H is endowed with a natural $\mathbb{Z}[\Gamma]$ -module structure.

In what follows, we denote by I_Γ the augmentation ideal in the group ring $\mathbb{Z}[\Gamma]$, i.e. the ideal of $\mathbb{Z}[\Gamma]$ generated by $\{\gamma - 1 \mid \gamma \in \Gamma\}$. Let $[G, G]$ denote

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the commutator subgroup of G , generated by the commutators $[x, y] = xyx^{-1}y^{-1}$ of all the elements $x, y \in G$. Since, by definition $(\gamma - 1) * h = \tilde{\gamma}h\tilde{\gamma}^{-1}h^{-1} = [\tilde{\gamma}, h]$, for all $\gamma \in \Gamma$ and $h \in H$, we have an inclusion

$$I_\Gamma \cdot H \subseteq [G, G].$$

In [6], the following question arises in the context of Tan’s approach to an important particular case (the so-called “ p -primary part in characteristic p ”-case) of a generalization of a conjecture of Gross [4].

Question. Under what conditions do we have an equality

$$[G, G] = I_\Gamma \cdot H ?$$

In §§2–3 below, we use class-field theory to show that the answer to the Question above depends on the p -torsion subgroup of a certain idèle-class group associated to K . In §3, we use class-field theory and Galois cohomology to calculate this p -torsion subgroup explicitly. Based on this calculation, in §4 we settle the Question stated above. In §5, we give a sufficient condition for the equality $[G, G] = I_\Gamma \cdot H$ to hold true, for abstract groups G, H , and Γ , not necessarily arising in the number-theoretical context described above.

2. Group theoretical considerations

Throughout this section, H, G , and Γ are arbitrary abstract groups, fitting into a short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1,$$

with H and Γ abelian. As above, Γ is viewed as acting on H via lift-and-conjugation, and this action endows H with a $\mathbb{Z}[\Gamma]$ -module structure.

Proposition 2.1. *If $H/I_\Gamma \cdot H$ has no torsion, then one has an equality*

$$[G, G] = I_\Gamma \cdot H.$$

Proof. We have a short exact sequence of groups

$$(2.1) \quad 1 \longrightarrow H/I_\Gamma \cdot H \longrightarrow G/I_\Gamma \cdot H \longrightarrow \Gamma \longrightarrow 1.$$

We will need the following lemma, which was suggested to the author by Tan [7].

Lemma 2.1. *Let $1 \longrightarrow \overline{H} \longrightarrow \overline{G} \xrightarrow{\pi} \overline{\Gamma} \longrightarrow 1$ be an exact sequence of groups, such that*

- (1) $\overline{\Gamma}$ is a torsion abelian group.
- (2) \overline{H} is a non-torsion abelian group.
- (3) $\overline{\Gamma}$ acts trivially on \overline{H} (via the usual lift-and-conjugation action).

Then, \overline{G} is abelian.

Proof of Lemma 2.1. Hypothesis (3) above implies right away that $\overline{H} \subseteq \mathcal{Z}(\overline{G})$, where $\mathcal{Z}(\overline{G})$ denotes the center of \overline{G} . Let $a, b \in \overline{G}$. We will show that $[a, b] = 1$. Since $\overline{\Gamma}$ is abelian, $\pi([x, y]) = 1$ and therefore $[x, y] \in \overline{H}$, for all $x, y \in \overline{G}$. This shows that, if we denote by B the subgroup of G generated by b , then we have a function

$$f_a : B \longrightarrow \overline{H},$$

defined by $f_a(x) = [a, x]$, for all $x \in B$. We claim that f_a is a group morphism. Indeed, if $x, y \in B$ we have the equalities

$$\begin{aligned} f_a(x) \cdot f_a(y) &= (axa^{-1}x^{-1})(aya^{-1})y^{-1} = (aya^{-1})(axa^{-1}x^{-1})y^{-1} = \\ &= ayxa^{-1}x^{-1}y^{-1} = f_a(yx) = f_a(xy), \end{aligned}$$

which prove our claim. The second equality above is a consequence of $[a, x] \in \overline{H}$ and $\overline{H} \subseteq \mathcal{Z}(\overline{G})$. The last equality follows from the fact that B is abelian. Since $\overline{H} \subseteq \mathcal{Z}(\overline{G})$, we have $B \cap \overline{H} \subseteq \ker(f_a)$. Therefore, the image $\mathfrak{S}(f_a)$ of f_a is isomorphic to a quotient of $B/B \cap \overline{H}$. Since $B/B \cap \overline{H}$ is isomorphic to a subgroup of $\overline{\Gamma}$, hypothesis (2) shows that $\mathfrak{S}(f_a)$ is a torsion subgroup \overline{H} . Hypothesis (1) implies that $\mathfrak{S}(f_a)$ is a torsion subgroup of \overline{H} , while hypothesis (2) implies that $\mathfrak{S}(f_a)$ is trivial. Therefore $f_a(b) = [a, b] = 1$. This concludes the proof of Lemma 2.1. \square

An alternative proof for Lemma 2.1. In what follows, we give a second, homological in nature and very enlightening proof for Lemma 2.1, suggested to us by the referee. Let $\overline{V} := \overline{H} \otimes_{\mathbb{Z}} \mathbb{Q}$. Hypothesis (1) implies that we have an exact sequence of $\mathbb{Z}[\overline{\Gamma}]$ -modules

$$0 \longrightarrow \overline{H} \longrightarrow \overline{V} \longrightarrow \overline{V}/\overline{H} \longrightarrow 0,$$

with $\overline{\Gamma}$ acting trivially on each term (see hypothesis (3)). Hypotheses (3) and (1) imply that

$$H^1(\overline{\Gamma}, \overline{V}) = \text{Hom}(\overline{\Gamma}, \overline{V}) = 0, \quad H^1(\overline{\Gamma}, \overline{H}) = \text{Hom}(\overline{\Gamma}, \overline{H}) = 0.$$

Consequently, the usual coboundary maps in the long-exact sequence of $\overline{\Gamma}$ -cohomology groups associated to the last short exact sequence induce a group isomorphism

$$H^1(\overline{\Gamma}, \overline{V}/\overline{H}) \xrightarrow{\sim} H^2(\overline{\Gamma}, \overline{H}).$$

Let $\mathfrak{h} \in H^1(\overline{\Gamma}, \overline{V}/\overline{H}) = \text{Hom}(\overline{\Gamma}, \overline{V}/\overline{H})$ correspond to the extension class of \overline{G} in $H^2(\overline{\Gamma}, \overline{H})$ via this coboundary isomorphism. Then \overline{G} is isomorphic to the pull-back of $\overline{V} \longrightarrow \overline{V}/\overline{H}$ along $\mathfrak{h} : \overline{\Gamma} \longrightarrow \overline{V}/\overline{H}$. Since $\overline{\Gamma}$ and \overline{V} are abelian, so is \overline{G} . \square

Now, we return to the proof of Proposition 2.1. The exact sequence (2.1) satisfies properties (1), (2), and (3) in Lemma 2.1, and therefore $G/I_\Gamma \cdot H$ is abelian. This shows that

$$[G, G] \subseteq I_\Gamma \cdot H.$$

As the reverse inclusion is obviously true, we obtain the desired equality

$$[G, G] = I_\Gamma \cdot H.$$

This concludes the proof of Proposition 2.1. □

If A is an (additive) abelian group, \widehat{A} denotes the pro- p completion of A , namely

$$\widehat{A} := \varprojlim A/p^n A,$$

where \varprojlim denotes the usual projective limit with respect to the canonical surjections $A/p^{n+1}A \rightarrow A/p^n A$. Also, $A[p^\infty]$ will denote the subgroup of A consisting of all its p -power order elements. In what follows, we refer to $A[p^\infty]$ as the p -torsion subgroup of A . We have the following.

Lemma 2.2. *Let A be an abelian group. Then, the following hold true.*

- (1) *The inclusion $A[p^\infty] \subseteq A$ induces a group isomorphism*

$$\widehat{A[p^\infty]}[p^\infty] \xrightarrow{\sim} \widehat{A}[p^\infty].$$

- (2) *If A has a trivial p -torsion subgroup, then \widehat{A} has a trivial p -torsion subgroup.*

Proof. First, we prove (2). Let $\alpha = (\widehat{a}_n)_n \in \widehat{A}$, where $a_n \in A$ and \widehat{a}_n is the class of a_n in $A/p^n A$. By definition, we have $\pi_n(\widehat{a}_n) = \widehat{a_{n-1}}$, where $\pi_n : A/p^n A \rightarrow A/p^{n-1} A$ is the canonical projection, for all n . Assume that $p \cdot \alpha = 0$ in \widehat{A} . This implies that $pa_n \in p^n A$, for all $n \geq 1$. Since A has no p -torsion, this implies that $a_n \in p^{n-1} A$, for all $n \geq 1$, which shows that $\widehat{a_{n-1}} = \widehat{0}$ in $A/p^{n-1} A$, for all $n \geq 2$. This implies that $\alpha = 0$ in \widehat{A} .

Next, we prove (1). Let $B := A/A[p^\infty]$. We have an exact sequence of groups.

$$0 \rightarrow A[p^\infty] \rightarrow A \rightarrow B \rightarrow 0.$$

We have an obvious equality $p^n A \cap A[p^\infty] = p^n A[p^\infty]$, for all n . This implies that, for every n , we obtain an exact sequence of groups.

$$0 \rightarrow A[p^\infty]/p^n A[p^\infty] \rightarrow A/p^n A \rightarrow B/p^n B \rightarrow 0.$$

Since the canonical projections are surjective, the projective limit of these exact sequences with respect to the canonical projections leads to the exact sequence

$$0 \rightarrow \widehat{A[p^\infty]} \rightarrow \widehat{A} \rightarrow \widehat{B} \rightarrow 0.$$

By definition, B has a trivial p -torsion subgroup. Statement (2) in the Lemma implies that \widehat{B} has a trivial p -torsion subgroup as well. Consequently, the last exact sequence leads to the desired isomorphism $\widehat{A[p^\infty]}[p^\infty] \xrightarrow{\sim} \widehat{A}[p^\infty]$. \square

3. Class-field theoretical considerations

Now, we return to the notations and definitions in §1. Proposition 2.1 above shows that, since H is a pro- p group, in order to settle the Question stated above, it is important to characterize the p -torsion subgroup of $H/I_\Gamma \cdot H$. In particular, if one shows that $H/I_\Gamma \cdot H$ has no p -torsion then Proposition 2.1 above leads to the desired equality $[G, G] = I_\Gamma \cdot H$. In the current section, we use class-field theory to identify $H/I_\Gamma H$ with the pro- p completion of an idèle-class group of K and fully describe its torsion subgroup. As usual, let C_K denote the idèle-class group of K , i.e.

$$C_K := J_K/K^\times,$$

where J_K is the group of idèles of K . For the definitions and properties of J_K and C_K , as well as class-field theory in idèlic and Galois-cohomological language, the reader is referred to the classical texts [1] and [3].

If restricted to the context of characteristic p global fields, global class-field theory (see [1], Chpt. VIII, §3, or [3], Chpt. VII, §5.5) shows that the global Artin map induces a topological group isomorphism between the profinite completion of C_K and the Galois group $\text{Gal}(K^{\text{ab}}/K)$ of the maximal abelian extension K^{ab} of K . For every prime w of K , U_w denotes the group of w -local units of the completion K_w of K with respect to w . Let $\prod U_w$ denote the (closed) subgroup of J_K , consisting of all those idèles with local component 1 at all $w \in S_K$ and local component belonging to U_w , for all $w \notin S_K$. Since $S \neq \emptyset$, we have $K^\times \cap \prod U_w = \{1\}$, and therefore we can view $\prod U_w$ as a subgroup of C_K , by identifying it with its image via the injective group morphism $\prod U_w \rightarrow C_K$.

By global class-field theory and the definition of H , the global Artin map induces a topological group isomorphism of Γ -modules between the pro- p completion of $C_K/\prod U_w$ and H . Consequently, the global Artin map induces a topological group isomorphism between the pro- p completion of the quotient $C_K/(I_\Gamma \cdot C_K)\prod U_w$ and the group $H/I_\Gamma \cdot H$,

$$(3.1) \quad C_K/(I_\Gamma \cdot \widehat{C_K}) \prod U_w \xrightarrow{\sim} H/I_\Gamma \cdot H.$$

Our next goal is to prove the following theorem which gives a full description of the p -torsion of the group $C_K/(I_\Gamma \cdot C_K)\prod U_w$.

Theorem 3.1. *The p -torsion subgroup of $C_K/(I_\Gamma \cdot C_K)\prod U_w$ is isomorphic to $\wedge^2 \Gamma^{(p)}$, where $\Gamma^{(p)}$ is the p -Sylow subgroup of Γ .*

Before proceeding to proving Theorem 3.1, we need to make several Galois-cohomological considerations. For a Γ -module A and an $i \in \mathbb{Z}$, we denote by $\widehat{H}^i(\Gamma, A)$ the i -th Tate cohomology group of Γ with coefficients in A (see [3], Chpt. IV, §6 for the definitions). Also, $A[N_\Gamma]$ denotes the subgroup of A annihilated by the norm element $N_\Gamma \in \mathbb{Z}[\Gamma]$, where $N_\Gamma := \sum_{\gamma \in \Gamma} \gamma$. Obviously, $I_\Gamma \cdot A \subseteq A[N_\Gamma]$. By definition, $\widehat{H}^{-1}(\Gamma, A) = A[N_\Gamma]/I_\Gamma \cdot A$. Also, $\widehat{H}^0(\Gamma, A) = A^\Gamma/N_\Gamma \cdot A$, where A^Γ is the maximal subgroup of A fixed by Γ . We have an exact sequence of abelian groups.

$$(3.2) \quad 1 \longrightarrow \frac{C_K[N_\Gamma] \cdot \prod U_w}{(I_\Gamma \cdot C_K) \cdot \prod U_w} \longrightarrow \frac{C_K}{(I_\Gamma \cdot C_K) \prod U_w} \longrightarrow \frac{C_K}{C_K[N_\Gamma] \cdot \prod U_w} \longrightarrow 1.$$

However, since K/k is unramified away from S , Shapiro’s Lemma ([3], Prop. 2, p. 99) combined with the cohomological triviality of groups of units in unramified Galois extensions of local fields (see [3], Chpt. VII, §7) implies that

$$(3.3) \quad \widehat{H}^i(\Gamma, \prod U_w) = \prod_{v \notin S} \widehat{H}^i(\Gamma_v, U_{w(v)}) = 0, \quad \text{for all } i \in \mathbb{Z},$$

where Γ_v denotes the decomposition group of v in K/k and $w(v)$ is a fixed prime in K dividing v , for all $v \notin S$. Equality (3.3) for $i = 1$ gives the following.

$$I_\Gamma \cdot \prod U_w = (\prod U_w)[N_\Gamma].$$

This implies immediately that the left-most nonzero term of the exact sequence (3.2) is in fact isomorphic to $C_K[N_\Gamma]/I_\Gamma \cdot C_K = \widehat{H}^{-1}(\Gamma, C_K)$. Therefore, we obtain an exact sequence of groups.

$$(3.4) \quad 1 \longrightarrow \widehat{H}^{-1}(\Gamma, C_K) \longrightarrow C_K/(I_\Gamma \cdot C_K) \cdot \prod U_w \longrightarrow C_K/C_K[N_\Gamma] \cdot \prod U_w \longrightarrow 1.$$

Consequently, proving Theorem 3.1 amounts to studying the p -torsion subgroups of the two end-terms of exact sequence (3.4). This is accomplished by the next two lemmas.

Lemma 3.1. *The p -torsion subgroup of $\widehat{H}^{-1}(\Gamma, C_K)$ is isomorphic to $\wedge^2 \Gamma^{(p)}$.*

Proof. Global class-field theory (see [3], Chpt. VII, §11.3) gives an isomorphism

$$\widehat{H}^{-1}(\Gamma, C_K) \xrightarrow{\sim} \widehat{H}^{-3}(\Gamma, \mathbb{Z}).$$

On the other hand, one has an equality

$$\widehat{H}^{-3}(\Gamma, \mathbb{Z}) = H_2(\Gamma, \mathbb{Z}).$$

Theorem 6.4 (iii) in [2] gives an isomorphism

$$H_2(\Gamma, \mathbb{Z}) \xrightarrow{\sim} \wedge^2 \Gamma,$$

where the exterior product is taken over \mathbb{Z} . Taking p -torsion of both sides in the last equality concludes the proof of Lemma 3.1. \square

In order to describe the p -torsion subgroup of the right-most nonzero term of exact sequence (3.4), we need the following result, proved by Kisilevsky in [5] and shown to imply the classical Leopoldt Conjecture for characteristic p function fields.

Theorem 3.2 (Kisilevsky). *Let k be an arbitrary characteristic p function field. Let v a prime in k and k_v the completion of k with respect to v . If $x \in k$ is the p -power of an element in k_v , then x is the p -power of an element in k .*

Proof. See [5] \square

Lemma 3.2. *The p -torsion subgroup of $C_K/C_K[N_\Gamma] \cdot \prod U_w$ is trivial.*

Proof. Let us assume that $C_K/C_K[N_\Gamma] \cdot \prod U_w$ has p -torsion. Let $j \in J_K$ such that its class $\widehat{j} \in C_K$ gives rise to an element of order p in the quotient $C_K/C_K[N_\Gamma] \cdot \prod U_w$. This means that there exist $\rho \in J_K$, $u \in \prod U_w$, $x \in K^\times$, and $y \in k^\times$, such that

- (1) $j^p = \rho \cdot u \cdot x$ in J_K .
- (2) $N_\Gamma(\rho) = y$.

By taking norms in (1) above, we obtain

$$N_\Gamma(j)^p = N_\Gamma(u) \cdot yN_\Gamma(x).$$

However, since $S \neq \emptyset$, this implies right away that $y \cdot N_\Gamma(x)$ is a p -power locally, at primes in S . Theorem 3.2 above implies that there exists $z \in k^\times$, such that $y \cdot N_\Gamma(x) = z^p$. This shows that $N_\Gamma(j)^p = N_\Gamma(u) \cdot z^p$, and therefore $N_\Gamma(u) = \theta^p$, for some $\theta \in \prod U_w$, where the product is taken over all primes v of k which are not in S , and U_w denotes the unit group of the completion k_v of k at v . However, equality (3.3) for $i = 0$ implies that $\widehat{H}^0(\Gamma, \prod U_w) = 0$ and therefore

$$\prod U_w = (\prod U_w)^\Gamma = N_\Gamma(\prod U_w).$$

This shows that there exists $u' \in \prod U_w$ such that $\theta = N_\Gamma(u')$. This obviously implies that

$$N_\Gamma(j) = N_\Gamma(u') \cdot z, \text{ with } u' \in \prod U_w \text{ and } z \in k^\times.$$

The last equality shows that $\widehat{j/u'} \in C_K[N_\Gamma]$ and therefore $\widehat{j} \in C_K[N_\Gamma] \cdot \prod U_w$. This shows that the element $\widehat{j} \in C_K$ gives rise to the trivial class in the quotient $C_K/C_K[N_\Gamma] \cdot \prod U_w$. \square

Proof of Theorem 3.1. This is a direct consequence of exact sequence (3.4), Lemma 3.1 and Lemma 3.2 above. \square

The next corollary fully describes the torsion subgroup $T(H/I_\Gamma \cdot H)$ of $H/I_\Gamma \cdot H$.

Corollary 3.1. (1) *One has an isomorphism of groups*

$$T(H/I_\Gamma \cdot H) \xrightarrow{\sim} \wedge^2 \Gamma^{(p)}.$$

(2) *$H/I_\Gamma \cdot H$ has no torsion if and only if $\Gamma^{(p)}$ is cyclic.*

Proof. Let us first remark that since H is a pro- p group, its torsion subgroup and p -torsion subgroup are identical. With this in mind, statement (1) is a direct consequence of isomorphism (2), Theorem 3.1, Lemma 2.2 applied to the group $A := C_K/(I_\Gamma \cdot C_K) \cdot \prod U_w$, and the finiteness of Γ . Statement (2) is a direct consequence of (1) □

4. The answer to the Question stated in §1

We work under the hypotheses and with the notations of §§1 and 3. The following theorem provides a full answer to the Question raised in §1.

Theorem 4.1. *The following statements are equivalent*

- (1) $[G, G] = I_\Gamma \cdot H$.
- (2) *The p -Sylow subgroup $\Gamma^{(p)}$ of Γ is cyclic.*

Proof. The implication (2) \implies (1) is a direct consequence of Corollary 3.1 (2) and Proposition 2.1. Now, let us assume that $[G, G] = I_\Gamma \cdot H$. As in §1, let $k_S^{\text{ab},p}$ be the maximal pro- p abelian extension of k , unramified outside of S . Let L' be the maximal subfield of $K_S^{\text{ab},p}$ fixed by $[G, G]$. Then, under the present hypothesis, we have

$$(4.1) \quad \text{Gal}(L'/K) \xrightarrow{\sim} H/[G, G] = H/I_\Gamma \cdot H.$$

By definition, L' is the maximal subfield of $K_S^{\text{ab},p}$ which is an abelian extension of k . Obviously, we have an inclusion $k_S^{\text{ab},p} \subseteq L'$. Also, if we denote by K' the maximal subfield of K fixed by $\Gamma^{(p)}$, then K'/k and $k_S^{\text{ab},p}/k$ are linearly disjoint extensions of k and their compositum $K' \cdot k_S^{\text{ab},p}$ inside L' equals L' . Consequently, we have a group isomorphism

$$\text{Gal}(L'/k) \xrightarrow{\sim} \Gamma/\Gamma^{(p)} \times \text{Gal}(k_S^{\text{ab},p}/k).$$

The isomorphism above combined with (4.1) and the fact that H is a pro- p group shows that $H/I_\Gamma \cdot H$ is isomorphic to a subgroup of $\text{Gal}(k_S^{\text{ab},p}/k)$ (via the usual map restricting automorphisms of $K_S^{\text{ab},p}$ to automorphisms of $k_S^{\text{ab},p}$). This shows that if we prove that $\text{Gal}(k_S^{\text{ab},p}/k)$ has a trivial torsion subgroup, then $H/I_\Gamma \cdot H$ has a trivial torsion subgroup which, via Corollary 3.1(2) implies that, indeed, $\Gamma^{(p)}$ is cyclic. However, the global Artin

map for k induces an isomorphism

$$J_k/k^\times \cdot \widehat{\prod U_v} \xrightarrow{\sim} \text{Gal}(k_S^{\text{ab},p}/k).$$

Therefore, Lemma 2.2 implies that it suffices to show that the group $J_k/k^\times \cdot \prod U_v$ has no p -torsion. This follows immediately by applying once again Kisilevsky's Theorem 3.2. Indeed, assume that $j \in J_k$ has the property that $j^p = x \cdot u$, where $x \in k^\times$ and $u \in \prod U_v$. Since S is non-empty, this implies that x is a p -power in k_v^\times , for all $v \in S$. Theorem 3.2 implies that $x = y^p$, for some $y \in k^\times$. This implies that $u = \theta^p$, with $\theta = j/x$, $\theta \in \prod U_v$. Consequently, $j = y \cdot \theta \in k^\times \cdot \prod U_v$, and therefore the class \hat{j} of j in the quotient $J_k/k^\times \cdot \prod U_v$ is trivial. This concludes the proof of the implication (1) \implies (2). □

5. More group theory (final thoughts)

We conclude with a short purely group-theoretical section providing a sufficient condition for the equality $[G, G] = I_\Gamma \cdot H$ to hold true, where G , H , and Γ are abstract groups.

Lemma 5.1. *Let us assume that we have an exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1,$$

with H and Γ abelian and H normal in G . Let us assume that π has a set-theoretic section $s : \Gamma \longrightarrow G$, such that $s(x) \cdot s(y) = s(y) \cdot s(x)$, for all x, y in Γ . Then, we have an equality $[G, G] = I_\Gamma \cdot H$.

Proof. It suffices to show that $[G, G] \subseteq I_\Gamma \cdot H$. Let α, β be two elements in G . Let $x, y \in \Gamma$, and $a, b \in H$, such that $\alpha = s(x)a$ and $\beta = s(y)b$. Since $s(y)^{-1}s(x)^{-1} = s(x)^{-1}s(y)^{-1}$, we have

$$\begin{aligned} [\alpha, \beta] &= s(x)as(y)ba^{-1}s(x)^{-1}b^{-1}s(y)^{-1} = \{s(x)as(x)^{-1}\} \cdot \\ &\quad \cdot \{s(x)s(y)bs(y)^{-1}s(x)^{-1}\} \cdot \{s(y)s(x)a^{-1}s(x)^{-1}s(y)^{-1}\} \cdot \\ &\quad \cdot \{s(y)b^{-1}s(y)^{-1}\}. \end{aligned}$$

Let us denote by m, n, p , and q respectively the elements appearing inside braces to the right of the second equality above. Since H is normal in G , we have $m, n, p, q \in H$. Since H is assumed to be abelian and $a, b \in H$, the equalities above give

$$\begin{aligned} [\alpha, \beta] &= \{ma^{-1}\} \cdot \{nb^{-1}\} \cdot \{pa\} \cdot \{qb\} = \\ &= [s(x), a] \cdot [s(x)s(y), b] \cdot [s(y)s(x), a^{-1}] \cdot [s(y), b^{-1}]. \end{aligned}$$

Let us now recall that s is a section of π . Therefore, there exists an element $\mu \in H$, such that $s(x)s(y) = s(xy)\mu$. Since H is abelian, this implies that

$[s(x)s(y), b] = [s(xy)\mu, b] = [s(xy), b]$ and $[s(y)s(x), a^{-1}] = [s(yx), a^{-1}]$. We obtain

$$\begin{aligned} [\alpha, \beta] &= [s(x), a] \cdot [s(xy), b] \cdot [s(yx), a^{-1}] \cdot [s(y), b] \\ &= \{(x-1) * a\} \cdot \{(xy-1) * b\} \cdot \{(yx-1) * a^{-1}\} \cdot \{(y-1) * b^{-1}\}. \end{aligned}$$

This shows that $[\alpha, \beta] \in I_\Gamma \cdot H$, which concludes the proof of Lemma 5.1. \square

Corollary 5.1. *Assume that we have an exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1,$$

with H and Γ abelian and H normal in G . Assume that either (1) or (2) below hold.

- (1) Γ is cyclic.
- (2) The exact sequence above is split.

Then, we have an equality $[G, G] = I_\Gamma \cdot H$.

Proof. It is very easy to check that if either one of the conditions above is satisfied, one can construct a set-theoretic section s for π , such that $s(x)s(y) = s(y)s(x)$, for all $x, y \in \Gamma$. (Under condition (2), one can actually find a group morphism section s . Such a section satisfies the commutativity property automatically, since Γ is assumed to be abelian.) The corollary is then a consequence of Lemma 5.1. \square

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