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# Gross' conjecture for extensions ramified over four points of $\mathbb{P}^1$

par Po-Yi HUANG

Résumé. Dans le papier ci-après, avec une hypothése modérée, nous prouvons une conjecture de Gross pour l'élément Stickelberger de l'extension abelienne maximale sur le corps des fonctions rationnelles non ramifiée en dehors d'un ensemble des quatre places de degré 1.

Abstract. In this paper, under a mild hypothesis, we prove a conjecture of Gross for the Stickelberger element of the maximal abelian extension over the rational function field unramified outside a set of four degree-one places.

## 1. Introduction

Let $K$ be a global function field, $S$ be a non-empty finite set of places of $K$. Consider the $S$-zeta function

$$\zeta_S(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_S} (\mathbf{N}\mathfrak{a})^{-s},$$

where $\mathcal{O}_S$ is the ring of $S$-integers, and the summation ranges over all ideals $\mathfrak{a}$ in this ring. It is well-known that this series converges for $\Re(s) > 1$ and has a meromorphic continuation to the whole complex plane, with at most a simple pole at $s = 1$ and no other singularities. The leading term of its Taylor expansion at $s = 0$ has a good formula (*Class Number Formula*):

$$\zeta_S(s) = -\frac{h_S R_S}{\omega_S} s^n + O(s^{n+1}), \quad \text{as } s \to 0.$$

Here $h_S$ is the class number of $\mathcal{O}_S$, $R_S$ is the $S$-regulator, $\omega_S$ is the number of roots of unity in $\mathcal{O}_S$, and $n = \#S - 1$, the rank of the units group, $\mathcal{O}_S^*$ ([5]).

Gross' Conjecture is a generalization of the above class number formula. In order to state it, we need to modify the above setting. We will follow the notations used in [5]. First we fix a non-empty finite set $T$ of places of $K$ such that $T \cap S = \emptyset$. Let $U_{S,T}$ be the subgroup of $\mathcal{O}_S^*$ consisting of units

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congruent to 1 modulo $T$, which is known to be a free $\mathbb{Z}$-module ([5]). The modified zeta function is defined as

$$\zeta_{S,T}(s) = \zeta_S(s) \prod_{\mathfrak{q}\in T}(1 - (\mathbf{N}\mathfrak{q})^{1-s}).$$

Then $\zeta_{S,T}(s)$ is an entire function and the Taylor expansion at $s = 0$ becomes

$$\zeta_{S,T}(s) = (-1)^{\#T-1} \cdot \frac{h_{S,T}R_{S,T}}{\omega_{S,T}} s^n + O(s^{n+1}), \quad \text{as } s \to 0.$$

Here $h_{S,T}$ is the order of ray class group modulo $T$, $R_{S,T}$ is the regulator of $U_{S,T}$, and $\omega_{S,T}$ is the number of roots of unity in $U_{S,T}$, which, in our case, equals to 1 ([5]).

Similarly, for a finite abelian extension $L/K$ unramified outside $S$, with $G = \mathrm{Gal}(L/K)$, and for each character $\chi \in \widehat{G}$, the modified $L$-function is defined as ([5])

$$L_{S,T}(\chi, s) = L_S(\chi, s) \prod_{\mathfrak{q}\in T}(1 - \chi(\phi_\mathfrak{q})(\mathbf{N}\mathfrak{q})^{1-s}),$$

where

$$L_S(\chi, s) = \sum_{\mathfrak{a}\subset\mathcal{O}_S} \widehat{\chi}(\mathfrak{a})(\mathbf{N}\mathfrak{a})^{-s}$$

is the usual $L$-function. Here for a prime ideal $\mathfrak{p}$, $\widehat{\chi}(\mathfrak{p}) = \chi(\phi_\mathfrak{p})$ where $\phi_\mathfrak{p}$ denotes the Frobenius element at $\mathfrak{p}$, and for an integral ideal $\mathfrak{a}$, if $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$, then $\widehat{\chi}(\mathfrak{a}) = \prod \widehat{\chi}(\mathfrak{p}_i)^{n_i}$.

Now we introduce the Stickelberger element $\theta_{S,T}$. It is an element of $\mathbb{C}[G]$, with the property that

$$\chi(\theta_{S,T}) = L_{S,T}(\chi, 0), \quad \forall\ \chi \in \widehat{G}.$$

In our case, $\theta_{S,T} \in \mathbb{Z}[G]$([5], Proposition 3.7). In the group ring $\mathbb{Z}[G]$, the augmentation ideal $I$ is the kernel of the homomorphism

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \\ \sum_{g\in G} \alpha_g g & \mapsto & \sum_{g\in G} \alpha_g. \end{array}$$

In other words, $I$ is generated by $\{g-1 \mid g \in G\}$. Through the isomorphism $g \mapsto g-1$, we can identify $G$ with $I/I^2$ ([5]). Suppose that $S = \{v_0, \ldots, v_n\}$. For each place $v_i$, let $r_{v_i} : K_{v_i}^* \longrightarrow G_{v_i} \subset G \cong I/I^2$ be the local reciprocity map. We choose a basis $u_1, \ldots, u_n$ of $U_{S,T}$ such that the sign of the determinant $\det(v_i(u_j))_{1\le i,j\le n}$ is positive and define the Gross regulator $\det_G(\lambda_{S,T})$ as the residue class modulo $I^{n+1}$ of the determinant $\det(r_{v_i}(u_j) - 1)_{1\le i,j\le n}$ ([5]).

Since $L/K$ is unramified outside $S$, $r_v(u_j) = 1$ for $v \notin S$, the product formula says that the above definition is well-defined and is independent of the choice of the basis $u_1, ..., u_n$.

Gross' Conjecture ([5]) says that

$$\theta_{S,T} \equiv (-1)^{\#T-1} h_{S,T} \det_G(\lambda_{S,T}) \pmod{I^{n+1}}. \tag{1}$$

Stickelberger elements and Gross regulators enjoy functorial properties. Regarding to this aspect, we quote the following simple lemma ([11], Proposition 1.4).

**Lemma 1.1.** *Suppose that Gross' Conjecture holds for the extension $L/K$ with respect to $S$ and $T$. Let $L'/K$ be a subextension of $L/K$, $S \subset S'$ and $T \subset T'$. Then the following are true.*

(1) Gross' Conjecture holds for $L'/K$ with respect to $S$ and $T$.
(2) Gross' Conjecture holds for $L/K$ with respect to $S'$ and $T$.
(3) Gross' Conjecture holds for $L/K$ with respect to $S$ and $T'$.

There are already many evidences of Gross' Conjecture over function fields as well as number fields, for instance, [1], [2], [3], [4], [6], [8], [9], [10], [12], [13], in which various different methods are used. However, in this note we take another approach and follow the method in [11]. In [11] M. Reid proves the following theorem.

**Theorem 1.2.** *Let $K = \mathbb{F}_q(x)$, $L$ be any abelian extension of $K$ unramified outside $S$ which is a set of three degree-one places of $K$, $T$ be a set of places such that the greatest common divisor of their degrees is relatively prime to $q-1$. Then Gross' Conjecture holds.*

We generalize the above theorem to the following.

**Theorem 1.3.** *Let $K = \mathbb{F}_q(x)$, $L$ be any abelian extension of $K$ unramified outside $S$ which is a set of four degree-one places of $K$, $T$ be a set of places such that the greatest common divisor of their degrees is relatively prime to $q-1$. Then Gross' Conjecture holds.*

This main theorem is proved in Section 3.1, as a consequence of Theorem 3.2 which is a special case and whose proof will be given at the end of this paper. The proof is based on an expressing of the difference of both sides of the conjecture as certain polynomial which, by a series of computations, is shown to equal to a sum of several products. The polynomial will contains about 300 terms if these products are expanded as sums of monomials. We first use the software "Maple" to do the expansions of these products as well as the cancelations of monomials with opposite signs and reduce it to one with only 70 terms. Then we use some congruence relations to show that it is actually zero.

It seems that one can use a similar method to deal with the case where $S$ contains $n$ degree-one places for any given $n$, but if one does so, one will also need to deal with computations whose complexity will increase rapidly with $n$. To have this kind of method work for all $n$ at one time, one needs

to introduce additional tool to overcome this difficulty. We will discuss this matter in the forthcoming paper [7].

## 2. Group Rings

Let $G$ be a finite abelian group, $\mathbb{Z}[G]$ its group ring, and $I_G$ its augmentation ideal. If there is no ambiguity, we use $I$ instead of $I_G$. In this section, we study some basic congruence relations modulo $I^2, I^3, I^4$. Lemma 2.1 can be proved by straightforward computations. Other lemmas except Lemma 2.3 (3) are from [11]. It is possible to generalize Lemma 2.3 to every $r$.

**Lemma 2.1.** *The following are true.*

(1) If $A, B \in I$, $A \equiv A' \pmod{I^2}$ and $B \equiv B' \pmod{I^2}$, then $AB \equiv A'B' \pmod{I^3}$.
(2) If $A, B, C \in I$, $A \equiv A' \pmod{I^2}$, $B \equiv B' \pmod{I^2}$ and $C \equiv C' \pmod{I^2}$, then $ABC \equiv A'B'C' \pmod{I^4}$.
(3) If $g_1, g_2 \in G$, then $g_1 g_2 - 1 \equiv (g_1 - 1) + (g_2 - 1) \pmod{I^2}$.

**Lemma 2.2.** *Let $g \in G$ be an element of order $n$. If $n$ is odd, then $n(g-1) \equiv 0 \pmod{I^3}$. If $n = 2m$ is even, then $n(g-1) \equiv m(g-1)^2 \pmod{I^3}$. In both cases, $n(g-1) \in I^2$. If $\#G = n$, then $n$ annihilates $I^r/I^{r+1}$.*

**Lemma 2.3.** *Let $G = G_1 \times \cdots \times G_r$, where every $G_i$ is a cyclic group of order $n$. Put $m = n/2$, if $n$ is even, and put $m = 0$, if $n$ is odd. Let $g_i$ be a generator of $G_i$, and $a_i = g_i - 1 \in \mathbb{Z}[G]$. Then the following are true.*

(1) If $r = 1$, $G = G_1$, then $\sum_{\sigma \in G}(\sigma - 1) \equiv m a_1 \pmod{I^2}$.
(2) If $r = 2$, $G = G_1 \times G_2$, then $\sum_{\sigma \in G}(\sigma - 1) \equiv m^2(a_1^2 + a_1 a_2 + a_2^2) \pmod{I^3}$.
(3) If $r = 3$, $G = G_1 \times G_2 \times G_3$, then $\sum_{\sigma \in G}(\sigma - 1) \equiv m^3(a_1^3 + a_2^3 + a_3^3 + a_1^2 a_2 + a_1^2 a_3 + a_2^2 a_3 + a_1 a_2 a_3) \pmod{I^4}$.

*Proof.* (of (3))
We have

$$g_1^i g_2^j g_3^k - 1 = (g_3^k - 1)(g_1^i g_2^j - 1) + (g_1^i g_2^j - 1) + (g_3^k - 1), \quad 0 \le i, j, k < n.$$

By Lemma 2.2, $m a_1^2 a_3 \equiv n a_1 a_3 \equiv m a_1 a_3^2 \pmod{I^4}$, and consequently we get

$$\sum_{\sigma \in G}(\sigma - 1) = \Big(\sum_{k=0}^{n-1}(g_3^k - 1)\Big)\Big(\sum_{\sigma \in G_1 \times G_2}(\sigma - 1)\Big) + n \sum_{\sigma \in G_1 \times G_2}(\sigma - 1) + n^2 \sum_{k=0}^{n-1}(g_3^k - 1)$$

$$\equiv ma_3 \cdot m^2(a_1^2 + a_1a_2 + a_2^2) + nm^2(a_1^2 + a_1a_2 + a_2^2) + n^2ma_3 \pmod{I^4}$$
$$\equiv m^3(a_1^2a_3 + a_1a_2a_3 + a_2^2a_3 + a_1^3 + a_1^2a_2 + a_2^3 + a_3^3) \pmod{I^4}.$$

□

**Lemma 2.4.** *Suppose that $g_1, g_2 \in G$ are of order $n_1, n_2$, and $(n_1, n_2) = 1$. Then for any $r$,*

$$g_1g_2 - 1 \equiv (g_1 - 1) + (g_2 - 1) \pmod{I^r}.$$

**Corollary 2.5.** *Let $G_1, G_2$ be finite groups of order $n_1, n_2$, and $(n_1, n_2) = 1$. Let $G = G_1 \times G_2$, and $\pi_i : G \to G_i$ be the natural projection. For $\eta \in \mathbb{Z}[G]$,*

$$\eta \in I_G^r \quad \Leftrightarrow \quad \pi_1(\eta) \in I_{G_1}^r \quad \text{and} \quad \pi_2(\eta) \in I_{G_2}^r.$$

## 3. Extensions Ramified over Four Points

**3.1. A Reduction of the Proof.** Let $K = \mathbb{F}_q(x)$ be the rational function field over the finite field with $q$ elements, and let $S = \{\infty, x, x-1, x-s\}$, a set of four degree-one places of $K$. Let $K_S^{tame}$ be the maximal abelian extension unramified outside $S$ and at worst tamely ramified over $S$.

The following lemma is from Class Field Theory.

**Lemma 3.1.** *We have $K_S^{tame} = \overline{\mathbb{F}}_q(\sqrt[q-1]{x}, \sqrt[q-1]{x-1}, \sqrt[q-1]{x-s})$, and $\mathrm{Gal}(K_S^{tame}/K) \cong \widehat{\mathbb{Z}} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$.*

**Theorem 3.2.** *Let $L = \mathbb{F}_{q^w}(\sqrt[q-1]{x}, \sqrt[q-1]{x-1}, \sqrt[q-1]{x-s})$, $S = \{\infty, x, x-1, x-s\}$, $s \in \mathbb{F}_q \setminus \{0, 1\}$. Let $T$ contain a single place, $T = \{f(x)\}$, where $f(x)$ is a monic irreducible polynomial and $\deg(f) = d$. Then Gross' Conjecture holds in this case when both sides are multiplied by $(1+q+q^2+\cdots+q^{d-1})^2$, in other words,*

$$(1 + q + q^2 + \cdots + q^{d-1})^2\theta_{S,T}$$
$$\equiv (1 + q + q^2 + \cdots + q^{d-1})^2 h_{S,T}\det_G(\lambda_{S,T}) \pmod{I^4}.$$

We will postpone the proof of Theorem 3.2 until Section 3.4. Here we use the theorem to prove Theorem 1.3. This proof is similar to the one in [11].

*Proof.* (of Theorem 1.3)

By Corollary 2.5, we may assume that $G$ is a $p$-group for some prime number $p$. If $p \mid q$, then the Conjecture is already true ([12]). Consequently, we may assume that $L/K$ is a subextension of $K_S^{tame}/K$ and $(p, q) = 1$.

If $(p, q-1) = 1$, then the $p$-part of $\mathrm{Gal}(K_S^{tame}/K)$ corresponds to a constant field extension, and Gross' Conjecture holds.

Now suppose that $p$ divides $q-1$. By our hypothesis, $T$ contains a place whose degree is not divisible by $p$. Let $\mathfrak{a}$ be such a place, and put $T_0 = \{\mathfrak{a}\}$.

Let $w = p^k$. Then for some $k$, $L/K$ is a subextension of $L_w/K$ where $L_w = \mathbb{F}_{q^w}(\sqrt[q-1]{x}, \sqrt[q-1]{x-1}, \sqrt[q-1]{x-s})$. By Theorem 3.2, Gross' Conjecture for $L_w/K$ holds when multiplied by the factor $(1+q+q^2+\cdots+q^{d-1})^2$. This implies that Gross' Conjecture also holds for $L/K$ when multiplied by the same factor (Lemma 1.1 (1)). Since $G$ is a $p$-group, the augmentation quotient $I^r/I^{r+1}$ is $p^\infty$-torsion (Lemma 2.2). However, since $(1+q+q^2+\cdots+q^{d-1})^2 \equiv d^2 \not\equiv 0 \pmod{p}$, the Conjecture for $S$ and $T_0$ holds. Using Lemma 1.1(3), we prove the theorem. □

**3.2. Notations and Pre-Computations.** For the rest of this paper, let $L = \mathbb{F}_{q^w}(\sqrt[q-1]{x}, \sqrt[q-1]{x-1}, \sqrt[q-1]{x-s})$ and $G = \mathrm{Gal}(L/K)$. We keep the notations in Theorem 3.2. Then $G = G_\infty \times G_0 \times G_1 \times G_s$, where $G_\infty = \mathrm{Gal}(\mathbb{F}_{q^w}/\mathbb{F}_q) \cong \mathbb{Z}/w\mathbb{Z}$, $G_i = \mathrm{Gal}(\mathbb{F}_q(\sqrt[q-1]{x-i})/\mathbb{F}_q(x)) \cong \mathbb{F}_q^*$, $i = 0, 1, s$. Denote

$$H = G_0 \times G_1 \times G_s.$$

**Definition.** Define the isomorphism $\tau : \mathbb{F}_q^* \times \mathbb{F}_q^* \times \mathbb{F}_q^* \longrightarrow H$ such that for $\alpha, \beta, \gamma \in \mathbb{F}_q^*$,

$$\begin{aligned}\tau(\alpha,\beta,\gamma)(\sqrt[q-1]{x}) &= \alpha \cdot \sqrt[q-1]{x},\\ \tau(\alpha,\beta,\gamma)(\sqrt[q-1]{x-1}) &= \beta \cdot \sqrt[q-1]{x-1},\\ \tau(\alpha,\beta,\gamma)(\sqrt[q-1]{x-s}) &= \gamma \cdot \sqrt[q-1]{x-s}.\end{aligned}$$

Also, let $F \in G_\infty$ be the Frobenius element:

$$\begin{array}{rccc} F: & \mathbb{F}_{q^w}^* & \longrightarrow & \mathbb{F}_{q^w}^* \\ & a & \mapsto & a^q. \end{array}$$

For the rest of this paper, we denote $G = G_\infty \cdot H$. Thus an element $g \in G$ can be expressed as the product of its $G_\infty$-part and its $H$-part.

**Lemma 3.3.** *If $\mathfrak{a} \notin S$ is a degree-$d'$ place, which corresponds to a monic irreducible polynomial $h(x)$, then the $G_\infty$-part (resp. the $H$-part) of the Frobenius element at $\mathfrak{a}$ is given by $F^{d'}$ (resp. $\tau((-1)^{d'}h(0), (-1)^{d'}h(1), (-1)^{d'}h(s))$).*

*Proof.* Similar to [11], Lemma 3.4. □

**Definition.** Define $\Lambda = \cup_{i=0}^{\infty}\Lambda_i$ where, for $i = 0, 1, 2, \ldots$,

$$\Lambda_i \stackrel{\text{def}}{=} \{h(x) \in \mathbb{F}_q[x] \mid h \text{ is monic, } \deg(h) = i, h(0) \neq 0, h(1) \neq 0, h(s) \neq 0\},$$

and define the map $\phi : \Lambda \to H$ such that if $\deg(h) = d'$ then

$$\phi(h) = \tau((-1)^{d'}h(0), (-1)^{d'}h(1), (-1)^{d'}h(s)).$$

Also, for $(\alpha, \beta, \gamma) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \times \mathbb{F}_q^*$, denote $\delta(\alpha, \beta, \gamma) = \tau(\alpha, \beta, \gamma) - 1 \in \mathbb{Z}[G]$.

The following result is similar to [11], Proposition 3.5, which is for the case where $\#S = 3$. Here we give a more straightforward proof which also works for the general case.

**Lemma 3.4.** *Let $L, S, T$ be as in* Theorem 3.2. *Then we can express the Stickelberger elements as*

$$\theta_{S,T} = (1 - q^d F^d \phi(f))(1 + F \sum_{h \in \Lambda_1} \phi(h) + F^2 \sum_{h \in \Lambda_2} \phi(h)) + F^3(1 + qF + q^2F^2 + \cdots + q^{d-1}F^{d-1}) \sum_{\sigma \in H} \sigma.$$

*Proof.* Put $Y = q^{-s}$ and let

$$L_S(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_S} \phi_{\mathfrak{a}}(\mathbf{N}\mathfrak{a})^{-s}, \quad L_{S,T}(s) = \prod_{\mathfrak{a} \in T} (1 - \phi_{\mathfrak{a}}(\mathbf{N}\mathfrak{a})^{1-s}) \cdot L_S(s).$$

Here, as before, $\phi_{\mathfrak{p}}$ is the Frobenius element at $\mathfrak{p}$ if $\mathfrak{p}$ is a prime ideal, and $\phi_{\mathfrak{a}} = \prod \phi_{\mathfrak{p}_i}^{n_i}$ if $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$. By Lemma 3.3, if $h(x)$ is a monic polynomial and $(h) = \mathfrak{a}$, then $\phi_{\mathfrak{a}} = F^{\deg(h)}\phi(h)$.

Now we consider $L_S(s)$ and $L_{S,T}(s)$ as elements of $\mathbb{Z}[G][[Y]]$. Then we have

$$L_S(s) = \sum_{h \in \Lambda} F^{\deg(h)} \phi(h) (q^{\deg(h)})^{-s} = \sum_{i=0}^{\infty} V_i F^i Y^i,$$

where

$$V_i = \sum_{h \in \Lambda_i} \phi(h).$$

Let us consider $V_3$. First, note that $\#\Lambda_3 = (q-1)^3 = \#H$. It is easy to show that the map $\phi|_{\Lambda_3} : \Lambda_3 \to H$ is injective, hence is surjective. Therefore,

$$V_3 = \sum_{\sigma \in H} \sigma.$$

Similarly, we have, for $i \geq 3$,

$$V_i = q^{i-3} \sum_{\sigma \in H} \sigma.$$

Now, as $T = \{f(x)\}$,

$$L_{S,T}(s) = (1 - q^d F^d \phi(f) Y^d) \sum_{i=0}^{\infty} V_i F^i Y^i.$$

The term $V_3 F^3 Y^3$ multiplying with $q^d F^d \phi(f) Y^d$ will cancel with the term $V_{d+3} F^{d+3} Y^{d+3}$, and, by this, the coefficient of $Y^{d+3}$ is zero. Similarly, the coefficients of higher degree terms are also zero. Consequently,

the degree of $L_{S,T}$ is at most $d+2$, and

$$L_{S,T}(s) = (1 - q^d F^d \phi(f) Y^d)(1 + V_1 FY + V_2 F^2 Y^2) + \sum_{i=3}^{d+2} V_i F^i Y^i.$$

Because $\theta_{S,T} = L_{S,T}(0)$, the lemma is proved. □

For future computations, we define the following data.

**Definition.** We define the set of data $t, \alpha, \beta, \gamma, a, b, c, k, l, m, v, g_0, g_1, g_s, A, B, C, D$ as following. The element $t$ is a fixed generator of the multiplicative group $\mathbb{F}_q^*$. The elements $\alpha, \beta, \gamma$ in $\mathbb{F}_q^*$ and the residue class $a, b, c, k, l, m$ in $\mathbb{Z}/(q-1)\mathbb{Z}$ are defined to satisfy the following

$$\alpha = t^a = (-1)^d f(0), \quad \beta = t^b = (-1)^d f(1), \quad \gamma = t^c = (-1)^d f(s).$$

$$t^k = s, \quad t^l = s - 1, \quad t^m = -1.$$

The elements $g_0, g_1, g_s$ in $H$ are such that

$$g_0 = \tau(t, 1, 1), \quad g_1 = \tau(1, t, 1), \quad g_s = \tau(1, 1, t).$$

Finally, define $A = F - 1, B = g_0 - 1, C = g_1 - 1, D = g_s - 1$, and $v = 1 + q + q^2 + \cdots + q^{d-1}$.

Note that $A, B, C, D$ generate $I$. An element is in $I^r$ if and only if it can be written as a finite sum of monomials in $A, B, C, D$, with each monomial of total degree at least $r$.

The following lemma is a direct consequence of Lemma 2.2.

**Lemma 3.5.** *We have the following identities:*

$$\begin{aligned} (q-1)B &\equiv mB^2 \pmod{I^3}, \quad 2mB \equiv mB^2 \pmod{I^3}, \\ (q-1)C &\equiv mC^2 \pmod{I^3}, \quad 2mC \equiv mC^2 \pmod{I^3}, \\ \textit{and} \quad (q-1)D &\equiv mD^2 \pmod{I^3}, \quad 2mD \equiv mD^2 \pmod{I^3}. \end{aligned}$$

Note that we do not have $(q-1)A \equiv mA^2 \pmod{I^3}$.

**3.3. The Computation of $\theta_{S,T}$.** In this section, we will find an expression (see Lemma 3.10) of $\theta_{S,T} \pmod{I^4}$ in terms of those quantities defined in Section 3.2. For this purpose, we need the following three technical lemmas, among which the first is in fact similar to a result in [11] and the others are crucial here but otherwise not needed if one is dealing with the case where $\#S \leq 3$. On the other hand, if one is going to treat the case where $\#S \geq 5$, then more formulae of this type are needed and their cardinality as well as their complexity increase with the number $\#S$.

**Lemma 3.6.** *We have*

$$\sum_{g\in\Lambda_1}(\phi(g)-1)\equiv (m-k)B-lC+(m-k-l)D \pmod{I^2}.$$

*Proof.* Using Lemma 2.1 (3) and definitions of the maps $\phi$ and $\delta$, we have

$$\begin{aligned}\sum_{g\in\Lambda_1}(\phi(g)-1) &\equiv \left(\prod_{g\in\Lambda_1}\phi(g)\right)-1 \pmod{I^2}\\ &\equiv \delta((-1)^{q-3}\prod_{g\in\Lambda_1}g(0),(-1)^{q-3}\prod_{g\in\Lambda_1}g(1),\\ &\qquad (-1)^{q-3}\prod_{g\in\Lambda_1}g(s)) \pmod{I^2}\end{aligned}$$

Note that we always have $(-1)^{q-3}=1$ for $q$ either even or odd. By Wilson's Theorem we have $\prod_{g\in\Lambda_1}g(0)=(-1)/((-1)\cdot(-s))=t^{m-k}$ and also $\prod_{g\in\Lambda_1}g(1)=(-1)/(1\cdot(1-s))=t^{-l}$ and $\prod_{g\in\Lambda_1}g(s)=(-1)/(s\cdot(s-1))=t^{m-k-l}$. The definition of the data $B,C,D$ says that

$$\delta(t^{m-k},t^{-l},t^{m-k-l})\equiv (m-k)B-lC+(m-k-l)D \pmod{I^2},$$

and the lemma is proved. □

**Lemma 3.7.** *We have*

$$\sum_{g\in\Lambda_2}(\phi(g)-1)\equiv (k-m)B+lC+(k+l-m)D \pmod{I^2}.$$

*Proof.* The proof is similar to that for Lemma 3.6. Let

$$S_1 \stackrel{\text{def}}{=} \{g(x)\in\mathbb{F}_q[x] \mid g \text{ is monic, } \deg(g)=2, g(0)\neq 0\},$$

$$S_2 \stackrel{\text{def}}{=} \{(x-1)(x-i)\,|\,i\in\mathbb{F}_q, i\neq 0\},$$

$$S_3 \stackrel{\text{def}}{=} \{(x-s)(x-i)\,|\,i\in\mathbb{F}_q, i\neq 0\}.$$

Then

$$\Lambda_2 = S_1\setminus(S_2\cup S_3),$$

hence, by Wilson's Theorem again,

$$\begin{aligned}\prod_{g\in\Lambda_2} g(0) &= \left(\prod_{g\in S_1} g(0)\right)\cdot\left(\prod_{g\in S_2} g(0)\right)^{-1}\\ &\qquad\cdot\left(\prod_{g\in S_3} g(0)\right)^{-1}\cdot\left(\prod_{g\in S_2\cap S_3} g(0)\right)\\ &= (-1)^q\cdot(-1)\cdot(-1)\cdot s\\ &= -s.\end{aligned}$$

Similarly, we have

$$\prod_{g\in\Lambda_2} g(1) = s-1, \quad \prod_{g\in\Lambda_2} g(s) = -s(s-1).$$

Therefore,

$$\begin{aligned}\sum_{g\in\Lambda_2}(\phi(g)-1) &\equiv \delta(\prod_{g\in\Lambda_2} g(0), \prod_{g\in\Lambda_2} g(1), \prod_{g\in\Lambda_2} g(s)) \pmod{I^2}\\ &= \delta(-s, s-1, -s(s-1)) \pmod{I^2}\\ &= \delta(t^{k-m}, t^l, t^{k+l-m}) \pmod{I^2}\\ &\equiv (k-m)B + lC + (k+l-m)D \pmod{I^2}.\end{aligned}$$

□

**Lemma 3.8.** *We have*

$$\begin{aligned}\sum_{g\in\Lambda_1\cup\Lambda_2}(\phi(g)-1) &\equiv m^2B^2 + (m^2+ml)C^2 + (lk-lm+km)D^2\\ &\quad + (m^2+ml)BC + (lk-lm+km)BD\\ &\quad + (lk+mk)CD \pmod{I^3}.\end{aligned}$$

*Proof.* The proof is similar to the previous one but involves more computations. Consider the projection $\eta$: $G_0\times G_1\times G_s\to G_0\times G_1$. Let

$$\Gamma_i = \{h(x)\in\mathbb{F}_q[x] \mid h \text{ is monic, } \deg(h)=i, h(0)\neq 0, h(1)\neq 0\}.$$

Let $\pi$: $\cup_{i=0}^{\infty}\Gamma_i \to G_0\times G_1$ be such that $\pi(g) = \tau((-1)^{\deg(g)}g(0), (-1)^{\deg(g)}g(1), 1)$. Then $\pi|_\Lambda = \eta\circ\phi$. The map $\pi|_{\Gamma_2}$ is injective. As $\#\Gamma_2 = |G_0|\cdot|G_1| = (q-1)^2$, it is also surjective. By Lemma 2.3, we have

$$\sum_{g\in\Gamma_2}(\pi(g)-1) = \sum_{\sigma\in G_0\times G_1}(\sigma-1) \equiv m^2(B^2+BC+C^2) \pmod{I^3}.$$

Put $\Omega = \mathbb{F}_q \setminus \{0,1\}$, $S_1 = \{(x-s)\}$ and $S_2 = \{(x-s)(x-i) \mid i \in \Omega\}$. We have $\Gamma_1 \supset S_1$, $\Gamma_2 \supset S_2$ and $\Lambda_1 \cup \Lambda_2 = (\Gamma_1 \setminus S_1) \sqcup (\Gamma_2 \setminus S_2)$. Therefore,

$$\begin{aligned}(2)\quad \sum_{g\in\Lambda_1\cup\Lambda_2}(\pi(g)-1) &= \sum_{g\in\Gamma_2}(\pi(g)-1) - \sum_{g\in S_2}(\pi(g)-1) + \sum_{g\in\Gamma_1}(\pi(g)-1)\\ &\quad - \sum_{g\in S_1}(\pi(g)-1)\\ &\equiv m^2(B^2+BC+C^2) - \mathbf{I} + \mathbf{II} - \mathbf{III} \pmod{I^3},\end{aligned}$$

where

$$\begin{aligned}\mathbf{I} &= \sum_{i\in\Omega}(\pi((x-s)(x-i))-1)\\ \mathbf{II} &= \sum_{i\in\Omega}(\pi(x-i)-1)\\ \mathbf{III} &= (\pi(x-s)-1).\end{aligned}$$

For each $i \in \Omega$, define $a_i, b_i \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that $i-0 = t^{a_i}$, $i-1 = t^{b_i}$. Then by Wilson's Theorem, we have

$$(3)\qquad \sum_{i\in\Omega} a_i \equiv m \pmod{q-1}, \quad \text{and} \quad \sum_{i\in\Omega} b_i \equiv 0 \pmod{q-1}.$$

We have $\pi(x-i) = g_0^{a_i} g_1^{b_i}$, $\pi((x-s)(x-i)) = g_0^{a_i+k} g_1^{b_i+l}$, and hence

$$\begin{aligned}\mathbf{II} - \mathbf{I} &= \sum_{i\in\Omega}(g_0^{a_i} g_1^{b_i} - 1) - (g_0^{a_i+k} g_1^{b_i+l} - 1)\\ &= \sum_{i\in\Omega}((B+1)^{a_i}(C+1)^{b_i} - 1) - ((B+1)^{a_i+k}(C+1)^{b_i+l} - 1)\\ &\equiv \sum_{i\in\Omega}(a_iB + b_iC + a_ib_iBC + \binom{a_i}{2}B^2 + \binom{b_i}{2}C^2)\\ &\quad - ((a_i+k)B + (b_i+l)C + (a_i+k)(b_i+l)BC\\ &\quad + \binom{a_i+k}{2}B^2 + \binom{b_i+l}{2}C^2) \pmod{I^3}.\end{aligned}$$

Note that the above congruence is from the binomial expansion. Now we have $\binom{\alpha+\beta}{2} - \binom{\alpha}{2} = \alpha\beta + \binom{\beta}{2}$ and $\#\Omega = q-2$. These together with Equation (3) and Lemma 3.5 imply

$$\begin{aligned}(4)\qquad \mathbf{II}-\mathbf{I} &\equiv -(q-2)(kB+lC)-(k\sum a_i+(q-2)\binom{k}{2})B^2\\ &\quad -(l\sum b_i+(q-2)\binom{l}{2})C^2\\ &\quad -(k\sum b_i+l\sum a_i+(q-2)kl)BC \pmod{I^3}\\ &\equiv -(q-2)(kB+lC)+(km+\binom{k}{2})B^2+\binom{l}{2}C^2\\ &\quad +(lm+kl)BC \pmod{I^3}.\end{aligned}$$

Also,

$$\begin{aligned}(5)\qquad \mathbf{III} &= g_0^k g_1^l-1\\ &= (B+1)^k(C+1)^l-1\\ &\equiv kB+lC+klBC+\binom{k}{2}B^2+\binom{l}{2}C^2 \pmod{I^3}.\end{aligned}$$

From (2), (4), (5), we get

$$\begin{aligned}\sum_{g\in\Lambda_1\cup\Lambda_2}(\pi(g)-1) &\equiv m^2(B^2+BC+C^2)-(q-2)(kB+lC)\\ &\quad +(km+\binom{k}{2})B^2+\binom{l}{2}C^2+(lm+kl)BC\\ &\quad -(kB+lC+klBC+\binom{k}{2}B^2+\binom{l}{2}C^2) \pmod{I^3}\\ &\equiv -(q-1)kB-(q-1)lC+(m^2+km)B^2+m^2C^2\\ &\quad +(m^2+ml)BC \pmod{I^3}.\end{aligned}$$

Finally, we use Lemma 3.5 to obtain

$$(6)\qquad \sum_{g\in\Lambda_1\cup\Lambda_2}(\pi(g)-1)\equiv m^2B^2+(m^2+ml)C^2+(m^2+ml)BC \pmod{I^3}.$$

Similar formulae can be obtained by using other projections. In fact, if we put $\eta'\colon G_0\times G_1\times G_s\to G_0\times G_s$ and $\pi'|_\Lambda=\eta'\circ\phi$, then we have

$$\begin{aligned}(7)\qquad \sum_{g\in\Lambda_1\cup\Lambda_2}(\pi'(g)-1) &\equiv m^2B^2+(lk-lm+km)BD\\ &\quad +(lk-lm+km)D^2 \pmod{I^3},\end{aligned}$$

and also, if $\eta''\colon G_0 \times G_1 \times G_s \to G_1 \times G_s$ and $\pi''|_\Lambda = \eta'' \circ \phi$, then

$$\sum_{g\in\Lambda_1\cup\Lambda_2} (\pi''(g)-1) \equiv (m^2+ml)C^2 + (lk-lm+km)D^2 + (lk+mk)CD \pmod{I^3}. \tag{8}$$

It is known that the kernel of the natural map

$$\mathbb{Z}[G_0\times G_1\times G_s] \to \mathbb{Z}[G_0\times G_1]\times\mathbb{Z}[G_0\times G_s]\times\mathbb{Z}[G_1\times G_s]$$

is contained in $I^3$ ([10], Lemma 2). Therefore, the congruences (6), (7), (8) together imply

$$\begin{aligned}\sum_{g\in\Lambda_1\cup\Lambda_2} (\phi(g)-1) \equiv{}& m^2B^2 + (m^2+ml)C^2 + (lk-lm+km)D^2 \\ &+ (m^2+ml)BC + (lk-lm+km)BD \\ &+ (lk+mk)CD \pmod{I^3}.\end{aligned}$$

□

**Lemma 3.9.**

$$\theta_{S,T} \equiv -vA^3 + \mathbf{IV}\cdot\mathbf{V} - (\phi(f)-1)\cdot\mathbf{VI} + v\sum_{\sigma\in H}(\sigma-1) \pmod{I^4},$$

*where*

$$\begin{aligned}\mathbf{IV} &= 1-q^d-(F^d-1)-(\phi(f)-1)\\ \mathbf{V} &= (F-1)\Big(\sum_{g\in\Lambda_1}(\phi(g)-1) + 2\sum_{g\in\Lambda_2}(\phi(g)-1)\Big) + \sum_{g\in\Lambda_1\cup\Lambda_2}(\phi(g)-1)\\ \mathbf{VI} &= (q-1)^2 + (2q^2-5q+3)(F-1) + (q^2-3q+3)(F-1)^2\end{aligned}$$

*Proof.* We shall compute the right-hand side of the formula in Lemma 3.4. First, we use the facts $\#\Lambda_1 = q-3$ and $\#\Lambda_2 = q^2-3q+3$ to get

$$\begin{aligned}1 + F\sum_{g\in\Lambda_1}\phi(g) + F^2\sum_{g\in\Lambda_2}\phi(g) = 1 &+ \Big((q-3)+(q-3)(F-1)\\ &+ \sum_{g\in\Lambda_1}(\phi(g)-1) + (F-1)\sum_{g\in\Lambda_1}(\phi(g)-1)\Big)\\ &+ \Big((q^2-3q+3)+(q^2-3q+3)(F^2-1)\\ &+ \sum_{g\in\Lambda_2}(\phi(g)-1) + (F^2-1)\sum_{g\in\Lambda_2}(\phi(g)-1)\Big)\end{aligned} \tag{9}$$

$$= \mathbf{VI} + \mathbf{VII},$$

where

$$\mathbf{VII} = \sum_{g\in\Lambda_1\cup\Lambda_2}(\phi(g)-1) + (F-1)\sum_{g\in\Lambda_1}(\phi(g)-1) + ((F-1)^2+2(F-1))\sum_{g\in\Lambda_2}(\phi(g)-1).$$

Those terms involved in **VII** are basically computed in Lemma 3.6, Lemma 3.7 and Lemma 3.8. In particular, we see that $\mathbf{VII}\in I^2$. We need to compute $(1-q^dF^d\phi(f))\cdot\mathbf{VII}$, and it is easy to see that this equals to

$$\Big(1-q^d-q^d(F^d-1)-q^d(\phi(f)-1)-q^d(F^d-1)(\phi(f)-1)\Big)\cdot\mathbf{VII}\,.$$

Lemma 3.5 allow us to make some simplification of this. For instance, since $q\equiv 1 \pmod{q-1}$ we have $q^d\cdot\mathbf{VII}\equiv\mathbf{VII} \pmod{I^3}$ and hence

$$(1-q^dF^d\phi(f))\cdot\mathbf{VII}\equiv\Big(1-q^d-(F^d-1)-(\phi(f)-1) -(F^d-1)(\phi(f)-1)\Big)\cdot\mathbf{VII} \pmod{I^4}. \tag{10}$$

Note that $(F^d-1)(\phi(f)-1)\in I^2$ and, consequently, $(F^d-1)(\phi(f)-1)\cdot\mathbf{VII}\in I^4$. Hence $(1-q^dF^d\phi(f))\cdot\mathbf{VII}\equiv\mathbf{IV}\cdot\mathbf{VII} \pmod{I^4}$. Also, $\mathbf{VII}-\mathbf{V}=(F-1)^2\cdot\sum_{g\in\Lambda_2}(\phi(g)-1)$ which is in $I^3$. By Lemma 3.5 again, the multiple of this term with the factor **IV** is actually in $I^4$. From these, we see that

$$(1-q^dF^d\phi(f))\cdot\mathbf{VII}\equiv\mathbf{IV}\cdot\mathbf{V} \pmod{I^4} \tag{11}$$

By definition, $\phi(f)-1\in\mathbb{Z}[H]\cap I=(B,C,D)$, and we can also apply Lemma 3.5 to get $q^d(\phi(f)-1)\equiv\phi(f)-1 \pmod{I^2}$ and $(B,C,D)^r\cdot\mathbf{VI}\in I^{r+2}$ (here we use the fact that $q-1\mid 2q^2-5q+3$). From these, we get

$$\begin{aligned}(1-q^dF^d\phi(f))\cdot\mathbf{VI} &= \Big(1-q^dF^d-q^d(\phi(f)-1) -q^d(F^d-1)(\phi(f)-1)\Big)\cdot\mathbf{VI}\\ &\equiv(1-q^dF^d)\cdot\mathbf{VI}-(\phi(f)-1)\cdot\mathbf{VI} \pmod{I^4}\end{aligned} \tag{12}$$

Note that we have $\#H=(q-1)^3$ and, by Lemma 2.3,

$$\sum_{\sigma\in H}(\sigma-1)\equiv m^3(B^3+C^3+D^3+B^2C+B^2D+C^2D+BCD) \pmod{I^4}. \tag{13}$$

Put

$$N=1+qF+q^2F^2+\cdots+q^{d-1}F^{d-1}.$$

Then it is easy to see

$$(14)\quad F^3N\sum_{\sigma\in H}\sigma = (q-1)^3F^3N + N\sum_{\sigma\in H}(\sigma-1) + N(F^3-1)\sum_{\sigma\in H}(\sigma-1)$$
$$\equiv (q-1)^3F^3N + v\sum_{\sigma\in H}(\sigma-1) \pmod{I^4}$$

We complete the proof by using Lemma 3.4, Equations (9), (11), (12), (14) and the following straightforward computation:

$$\begin{aligned}
&(1-q^dF^d)\cdot\mathbf{VI} + (q-1)^3F^3N\\
&= N(1-qF)(1+(q-3)F+(q^2-3q+3)F^2) + (q-1)^3F^3N\\
&= N(1-3F+3F^2-F^3)\\
&= N(1-F)^3\\
&\equiv -vA^3 \pmod{I^4}.
\end{aligned}$$

□

**Lemma 3.10.** *We have*

$$\begin{aligned}
\theta_{S,T} \equiv\ & -vA^3 + mvA\Big((k-m)B^2 + lC^2 + (k+l-m)D^2\Big)\\
& - A(dA+aB+bC+cD)\Big((k-m)B + lC + (k+l-m)D\Big)\\
& + mv\Big(m^2B^3 + (m^2+ml)C^3 + (lk-lm+km)D^3\\
& \qquad + (m^2+ml)B^2C + (lk-lm+km)B^2D + (lk+mk)C^2D\Big)\\
& - (dA+aB+bC+cD)\Big(m^2B^2 + (m^2+ml)C^2\\
& \qquad + (lk-lm+km)D^2 + (m^2+ml)BC\\
& \qquad + (lk-lm+km)BD + (lk+mk)CD\Big)\\
& + m^2(aB^3+bC^3+cD^3) + mA(aB^2+bC^2+cD^2)\\
& - A^2(aB+bC+cD) + vm^3\Big(B^3+C^3+D^3+B^2C\\
& \qquad + B^2D + C^2D + BCD\Big) \pmod{I^4}.
\end{aligned}$$

*Proof.* We shall express the right-hand side of the formula in Lemma 3.9 as a polynomial in the data $A, B, C, D, a, b, c, d, k, l, m, v$. To do so, we first apply Lemma 3.6, Lemma 3.7, Lemma 3.8, together with the facts that $F^d - 1 \equiv dA \pmod{I^2}$ and $\phi(f) - 1 \equiv aB + bC + cD \pmod{I^2}$. Using these, we are able to write the product

$$\Big(-(F^d-1) - (\phi(f)-1)\Big)\cdot\mathbf{V}$$

as such kind of polynomial. In order to deal with the expression of the product $(1-q^d)\cdot\mathbf{V}$ as well as that of $-(\phi(f)-1)\cdot\mathbf{VI}$, we also need to use Lemma 3.5. Then we actually get

$$\begin{aligned}\mathbf{IV}\cdot\mathbf{V} \equiv\ & mvA\Big((k-m)B^2+lC^2+(k+l-m)D^2\Big)\\ & -A(dA+aB+bC+cD)\Big((k-m)B+lC+(k+l-m)D\Big)\\ & +mv\Big(m^2B^3+(m^2+ml)C^3+(lk-lm+km)D^3\\ & \qquad +(m^2+ml)B^2C+(lk-lm+km)B^2D+(lk+mk)C^2D\Big)\\ & -(dA+aB+bC+cD)\Big(m^2B^2+(m^2+ml)C^2\\ & \qquad +(lk-lm+km)D^2+(m^2+ml)BC\\ & \qquad +(lk-lm+km)BD+(lk+mk)CD\Big)\pmod{I^4},\end{aligned}$$

$$\begin{aligned}-(\phi(f)-1)\cdot\mathbf{VI} \equiv\ & m^2(aB^3+bC^3+cD^3)+mA(aB^2+bC^2+cD^2)\\ & -A^2(aB+bC+cD)\pmod{I^4}.\end{aligned}$$

For the expression of $v\sum_{\sigma\in H}(\sigma-1)$, we just apply Equation (13). □

**3.4. The Proof.** In this section, we finish the proof of Theorem 3.2. To do so, we first compute the Gross regulator by using a method similar to the one used in [11].

It is difficult to find a basis of $U_{S,T}$. Instead of doing so, we consider the following. The group of units $U_S$ is generated by $\mathbb{F}_q^*, x, x-1, x-s$. Put

$$\begin{aligned}u_0 &= x^{1+q+q^2+\cdots+q^{d-1}}/((-1)^d f(0)) = x^v/\alpha,\\ u_1 &= (x-1)^{1+q+q^2+\cdots+q^{d-1}}/((-1)^d f(1)) = (x-1)^v/\beta,\\ u_s &= (x-s)^{1+q+q^2+\cdots+q^{d-1}}/((-1)^d f(s)) = (x-s)^v/\gamma.\end{aligned}$$

They are linearly independent. Let $V$ be the group, generated by $u_0, u_1, u_s$. The index $(U_S : V)$ is $(q-1)(1+q+q^2+\cdots+q^{d-1})^3$, and $(U_S : U_{S,T}) = (q^d-1)/h_{S,T}$. Therefore $(U_{S,T} : V) = (1+q+q^2+\cdots+q^{d-1})^2 h_{S,T}$, and

$$\begin{aligned}&(1+q+q^2+\cdots+q^{d-1})^2 h_{S,T}\det{}_G(\lambda_{S,T})\\ &\qquad\equiv \det\begin{pmatrix} r_0(u_0)-1 & r_0(u_1)-1 & r_0(u_s)-1\\ r_1(u_0)-1 & r_1(u_1)-1 & r_1(u_s)-1\\ r_s(u_0)-1 & r_s(u_1)-1 & r_s(u_s)-1\end{pmatrix}\pmod{I^4}\end{aligned}$$

where $r_0, r_1, r_s$ are the local reciprocity maps at $0, 1, s$ respectively.

**Lemma 3.11.** *The values of the local reciprocity maps are:*

$$r_0(u_0) = F^{-v}\tau((-1)^d\alpha^{-1}, (-1)^d, (-s)^{-d}),$$
$$r_1(u_1) = F^{-v}\tau(1, (-1)^d\beta^{-1}, (1-s)^{-d}),$$
$$r_s(u_s) = F^{-v}\tau(s^{-d}, (s-1)^{-d}, (-1)^d\gamma^{-1}),$$

$$r_0(u_1) = \tau((-1)^d\beta^{-1}, 1, 1), \qquad r_0(u_s) = \tau((-s)^d\gamma^{-1}, 1, 1),$$
$$r_1(u_0) = \tau(1, \alpha^{-1}, 1), \qquad r_1(u_s) = \tau(1, (1-s)^d\gamma^{-1}, 1),$$
$$r_s(u_0) = \tau(1, 1, s^d\alpha^{-1}), \qquad r_s(u_1) = \tau(1, 1, (s-1)^d\beta^{-1}).$$

*Proof.* Similar to [11], Proposition 3.7. □

**Corollary 3.12.** *We have*

$$(1+q+q^2+\cdots+q^{d-1})^2 h_{S,T}\det{}_G(\lambda_{S,T}) \equiv \det\begin{pmatrix} \lambda_{11} & (dm-b)B & (dm+dk-c)B \\ -aC & \lambda_{22} & (dm+dl-c)C \\ (dk-a)D & (dl-b)D & \lambda_{33} \end{pmatrix} \pmod{I^4},$$

*where* $\lambda_{11} = -vA + (dm-a)B + dmC + (dm-dk)D$, $\lambda_{22} = -vA + (dm-b)C + (dm-dl)D$, $\lambda_{33} = -vA - dkB - dlC + (dm-c)D$.

Now we can prove Theorem 3.2.

*Proof.* (of Theorem 3.2)

By Lemma 3.10 and Corollary 3.12, the residue class

$$v^2\theta_{S,T} - v^2 h_{S,T}\det{}_G(\lambda_{S,T}) \pmod{I^4}$$

can be expressed as a polynomial $Z(A,B,C,D,a,b,c,d,k,l,m,v)$. Then we use the software "Maple" to expand those products into sums of monomials as well as to do the cancelations of monomials with opposite signs. The output is the following expression of $Z$. Note that the $A^3$ term in $Z$ vanishes, and since $v \equiv d \pmod{q-1}$, we have $(v-d)(B,C,D) \subset I^2$. Therefore we can replace $v$ by $d$ in our expression of $Z$.

```
   3 2   2    3 2   2    3 2       3 2 3      3 2 3      3 3 2    3 3 2
-2 d  m  A B  + 2 d  m  A D  - 2 d  A  D m + 2 d  m  C  l + 2 d  m  D  k + 2 d  m  B  C + d  m  B  D


    2   3 2    2   3 2    3 3 3    3 3 3    2    2    2    2 2  2  2  2
- 2 d  a B  m  - 2 d  b C  m  + 2 d  m  B  + 2 d  m  C  + 2 d  A a B  m - d  a B C  m - d  a B  C m

  2    2 2    2   3      2   2   2  2    2 2    2   3      3 2 2      3 2 2
- d  b C m  B - 2 d  b C  m l - d  b C  B m - d  c D m  B - 2 d  c D  k m + d  m  B  C l - d  m  B  D l

    3 2 2      3 2 2       3           3           3          2    2
+ 2 d  m  B  D k + d  m  C  D k - 2 d  A B C m l - 4 d  A B D k m - 2 d  A C D k m - d  a B C  m l

  2    2       2    2      2   2       2   2         2   2         2
+ d  a B D  m l - 2 d  a B D  k m - d  a B  C m l + d  a B  D m l - 2 d  a B  D k m - 2 d  a B C D k m

  2    2       2   2         2            2   2         2    2        2
- d  b C D  k m - 2 d  b C  B m l - 2 d  b C B D k m - d  b C  D k m - 2 d  c D C  m l - 2 d  c D B C m l
```

$$
\begin{aligned}
&\quad 2\quad 2\qquad 2\quad 2\qquad 3\ 2\qquad 3\ 3\ 2\quad 3\ 3\quad 2\quad 3\ 2\ 2\qquad 3\\
&-2d\ cD\ Bkm-2d\ cD\ Ckm-2d\ A\ mC-Bd\ m\ D\ -2d\ m\ CD\ +d\ m\ C\ kB-2d\ Am\,Ck\,B\\
&\quad 3\quad 2\quad 3\quad 2\quad 2\quad 3\quad 3\ 2\quad 2\ 3\ 2\quad 3\ 2\ 2\\
&-2d\ Am\,C\ l+4d\ Am\ CD-2d\ Am\,Cc\,D-4d\ AD\,m\,l\,C+2Bd\ m\ AD+B\ d\ m\ Ck+Bd\ m\ C\ l\\
&\quad 3\ 2\qquad 2\ 2\ 2\quad 2\quad 2\qquad 3\ 2\qquad 3\ 2\ 2\qquad 2\ 2\quad 2\\
&+2Bd\ m\ D\,l\,C+Bd\ m\ D\ c-d\ mC\ bkB+2d\ m\ CD\,kB+3d\ m\ C\ D\,l+2d\ m\ CD\ c\\
&\quad 2\ 3\qquad 2\ 3\qquad 2\ 3\qquad 2\ 3\ 2\qquad 2\ 3\ 2\qquad 2\ 2\\
&-2D\ d\ m\,l\,kB-2D\ d\ kAm-2D\ d\ km\,l\,C+2D\ d\ m\ kB+3D\ d\ m\ l\,C-B\ d\ m\,b\,Ck\\
&\quad 3\ 2\ 2\quad 3\qquad 2\quad 2\qquad 2\quad 2\quad 2\ 3\quad 2\\
&+Bd\ m\ D\ l-2d\ mCD\,lkB-2d\ mCD\ l\,c-2D\,d\ m\,b\,C\ l+D\ d\ km\,C
\end{aligned}
$$

Theorem 3.2 is proved by checking that $Z \equiv 0 \pmod{I^4}$. To do so, we use the obvious congruences

$$mB^2C \equiv mBC^2, \quad mB^2D \equiv mBD^2, \quad mC^2D \equiv mCD^2 \pmod{I^4},$$

which are consequences of Lemma 3.5. □

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