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The level of distribution of the sum-of-digits function of linear recurrence number systems

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To the memory of Christian Mauduit

Résumé. Soit $G=(G_j)_{j\geq 0}$ une suite strictement croissante des entiers définie par récurrence et telle que $G_0=1$. Soit $X^d-a_1X^{d-1}-\cdots-a_{d-1}X-a_d$ son polynôme caractéristique. Il est bien connu que tout entier positif ν possède une *écriture glouton* unique telle que $\nu=\varepsilon_0(\nu)G_0+\cdots+\varepsilon_\ell(\nu)G_\ell$ pour $\ell\in\mathbb{N}$ qui satisfait $G_\ell\leq\nu<G_{\ell+1}$. Ici les *chiffres* sont définis de manière récursive par la relation $0\leq\nu-\varepsilon_\ell(\nu)G_\ell-\cdots-\varepsilon_j(\nu)G_j<G_j$, où $0\leq j\leq\ell$. Dans cet article, nous étudions la *somme des chiffres* $s_G(\nu)=\varepsilon_0(\nu)+\cdots+\varepsilon_\ell(\nu)$ sous certains conditions naturelles sur la suite G. En particulier, nous déterminons son *niveau de distribution*. Pour être plus précis, nous montrons que pour $r,s\in\mathbb{N}$ avec $\gcd(a_1+\cdots+a_d-1,s)=1$ on a

$$\sum_{q<x^{\vartheta-\varepsilon}}\max_{z<x}\max_{1\leq h\leq q}\left|\sum_{\substack{k<z,s_G(k)\equiv r \bmod s\\ k\equiv h \bmod q}}1-\frac{1}{q}\sum_{k<z,s_G(k)\equiv r \bmod s}1\right| \ll x(\log 2x)^{-A}$$

pour tous $x\geq 1$ et $A,\varepsilon\in\mathbb{R}_{>0}$. Dans ce cas, $\vartheta=\vartheta(G)\geq\frac{1}{2}$ peut être calculé explicitement, et on obtient $\vartheta(G)\to 1$ pour $a_1\to\infty$. Comme application nous montrons que si le coefficient a_1 n'est pas trop petit, alors $\#\{k\leq x: s_G(k)\equiv r \pmod s,\ k \text{ a au plus deux facteurs premiers}\}\gg x/\log x$. En outre, en utilisant le crible de Bombieri, on en déduit un théorème des nombres presque premiers pour s_G.

Notre travail étend les résultats antérieurs sur la fonction somme des chiffres classique en base q obtenus par Fouvry and Mauduit.

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Mots-clefs. Sum of digits, linear recurrence number system, level of distribution, almost prime.

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Abstract. Let $G = (G_j)_{j\geq 0}$ be a strictly increasing linear recurrent sequence of integers with $G_0 = 1$ having characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$. It is well known that each positive integer ν can be uniquely represented by the so-called *greedy expansion* $\nu = \varepsilon_0(\nu)G_0 + \cdots + \varepsilon_\ell(\nu)G_\ell$ for $\ell \in \mathbb{N}$ satisfying $G_\ell \leq \nu < G_{\ell+1}$. Here the *digits* are defined recursively in a way that $0 \leq \nu - \varepsilon_\ell(\nu)G_\ell - \cdots - \varepsilon_j(\nu)G_j < G_j$ holds for $0 \leq j \leq \ell$. In the present paper we study the *sum-of-digits function* $s_G(\nu) = \varepsilon_0(\nu) + \cdots + \varepsilon_\ell(\nu)$ under certain natural assumptions on the sequence G. In particular, we determine its *level of distribution* x^ϑ. To be more precise, we show that for $r, s \in \mathbb{N}$ with $\gcd(a_1 + \cdots + a_d - 1, s) = 1$ we have for each $x \geq 1$ and all $A, \varepsilon \in \mathbb{R}_{>0}$ that

$$\sum_{q<x^{\vartheta-\varepsilon}} \max_{z<x} \max_{1\leq h\leq q} \left| \sum_{\substack{k<z, s_G(k)\equiv r \bmod s\\ k\equiv h \bmod q}} 1 - \frac{1}{q} \sum_{k<z, s_G(k)\equiv r \bmod s} 1 \right| \ll x(\log 2x)^{-A}.$$

Here $\vartheta = \vartheta(G) \geq \frac{1}{2}$ can be computed explicitly and we have $\vartheta(G) \to 1$ for $a_1 \to \infty$. As an application we show that $\#\{k \leq x : s_G(k) \equiv r \pmod{s},\ k$ has at most two prime factors$\} \gg x/\log x$ provided that the coefficient a_1 is not too small. Moreover, using Bombieri's sieve an "almost prime number theorem" for s_G follows from our result.

Our work extends earlier results on the classical q-ary sum-of-digits function obtained by Fouvry and Mauduit.

1. Introduction

The present paper is devoted to arithmetic properties of the sum-of-digits function s_G of a digit expansion with respect to a sequence $G = (G_j)_{j\geq 0}$ which is defined in terms of a linear recurrence relation. We establish a version of the theorem of Bombieri and Vinogradov for s_G (for the classical version of this theorem we refer e.g. to [14, Theorem 9.18]). In other words, we provide a result on the *level of distribution* of s_G (see for instance [14, Chapters 3, 22, and 25] or Greaves [19, Chapter 5] for information on this notion). From this result we derive distribution results for s_G on the set of integers having few prime factors. Our tools comprise exponential sum estimates and sieve methods. What we do here extends results of Fouvry and Mauduit [12, 13], where the level of distribution of the q-ary sum-of-digits function is investigated (see also the recent preprint of Spiegelhofer [30]). Our results provide a first step towards a generalization of the beautiful work of Mauduit and Rivat [22] on the q-ary sum of digits of primes to digit expansions w.r.t. a linear recurrent sequence. We mention that new ideas are needed in our setting in order to establish the exponential sum estimates necessary for proving our main results.

1.1. Linear recurrence number systems. We start with the definition of digit expansions w.r.t. a sequence of integers. Let $G = (G_j)_{j\ge 0}$ be a strictly increasing sequence of positive integers and suppose that $G_0 = 1$. Using the *greedy algorithm* one can associate a unique digit expansion to each positive integer ν w.r.t. this sequence G. Indeed, for each integer $\nu \ge 1$ there exists a unique $\ell \in \mathbb{N}$ such that $G_\ell \le \nu < G_{\ell+1}$. With this number ℓ we can define the *digits* $\varepsilon_\ell(\nu), \dots, \varepsilon_0(\nu)$ recursively in a way that

$$0 \le \nu - \varepsilon_\ell(\nu)G_\ell - \cdots - \varepsilon_j(\nu)G_j < G_j \qquad (0 \le j \le \ell).$$

This leads to the digit expansion

$$\nu = \varepsilon_0(\nu)G_0 + \cdots + \varepsilon_\ell(\nu)G_\ell \tag{1.1}$$

for ν w.r.t. the sequence G. It is easy to check that we have $0 \le \varepsilon_j(\nu) < \frac{G_{j+1}}{G_j}$ for each $0 \le j \le \ell$ and that this expansion is unique with the property that

$$0 \le \varepsilon_0(\nu)G_0 + \cdots + \varepsilon_j(\nu)G_j < G_{j+1}$$

for $0 \le j \le \ell$. Using the greedy expansion for the sequence G, we define the sum-of-digits function of ν w.r.t. G by

$$s_G(\nu) = \varepsilon_0(\nu) + \cdots + \varepsilon_\ell(\nu) \qquad (\nu \ge 1) \tag{1.2}$$

and by setting $s_G(0) = 0$ for convenience. In the present paper we deal with sequences G that are defined in terms of linear recurrences. This idea goes back to Zeckendorf [35] for the case of Fibonacci numbers (see e.g. [9, 21, 26] for the general case). We recall the following definition.

Definition 1.1 (Linear recurrence base)**.** We will refer to a strictly increasing sequence $G = (G_j)_{j\ge 0}$ as a *linear recurrence base*, if there exist $a_1, \dots, a_d \in \mathbb{N}$ with $a_d > 0$ such that the following conditions hold:

(1) $G_0 = 1$ and $a_1G_{k-1} + \cdots + a_kG_0 < G_k$ for $1 \le k < d$.
(2) $G_{n+d} = a_1G_{n+d-1} + \cdots + a_dG_n$ holds for each $n \in \mathbb{N}$.
(3) $(a_k, a_{k+1}, \dots, a_d) \preceq (a_1, a_2, \dots, a_{d-k+1})$ for $1 < k \le d$, where "$\prec$" indicates the lexicographic order.

The polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ is called the *characteristic polynomial* of the linear recurrence base G. Its dominant root (which is a positive real number) is called α.

We want to make some comments on this definition which is the same as the one used in Lamberger and Thuswaldner [21]. Item (3) immediately yields that $a_1 \ge \max\{a_2, \dots, a_d\}$. Moreover, our conditions imply with the same proof as Steiner [31, Lemma 2.1] that

$$G_{n+d-k} > a_{k+1}G_{n+d-k-1} + \cdots + a_dG_n \qquad (n \in \mathbb{N},\ 1 \le k \le d-1),$$

a condition that was used for instance in Drmota and Gajdosik [8, 9]. In [8, Lemma 3.1] it is proved (under milder conditions than ours) that the characteristic polynomial of G has a dominant root $\alpha > 1$ and, because all coefficients of the recurrence satisfy $0 \le a_j \le a_1$ in our case, we even have

$$\alpha \in [a_1, a_1 + 1). \tag{1.3}$$

The fact that α is dominant yields that there are constants $c, \delta \in \mathbb{R}_{>0}$ such that

$$G_n = c\alpha^n + \mathcal{O}(\alpha^{(1-\delta)n}) \qquad (n \ge 0). \tag{1.4}$$

If item (1) is strengthened to $G_0 = 1$ and $a_1G_{k-1}+\cdots+a_kG_0+1 = G_k$ for $1 \le k < d$, according to [31, Proposition 2.1] the string $\varepsilon_0, \dots, \varepsilon_\ell$ can occur as a digit string in (1.1) if and only if $(\varepsilon_j, \dots, \varepsilon_{j+d-1}) \prec (a_1, a_2, \dots, a_d)$ holds for $0 \le j \le \ell$ (here we have to pad $\varepsilon_0, \dots, \varepsilon_\ell$ with $d-1$ zeros). This is called the *Parry condition* and goes back to Parry [25] where it was introduced in the context of beta-numeration. We also mention that in some earlier papers on linear recurrence bases instead of item (3) the stronger condition $a_1 \ge a_2 \ge \cdots \ge a_d > 0$ is assumed (see e.g. [17, 26]).

A linear recurrence base together with the associated digit expansions (1.1) will be called a *linear recurrence number system.*

1.2. Previous results. The most prominent example of a linear recurrence base is the *Fibonacci sequence* $F = (F_j)_{j\ge 2}$ defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1}+F_n$ for $n \ge 0$ (note that we have to start with index $j = 2$ in the sequence F to meet the conditions of Definition 1.1). The associated linear recurrence number system was first studied by Zeckendorf [35]. For this reason expansions of the shape (1.1) are called *Zeckendorf expansions* in this case. In the meantime linear recurrence number systems received a lot of attention and have been studied by many authors. Without making an attempt to be complete we mention a few results on linear recurrence number systems with special emphasis on the sum-of-digits function s_G defined in (1.2).

Pethő and Tichy [26] provide an asymptotic formula of the summatory function of s_G. Using analytic methods and results from Coquet, Rhin, and Toffin [7], Grabner and Tichy [17] prove that $(zs_G(n))_{n\in\mathbb{N}}$ is equidistributed modulo 1 for each $z \in \mathbb{R} \setminus \mathbb{Q}$. By elementary exponential sum estimates Lamberger and Thuswaldner [21] establish distribution results of $s_G(n)$ in residue classes and derive some consequences including a Barban–Davenport–Halberstam type theorem for s_G. Distribution functions for so-called *G-additive functions* (a natural generalization of s_G analogous to the well-known q-additive functions) are investigated by Barat and Grabner [2]. In [2] the authors also provide a dynamic approach to linear recurrence number systems on the *G-compactification* $\mathcal{K}_G$ on which a dynamical system can be defined in terms of the addition of 1; this *G-odometer* goes back to Grabner et al. [16] (see also [3] for a more recent study of this object). A local limit law for s_G is proved by Drmota and Gajdosik [8].

In [9] the same authors consider sums of the shape $\sum_{\nu<N}(-1)^{s_G(\nu)}$. Drmota and Steiner [11, 32] establish a central limit theorem for G-additive functions along polynomial sequences, and Wagner [34] studies properties of sets of numbers $\nu < N$ characterized by the fact that $s_G(\nu) = k$ for some fixed positive integer k. Recently, Miller and his co-authors proved further distribution results related to linear recurrence number systems. See for example [4], where run lengths of zeros in Zeckendorf expansions are studied, or [6], which is concerned with the number of nonzero digits in Zeckendorf expansions. Motivated by the proof of Gelfond's old conjecture on the distribution of the sum-of-digits function of primes in residue classes by Mauduit and Rivat [22] and, more generally, by Sarnak's conjecture [28], the question whether s_G has nice distribution properties for prime arguments came into the focus of research. We mention that Möbius orthogonality of s_F is proved in the Zeckendorf case by Drmota et al. [10]. The exponential sum methods developed in [22] also led to a wealth of new results on sum-of-digits functions. In the context of Zeckendorf expansions the joint distribution of the ordinary q-ary sum-of-digits function and s_F is investigated by Spiegelhofer [29] by using methods in the spirit of [22]. Finally, we note that, starting with Barat and Grabner [2], van der Corput and Halton type sequences using linear recurrence bases are investigated. Work on this topic can be found in Ninomiya [24], Hofer et al. [20], and Thuswaldner [33].

We mention that Ostrowski expansions [5] as well as beta-expansions [15, 25, 27] are related to linear recurrence number systems.

1.3. Statement of results and associated exponential sums. Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1. The aim of the present article is to study the level of distribution $x^{\vartheta(G)}$ of the sum-of-digits function s_G. In other words, our main result is the extension of [12, Théorème] to linear recurrence bases.

Theorem 1.2. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ satisfying the conditions of Definition 1.1. Let $r, s \in \mathbb{N}$ with $\gcd(a_1 + \cdots + a_d - 1, s) = 1$. Then for each $x \geq 1$and all $A, \varepsilon \in \mathbb{R}_{>0}$, we have*

$$\sum_{q<x^{\vartheta-\varepsilon}} \max_{z<x} \max_{1\leq h\leq q} \left| \sum_{\substack{k<z \\ s_G(k)\equiv r \bmod s \\ k\equiv h \bmod q}} 1 - \frac{1}{q} \sum_{\substack{k<z \\ s_G(k)\equiv r \bmod s}} 1 \right| \ll x(\log 2x)^{-A}, \tag{1.5}$$

where the implied constant depends on ε and A. Here $\vartheta = \vartheta(G) \geq \frac{1}{2}$ can be computed explicitly and we have $\vartheta(G) \to 1$ for $a_1 \to \infty$.

Remark 1.3. We are able to give concrete values for $\vartheta(G)$. Let α be the dominant root of the characteristic polynomial of G. We show that $\vartheta(G) \geq \max\{\frac{1}{2}, 1 - \log_\alpha(m_G + 3)\}$ for m_G as in (3.7). Since $m_G \ll \log a_1 \ll \log \alpha$ by Lemma 3.7 this already implies that $\vartheta(G) \to 1$ for $a_1 \to \infty$. On top of this, in Lemma 5.2 we give better estimates for $\vartheta(G)$ for small values of a_1. These estimates are needed in order to prove Corollary 1.4 below.

Similarly as Fouvry and Mauduit [12] we can deduce two applications of Theorem 1.2. The first one deals with the distribution of the sum-of-digits function s_G evaluated along almost primes.

Corollary 1.4. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ satisfying the conditions of Definition 1.1. Let $r, s \in \mathbb{N}$ with $\gcd(a_1 + \cdots + a_d - 1, s) = 1$. Then for $a_1 \geq 59$ we have*

$$(1.6)\quad \#\{k \leq x : s_G(k) \equiv r \pmod{s},\ k = p_1 \text{ or } k = p_1p_2 \text{ with } p_1, p_2 \text{ prime}\} \gg \frac{x}{\log x}$$

for $x \to \infty$. If the characteristic polynomial of G has the particular form $X^2 - a_1X - 1$ then this result even holds for $a_1 \geq 15$.

It is well known (see for instance Greaves [19, Chapter 5]) that results on the level of distribution of a set $A(x)$ of positive integers less than x can be used to get results on the number of almost primes contained in $A(x)$. In particular, if the level of distribution of $A(x)$ is $x^{\vartheta-\varepsilon}$ with ϑ large enough to satisfy $\frac{1}{\vartheta} < 2 - \delta_2$ for a certain constant δ_2, then the number of almost primes in $A(x)$ can be estimated from below by a constant times $\frac{x}{\log x}$. There has been a lot of effort to get the constant δ_2 as small as possible. To our knowledge, currently the best value is $\delta_2 = 0.044560$ and this is due to Greaves [18] (although δ_2 is conjectured to be equal to 0). Thus in order to prove Corollary 1.4 we need to make sure that

$$(1.7)\qquad \vartheta(G) > 0.5113938\cdots = 1 - 0.4886061\ldots$$

for the linear recurrence bases indicated in its statement. The lower bound 59 (resp. 15) for a_1 is an artifact of the methods we are using in the proof. However, in principle our methods allow (with sufficient computer power) to extend the result to smaller values of a_1 (see Section 4 for details on this).

In the last section we introduce a block method where we do two steps at once. This reduces the lower bound on a_1 from 59 in the general case to 15 in the special case in Corollary 1.4. This seems a big step and if one uses blocks of larger size, one might reduce the lower bound further. However, this is in a special case and already here computation is quite involved and with each step it gets longer. Summing up we do not think that it is feasible to get the result for $a_1 = 1$ with present time computers.

Our second corollary provides a prime number theorem for numbers whose sum-of-digits function s_G lies in a prescribed residue class. Analogously to the case of the ordinary q-ary sum-of-digits function (see [12, Corollaire 2]) this corollary gives a nontrivial result only for large values of a_1. In the following statement $\Lambda_\ell = \mu * \log^\ell$ denotes the generalized von Mangoldt function ($\ell \geq 1$; here μ is the Möbius function and "$*$" denotes Dirichlet convolution).

Corollary 1.5. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ satisfying the conditions of Definition 1.1. Let $\ell, r, s \in \mathbb{N}$ with $\ell \geq 2$ and $\gcd(a_1 + \cdots + a_d - 1, s) = 1$. Then there is $x_0 = x_0(G, s, \ell)$ such that for $x \geq x_0$ we have*

$$\sum_{\substack{k<x\\ s_G(k)\equiv r \bmod s}} \Lambda_\ell(k) = \frac{\ell}{s} x (\log x)^{\ell-1} \left(1 + \mathcal{O}\left(\frac{(\log\log a_1)^5}{\log a_1}\right)\right),$$

where the implied constant depends only on s and ℓ.

Corollary 1.5 follows from Theorem 1.2 by an application of the sieve of Bombieri (cf. [14, Theorem 3.5]). Since the proof of Corollary 1.5 is *verbatim* the same as the one of [12, Corollaire 2] in [12, Section VII] we do not reproduce it here.

The paper is organized as follows. In Section 2 we reduce the problem of proving Theorem 1.2 to an exponential sum estimate and provide some preliminaries. Section 3 is devoted to the estimate of the exponential sums needed in the proof. In Section 4 we give a computer assisted improvement for these estimates to make them applicable for small values of the coefficient a_1. Using these preparations in Section 5 we provide the proof of Theorem 1.2 and of Corollary 1.4. Moreover, we establish an estimate for $\vartheta(G)$ for small values of a_1.

2. Rewriting the problem

Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ having dominant root α. The proof of Theorem 1.2 relies on exponential sums. Setting $e(z) = \exp(2\pi\sqrt{-1}z)$ we get for integers a, b, c with $c \geq 1$ that

$$\frac{1}{c}\sum_{h=1}^{c} e\left(\frac{h}{c}(a-b)\right) = \begin{cases} 1 & \text{if } a \equiv b \bmod c,\\ 0 & \text{otherwise.}\end{cases}$$

Thus the difference inside the absolute value of (1.5) may be written as

$$R(z)=R(z;u,q,r,s)=\sum_{\substack{k<z\\ s_G(k)\equiv r \bmod s\\ k\equiv u \bmod q}}1-\frac{1}{q}\sum_{\substack{k<z\\ s_G(k)\equiv r \bmod s}}1$$

$$=\frac{1}{sq}\sum_{b=1}^{s}\sum_{h=1}^{q-1}\sum_{k<z}e\left(\frac{b}{s}(s_G(k)-r)+\frac{h}{q}(k-u)\right).$$

Splitting the contribution of $b=s$ apart we get that

$$R(z)=\frac{1}{sq}\sum_{b=1}^{s-1}\sum_{h=1}^{q-1}e\left(-\frac{br}{s}-\frac{uh}{q}\right)\sum_{k<z}e\left(\frac{b}{s}s_G(k)+\frac{h}{q}k\right)+\mathcal{O}\left(\frac{q\log q}{s}\right). \tag{2.1}$$

In view of (2.1) the proof of Theorem 1.2 boils down to showing that

$$\sum_{Q<q\le 2Q}\sum_{h=1}^{q-1}\left|\sum_{k<z}e\left(\frac{r}{s}s_G(k)+\frac{h}{q}k\right)\right|\ll Qx(\log 2x)^{-A}$$

holds for each $A>0$ if $1\le r\le s-1$, $Q\le x^{\vartheta(G)-\varepsilon}$, and $z<x$. To make our proofs easier we want to subdivide the sum over k according to the greedy expansion (1.1) of z (of course we may assume w.l.o.g. that z is a positive integer). Since $z<x$ there is $N\le\log_\alpha x+C$ (for some constant C depending on G) such that

$$z=\sum_{0\le n\le N}\varepsilon_n(z)G_n. \tag{2.2}$$

For $y,\beta\in[0,1]$ we define the following exponential sum

$$S_n(y,\beta):=\sum_{k<G_n}e\left(\beta s_G(k)+yk\right).$$

Using (2.2) we gain by splitting off one digit of z after the other (like it is done for instance in the proof of [17, Lemma 1]),

$$\left|\sum_{k<z}e\left(\frac{r}{s}s_G(k)+\frac{h}{q}k\right)\right|\le\sum_{n=0}^{N}\left|\varepsilon_n(z)S_n\left(\frac{h}{q},\frac{r}{s}\right)\right|\ll\sum_{n=0}^{N}\left|S_n\left(\frac{h}{q},\frac{r}{s}\right)\right|.$$

Thus, since $N\le\log_\alpha x+C$, Theorem 1.2 follows if we prove

$$\sum_{Q<q\le 2Q}\sum_{h=1}^{q-1}\left|S_n\left(\frac{h}{q},\frac{r}{s}\right)\right|\ll Qx(\log 2x)^{-A} \tag{2.3}$$

for each $A>0$ if $1\le r\le s-1$, $Q\le x^{\vartheta(G)-\varepsilon}$, and $n\le\log_\alpha x+C$.

We start by setting up a recurrence relation for $S_n(y,\beta)$. Let

$$\mathcal{I}:=\{1\le j\le d\colon a_j\neq 0\} \tag{2.4}$$

be the set of indices corresponding to non-vanishing coefficients of the characteristic polynomial of G. As $a_d > 0$, item (3) of Definition 1.1 implies that $\{1, d\} \subset \mathcal{I}$. Then the exponential sum $S_n(y, \beta)$ satisfies the recurrence

$$S_n(y,\beta) = \sum_{j\in\mathcal{I}} A_{n,j}(y,\beta) S_{n-j}(y,\beta) \tag{2.5}$$

with

$$A_{n,j}(y,\beta) = \sum_{\ell=0}^{a_j-1} e\left(y\left(\sum_{k=1}^{j-1} a_k G_{n-k} + \ell G_{n-j}\right) + \beta\left(\sum_{k=1}^{j-1} a_k + \ell\right)\right) \tag{2.6}$$

for $1 \le j \le d$ (see [21, Equation (3)]). Iterating this recurrence relation we obtain

$$\begin{aligned} S_n(y,\beta) &= \sum_{j\in\mathcal{I}} A_{n,j}(y,\beta) S_{n-j}(y,\beta) \\ &= \sum_{j_1,j_2\in\mathcal{I}} A_{n,j_1}(y,\beta) A_{n-j_1,j_2}(y,\beta) S_{n-j_1-j_2}(y,\beta) \\ &= \sum_{j_1,\dots,j_k\in\mathcal{I}} A_{n,j_1}(y,\beta) \cdots A_{n-j_1-\cdots-j_{k-1},j_k}(y,\beta) S_{n-j_1-\cdots-j_k}(y,\beta), \end{aligned}$$

which makes sense as long as $n-j_1-\cdots-j_{k-1} \ge d$ holds for all constellations $(j_1, \dots, j_{k-1}) \in \mathcal{I}^{k-1}$. For $d \le n_0 < n$ and $1 \le k < n$ let

$$J_k(n_0) = \left\{ \mathbf{j} = (j_1, \dots, j_k) \in \mathcal{I}^k \colon n - \sum_{\ell=1}^{k-1} j_\ell > n_0 \ge n - \sum_{\ell=1}^{k} j_\ell \right\}. \tag{2.7}$$

Then

$$\begin{aligned} &|S_n(y,\beta)| \\ &\le \sum_{k=1}^{n-n_0} \sum_{(j_1,\dots,j_k)\in J_k(n_0)} \prod_{\ell=1}^{k} \left| A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta) \right| \cdot \left| S_{n-\sum_{r=1}^{k} j_r}(y,\beta) \right|. \end{aligned} \tag{2.8}$$

The central idea in proving (2.3) is a combination of maximum- and 1-norm estimates of $S_n(y, \beta)$ and related expressions.

3. Estimates of exponential sums related to $S_n(y, \beta)$

We subdivide this section into three parts. First we consider the 1-norm of $S_n(\,\cdot\,, \beta)$ and of its derivative. These 1-norms play a role in the proof of Theorem 1.2 after an application of an inequality due to Sobolev and Gallagher which is an important tool in the context of the large sieve (see Lemma 5.1 below for its statement). In the second part we estimate the maximum-norm of sums of certain products related to $S_n(y, \beta)$. Finally the third part deals with an estimation of a parameter which occurs in our estimate of the 1-norm of $S_n(\,\cdot\,, \beta)$.

3.1. The 1-norm of $S_n(\,\cdot\,,\beta)$. Let $G=(G_j)_{j\ge 0}$ be a linear recurrence base with characteristic polynomial $X^d-a_1X^{d-1}-\cdots-a_{d-1}X-a_d$ satisfying the conditions of Definition 1.1. We set $a=a_1$ and let $\beta\in\mathbb{R}$ be fixed. Define for $k\in\mathbb{N}$, $j\in\mathcal{I}$ with $k\ge j$, and $y\in\mathbb{R}$ the functions

$$f_{k,j}(y)=\begin{cases}\left|\frac{\sin\left(\pi a_j(\beta+yG_{k-j})\right)}{\sin\left(\pi(\beta+yG_{k-j})\right)}\right| & \text{if } \beta+yG_{k-j}\notin\mathbb{Z},\\ a_j & \text{if } \beta+yG_{k-j}\in\mathbb{Z}.\end{cases}$$

This permits us to write the modulus of the sums $A_{k,j}(y,\beta)$ in (2.6) as

$$|A_{k,j}(y,\beta)|=\left|\sum_{\ell=0}^{a_j-1}e\left(\ell(\beta+yG_{k-j})\right)\right|=f_{k,j}(y). \tag{3.1}$$

We note that the numerator of $f_{k,j}(y)$ has period $(a_jG_{k-j})^{-1}$.

For each $k\in\mathbb{N}$ we subdivide the interval $\left[-\frac{\beta}{G_k},1-\frac{\beta}{G_k}\right)$ (which is the same as $[0,1)$ modulo 1) into aG_k parts

$$I_k(b)=\left[\frac{b-a\beta}{aG_k},\frac{b+1-a\beta}{aG_k}\right)\qquad(0\le b<aG_k) \tag{3.2}$$

of equal length $(aG_k)^{-1}$. In each of the intervals $I_{k-j}(b)$ the supremum of $f_{k,j}(y)$ satisfies

$$\sup_{y\in I_{k-j}(b)}f_{k,j}(y)=m(j,b)\qquad(0\le b<aG_{k-j}), \tag{3.3}$$

with

$$m(j,b)=m_G(j,b)=\sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)}\left|\frac{\sin\pi a_jy}{\sin\pi y}\right|\qquad(j\in\mathcal{I},\ b\in\mathbb{Z}). \tag{3.4}$$

Thus the supremum of $|A_{k,j}(y,\beta)|=f_{k,j}(y)$ in (3.3) is independent of k. It is immediate that for $b\equiv 0,a-1\pmod a$ this supremum is equal to a_j (it is attained for $b\equiv 0\pmod a$ on the left endpoint of $I_{k-j}(b)$, and for $b\equiv a-1\pmod a$ for the limit towards the right endpoint of $I_{k-j}(b)$). If $j=1$ and $b\not\equiv 0,a-1\pmod a$ then $f_{k,1}(y)$ is a unimodal function on $I_{k-1}(b)$ which is equal to zero at its endpoints and whose global maximum is the unique local maximum in that interval.

We define the piecewise constant function

$$F_{k,j}(y)=m(j,b)\quad\text{for}\quad y\in I_{k-j}(b)\qquad(0\le b<aG_{k-j}), \tag{3.5}$$

which forms an upper bound for $f_{k,j}(y)$. The functions $f_{k,1}(y)$ and $F_{k,1}(y)$ are plotted in Figure 3.1 for a special set of parameters.

We will also need the following notations. With $m(j,b)$ as in (3.4) we define

$$m(j)=m_G(j)=\frac{1}{a}\sum_{b=0}^{a-1}m_G(j,b)\qquad(j\in\mathcal{I}) \tag{3.6}$$

and finally

$$m = m_G = \max_{j \in \mathcal{I}} m_G(j). \tag{3.7}$$

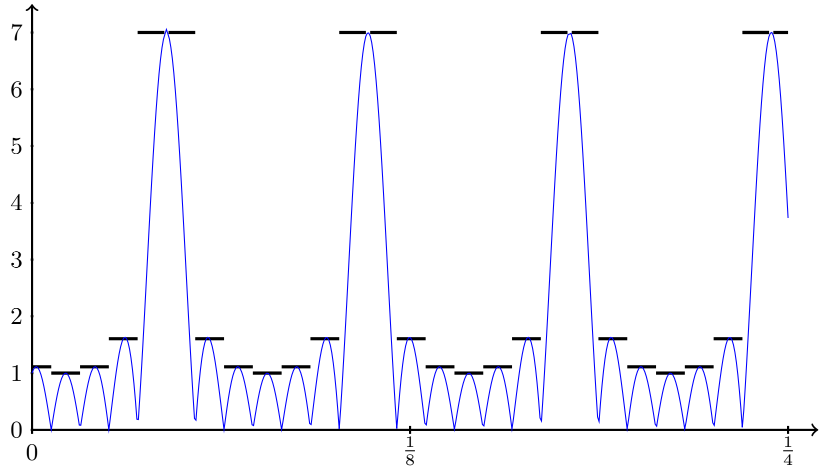

FIGURE 3.1. For the linear recurrence base $(G_j)_{j\geq 0}$ defined by $G_0 = 1$, $G_1 = 8$, and $G_{n+2} = 7G_{n+1} + G_n$ for $n \geq 0$ this image shows the function $f_{3,1}(y)$ together with its piecewise constant upper bound $F_{3,1}(y)$ in the interval $y \in [0, \frac{1}{4}]$ (here we chose $\beta = \frac{1}{3}$).

It will turn out that the 1-norm of $S_n(\,\cdot\,, \beta)$ can be estimated in terms of an integral over products of the functions $F_{k,j}(y)$. Thus we deal with such products in our first proposition.

Proposition 3.1. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ satisfying the conditions of Definition 1.1. Fix $k \in \mathbb{N}$ and let $n_0, n_1, \ldots, n_k$ be a strictly increasing sequence of integers satisfying $j_\ell := n_\ell - n_{\ell-1} \in \mathcal{I}$ for $1 \leq \ell \leq k$. Then*

$$\int_0^1 \prod_{\ell=1}^{k} f_{n_\ell, j_\ell}(y)\mathrm{d}y \leq \int_0^1 \prod_{\ell=1}^{k} F_{n_\ell, j_\ell}(y)\mathrm{d}y \ll (m+2)^k, \tag{3.8}$$

where $m = m_G$ is given by (3.7).

Proof. Since the first inequality in (3.8) is obvious it remains to prove the second one. Throughout this proof we set $g_\ell = \lfloor G_{n_\ell}/G_{n_{\ell-1}} \rfloor$. From the definition of the intervals $I_k(b)$ it is clear that each interval of the form $I_{n_{\ell-1}}(b)$ can be covered by $g_\ell + 2$ adjacent intervals of the form $I_{n_\ell}(b')$. To be more precise, there is $c \in \mathbb{N}$ such that

$$I_{n_{\ell-1}}(b) \subset I_{n_\ell}(c) \cup I_{n_\ell}(c+1) \cup \cdots \cup I_{n_\ell}(c + g_\ell + 1). \tag{3.9}$$

In the first step of our proof we subdivide $[-\beta/G_k, 1-\beta/G_k)$ ($\equiv [0,1) \bmod 1$) into intervals of the form $I_{n_0}(b)$ to obtain (recall that $a = a_1$)

$$J := \int_0^1 \prod_{\ell=1}^{k} F_{n_\ell,j_\ell}(y)\mathrm{d}y = \sum_{b_0=0}^{aG_{n_0}-1} \int_{I_{n_0}(b_0)} \prod_{\ell=1}^{k} F_{n_\ell,j_\ell}(y)\mathrm{d}y.$$

Since $n_0 = n_1 - j_1$, by definition, we have $F_{n_1,j_1}(y) = m(j_1, b_0)$ for $y \in I_{n_0}(b_0)$. Thus we may pull this constant out of the integral yielding

$$J \leq \sum_{b_0=0}^{aG_{n_0}-1} m(j_1,b_0) \int_{I_{n_0}(b_0)} \prod_{\ell=2}^{k} F_{n_\ell,j_\ell}(y)\mathrm{d}y.$$

Now we use (3.9) to cover each $I_{n_0}(b_0)$ by $g_1 + 2$ adjacent intervals of the form $I_{n_1}(b')$. More precisely, to each b_0 there is an integer $c_1(b_0)$ such that

$$I_{n_0}(b_0) \subset I_{n_1}(c_1(b_0)) \cup I_{n_1}(c_1(b_0)+1) \cup \cdots \cup I_{n_1}(c_1(b_0)+g_1+1).$$

Since the integrand is nonnegative this yields the estimate

$$J \leq \sum_{b_0=0}^{aG_{n_0}-1} m(j_1,b_0) \sum_{b_1=0}^{g_1+1} \int_{I_{n_1}(c_1(b_0)+b_1)} \prod_{\ell=2}^{k} F_{n_\ell,j_\ell}(y)\mathrm{d}y.$$

As before we have $F_{n_2,j_2}(y) = m(j_2, c_1(b_0)+b_1)$ for $y \in I_{n_1}(c_1(b_0)+b_1)$ and we may pull this constant out of the integral again to get

$$J \leq \sum_{b_0=0}^{aG_{n_0}-1} m(j_1,b_0) \sum_{b_1=0}^{g_1+1} m(j_2, c_1(b_0)+b_1) \int_{I_{n_1}(c_1(b_0)+b_1)} \prod_{\ell=3}^{k} F_{n_\ell,j_\ell}(y)\mathrm{d}y.$$

We may iterate this procedure $k-1$ times to subsequently pull out all factors from the integral. After this we end up with (the functions $c_2, \ldots, c_{k-1}$ are chosen in accordance with (3.9))

$$\begin{aligned} J \leq & \sum_{b_0=0}^{aG_{n_0}-1} m(j_1,b_0) \sum_{b_1=0}^{g_1+1} m(j_2, c_1(b_0)+b_1) \cdots \\ & \cdots \sum_{b_{k-1}=0}^{g_{k-1}+1} m(j_k, c_{k-1}(b_0,\ldots,b_{k-2})+b_{k-1}) \\ & \cdot \int_{I_{n_{k-1}}(c_{k-1}(b_0,\ldots,b_{k-2})+b_{k-1})} \mathrm{d}y \\ = & \frac{1}{aG_{n_{k-1}}} \sum_{b_0=0}^{aG_{n_0}-1} m(j_1,b_0) \sum_{b_1=0}^{g_1+1} m(j_2, c_1(b_0)+b_1) \cdots \\ & \cdots \sum_{b_{k-1}=0}^{g_{k-1}+1} m(j_k, c_{k-1}(b_0,\ldots,b_{k-2})+b_{k-1}). \end{aligned} \tag{3.10}$$

Thus we have to deal with sums of the form

$$K_\ell = \sum_{b=0}^{g_\ell+1} m(j_{\ell+1}, c+b) \qquad (c \in \mathbb{Z},\ \ell \in \{1,\dots,k-1\}). \tag{3.11}$$

We distinguish two cases according to whether $j_\ell = n_\ell - n_{\ell-1} = 1$ or not. If $j_\ell = n_\ell - n_{\ell-1} = 1$ then, for ℓ sufficiently large, $g_\ell = a$ (because (1.4) holds, where the dominant root α of the characteristic polynomial satisfies (1.3)) and, hence, for $j = j_{\ell+1}$ we get

$$\begin{aligned} K_\ell = \sum_{b=0}^{a+1} m(j, c+b) &\le am(j) + 2 \max_{0\le b<a} m(j,b) \\ &\le am(j) + 2a_j \le a(m(j)+2) \le \frac{G_{n_\ell}}{G_{n_{\ell-1}}}(m(j)+2). \end{aligned} \tag{3.12}$$

If $j_\ell = n_\ell - n_{\ell-1} > 1$ for ℓ sufficiently large, $g_\ell \ge a^2$ holds again by (1.4) and (1.3), and we may write $g_\ell + 1 = ha + r$ with $h \ge a$ and $0 \le r < a$ yielding (for $j = j_{\ell+1}$)

$$\begin{aligned} K_\ell &= \sum_{t=0}^{h-1}\sum_{u=0}^{a-1} m(j, c+ta+u) + \sum_{u=0}^{r} m(j, c+ha+u) \\ &\le ha\, m(j) + (r+1)a_j \le ha(m(j)+1) \le \frac{G_{n_\ell}}{G_{n_{\ell-1}}}(m(j)+1). \end{aligned}$$

Inserting this in (3.10) for all sufficiently large ℓ and observing that

$$\sum_{b_0=0}^{aG_{n_0}-1} m(j_1, b_0) \le a^2 G_{n_0}$$

we get the result. □

Proposition 3.2. *Let $G = (G_j)_{j\ge1}$ be a linear recurrence base with characteristic polynomial $X^d - a_1X^{d-1} - \dots - a_{d-1}X - a_d$ satisfying the conditions of Definition 1.1. Fix $k, n \in \mathbb{N}$ and let $j_1, j_2, \dots, j_k \in \mathcal{I}$. Then*

$$\int_0^1 \prod_{\ell=1}^{k} \left| A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right| \mathrm{d}y \ll (m+2)^k,$$

where $m = m_G$ is as in (3.7).

Proof. Using (3.1) the product may be rewritten as

$$\prod_{\ell=1}^{k} \left| A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right| = \prod_{\ell=1}^{k} f_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y).$$

The last product satisfies the conditions of Proposition 3.1 and we obtain our result by applying this proposition. □

We now state our estimate for the 1-norm of $S_n(\cdot,\beta)$. Note that in the following result the estimate $\|S_n(\cdot,\beta)\|_1 \ll \alpha^{\frac{n}{2}}$ is derived by easy general arguments (as in the classical case, see [22, Lemme 7] and [13, remarks after Théorème 2 and the beginning of Section IV]).

Proposition 3.3. *Let $G=(G_j)_{j\geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ having dominant root α. Then*

$$\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y \ll \min\{\alpha^{\frac{1}{2}},(m+3)\}^n,$$

where $m=m_G$ is as in (3.7).

Proof. We first show that $\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y \ll \alpha^{\frac{n}{2}}$. As in [22, Lemme 7], this immediately follows by applying the Cauchy–Schwarz inequality and Parseval's identity. Indeed, using (1.4) we obtain

$$\begin{aligned}\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y &\leq \left(\int_0^1 |S_n(y,\beta)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\\ &= \left(\int_0^1 \left|\sum_{k<G_n} e\left(\beta s_G(k)\right)e(yk)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}} \ll \alpha^{\frac{n}{2}}.\end{aligned}$$

It remains to prove that $\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y \ll (m+3)^n$. In view of (2.8) we have to deal with the cardinality of $J_k(d)$ before we can apply Proposition 3.2. To this matter let

$$C_{k,d}(n) = \left\{(j_1,\ldots,j_k)\in\{1,\ldots,d\}^k \;:\; n = j_1+\cdots+j_k\right\}.$$

It easy to see that $\#C_{k,d}(n) \leq \binom{n-1}{k-1}$ (there exist exact formulas, see e.g. Abramson [1]). Since $J_k(d)\subset\bigcup_{j=1}^d C_{k,d}(n-j)$ we gain $\#J_k(d) \ll \binom{n}{k}$. Using this in (2.8) together with Proposition 3.2 and the binomial theorem yields

$$\begin{aligned}\|S_n(y,\beta)\|_1 &\leq \sum_{k=1}^{n-d}\sum_{(j_1,\ldots,j_k)\in J_k(d)}\int_0^1\prod_{\ell=1}^k\left|A_{n-\sum_{r=1}^{\ell-1}j_r,j_\ell}(y,\beta)\right|\\ &\qquad\cdot|S_{n-j_1-\cdots-j_k}(y,\beta)|\,\mathrm{d}y\\ &\ll \sum_{k=1}^{n-d}\sum_{(j_1,\ldots,j_k)\in J_k(d)}\int_0^1\prod_{\ell=1}^k\left|A_{n-\sum_{r=1}^{\ell-1}j_r,j_\ell}(y,\beta)\right|\mathrm{d}y\\ &\ll \sum_{k=1}^{n-d}\sum_{(j_1,\ldots,j_k)\in J_k(d)}(m+2)^k \ll \sum_{k=1}^{n-d}\binom{n}{k}(m+2)^k\\ &\ll (m+3)^n.\end{aligned}$$

□

As mentioned at the beginning of Section 3 we also need the 1-norm of the derivative of $S_n(y,\beta)$ with respect to the first variable.

Proposition 3.4. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ having dominant root α. Then*

$$\int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \mathrm{d}y \ll \alpha^n \min\{\alpha^{\frac{n}{2}}, (m+3)^n\},$$

where $m = m_G$ is as in (3.7).

Proof. Again we use the Cauchy–Schwarz inequality, Parseval identity, and (1.4) to show that

$$\begin{aligned}\int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right|\mathrm{d}y &\leq \left(\int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}}\\ &= \left(\int_0^1 \left|\sum_{k<G_n} 2\pi k e\left(\beta s_G(k)\right)e(yk)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}}\\ &= \left(\sum_{k<G_n}(2\pi k)^2\right)^{\frac{1}{2}} \ll \alpha^{\frac{3n}{2}}.\end{aligned}$$

It remains to show that $\int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \mathrm{d}y \ll \alpha^n(m+3)^n$. Using Equation (2.8) we obtain for the 1-norm of the derivative of S_n that

$$\left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \ll \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_k)\in J_k(d)} \sum_{1\leq i\leq k} G_{n-\sum_{r=1}^{i} j_r} \prod_{\substack{\ell=1\\ \ell\neq i}}^{k} \left|A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right|.$$

Since $G_k \ll \alpha^k$ by (1.4) and $|A_{k,j}(y,\beta)| \leq a_j < \alpha$ by (3.1) and (1.3) we obtain

$$\begin{aligned}\left\|\frac{\partial S_n}{\partial y}(y,\beta)\right\|_1 \ll \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_k)\in J_k(d)} \sum_{1\leq i\leq k} \alpha^{n-\sum_{r=1}^{i} j_r} \alpha^{\sum_{r=1}^{i} j_r}\\ \cdot \int_0^1 \prod_{\ell=i+1}^{k} \left|A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right| \mathrm{d}y.\end{aligned}$$

Now an application of Proposition 3.2 yields

$$\left\|\frac{\partial S_n}{\partial y}(y,\beta)\right\|_1 \ll \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_k)\in J_k(d)} \alpha^n \sum_{1\leq i\leq k} (m+2)^{k-i} \ll \alpha^n (m+3)^n,$$

where we once more used that $J_k(d) \ll \binom{n}{k}$ and the binomial theorem. □

Remark 3.5. If we deal with particular cases of linear recurrences it is possible to improve the estimate in Propositions 3.3 and 3.4 slightly by the following consideration. Let $\varepsilon > 0$ be arbitrary. Then by (1.4) there is $N \in \mathbb{N}$ such that $\lfloor G_n/G_{n-1} \rfloor + 1 - G_n/G_{n-1} \geq \lfloor \alpha \rfloor + 1 - \alpha - \varepsilon := u$ holds for all $n \geq N$. Let r be the smallest positive integer satisfying $r^{-1} < u$. Since $I_n(c)$ is an interval of length $1/aG_n$ for each b there is $t \in R_r := \{0, r^{-1}, \dots, (r-1)r^{-1}\}$ and $c \in \mathbb{N}$ such that

$$I_{n-1}(b) \subset I_n(c+t) \cup I_n(c+1+t) \cup \ldots \cup I_n(c + \lfloor G_n/G_{n-1} \rfloor + t).$$

We use this instead of (3.9) in the proof of Proposition 3.1 whenever $n_\ell - n_{\ell-1} = 1$ and replace the maxima $m(a,b)$ by

$$m_G^{(t)}(j,b) = m^{(t)}(j,b) = \sup_{y \in (\frac{b+t}{a}, \frac{b+t+1}{a})} \left| \frac{\sin \pi a_j y}{\sin \pi y} \right| \qquad (j \in \mathcal{I}, b \in \mathbb{Z})$$

in these cases. Moreover, we set

$$m_G^{(r)} = m^{(r)} = \max_{j \in \mathcal{I}} m^{(r)}(j), \tag{3.13}$$

where $m_G^{(r)}(j) = \max_{t \in R_r} \{ \frac{1}{a} \sum_{b=0}^{a-1} m_G^{(t)}(j,b) \}$ for $j \in \mathcal{I}$. With these modifications the sum in the definition of K_ℓ in (3.11) runs only from 0 to g_ℓ and, hence, we get the better estimate $K_\ell \leq \frac{G_{n_\ell}}{G_{n_{\ell-1}}} (m_G^{(r)}(j) + 1)$ in (3.12). This entails that

$$\int_0^1 \prod_{\ell=1}^{k} f_{n_\ell, i_\ell}(y) \mathrm{d}y \ll (m^{(r)} + 1)^k. \tag{3.14}$$

Applying (3.14) in Proposition 3.2 instead of Proposition 3.1 we gain with the same proofs as the ones of Propositions 3.3 and 3.4 that

$$\begin{aligned} &\int_0^1 |S_n(y,\beta)| \, \mathrm{d}y \ll (m^{(r)} + 2)^n \\ \text{and} \quad &\int_0^1 \left| \frac{\partial S_n}{\partial y}(y,\beta) \right| \mathrm{d}y \ll \alpha^n (m^{(r)} + 2)^n. \end{aligned} \tag{3.15}$$

3.2. The maximum norm of sums related to $S_n(y,\beta)$. For certain $\beta \in \mathbb{Q}$ the maximum norm of $S_n(y,\beta)$ has been estimated by Lamberger and Thuswaldner [21]. However, for our purposes we require a variant of their estimate. To establish this variant we need some notation and some results from [21]. Let $G = (G_j)_{j \geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d - a_1 X^{d-1} - \cdots - a_{d-1} X - a_d$ having dominant root α. Fix $r, s \in \mathbb{N}$ and $y \in \mathbb{R}$ in a way that $\gcd(a_1 + \cdots + a_d - 1, s) = 1$ and $r \not\equiv 0 \pmod{s}$. According to [21], by iterating (2.5) in an appropriate way we can obtain a recurrence

$$S_n\Big(y,\frac{r}{s}\Big)=\sum_{j=1}^{D}B_{n,j}\Big(y,\frac{r}{s}\Big)S_{n-j}\Big(y,\frac{r}{s}\Big)\qquad (n\ge D) \tag{3.16}$$

for $S_n(y,\frac{r}{s})$ of order $D>d$ with coefficient functions $B_{n,j}(y,\frac{r}{s})$ having the following properties (for $a_1>1$ this recurrence is written explicitly in [21, Equation (5)] and for $a_1=1$ it is written in [21, Equation (12)]; however, we do not need these formulas here):

By [21, Proposition 1] there exist $b_1,\dots,b_D\in\mathbb{R}$ with $b_j\ge |B_{n,j}(y,\frac{r}{s})|$ for all $1\le j\le D$ and all $n\in\mathbb{N}$ such that the linear recurrent sequence

$$T_{n+D}=\sum_{j=1}^{D}b_jT_{n+D-j}\qquad (n\ge 0) \tag{3.17}$$

satisfies

$$\Big|S_n\Big(y,\frac{r}{s}\Big)\Big|<T_n\qquad (n\in\mathbb{N}) \tag{3.18}$$

for certain initial values $T_0,\dots,T_{D-1}\in\mathbb{R}_{>0}$. Moreover, from [21, Section 4.1] we see that there is a constant $\lambda=\lambda(G,s)<1$ such that

$$\alpha^{\lambda n}\ll T_n\ll\alpha^{\lambda n}\qquad (n\in\mathbb{N}). \tag{3.19}$$

We also need an analog of $J_k(n_0)$ from (2.7). For $D\le n_0<n$ and $1\le k<n$ let

$$\begin{aligned}K_k(n_0)&\\ &=\Big\{\mathbf{j}=(j_1,\dots,j_k)\in\{1,\dots,D\}^k\colon n-\sum_{\ell=1}^{k-1}j_\ell>n_0\ge n-\sum_{\ell=1}^{k}j_\ell\Big\}.\end{aligned} \tag{3.20}$$

Proposition 3.6. *Let $G=(G_j)_{j\ge0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d-a_1X^{d-1}-\cdots-a_{d-1}X-a_d$ having dominant root α. Let $n,r,s\in\mathbb{N}$ and $y\in\mathbb{R}$ be given in a way that* $\gcd(a_1+\cdots+a_d-1,s)=1$ *and* $r\not\equiv 0 \pmod s$. *Then for each $n_1\in\{D,\dots,n-1\}$ we have*

$$\sum_{k=1}^{n-n_1}\sum_{(j_1,\dots,j_k)\in K_k(n_1)}\prod_{\ell=1}^{k}\Big|B_{n-\sum_{r=1}^{\ell-1}j_r,j_\ell}\Big(y,\frac{r}{s}\Big)\Big|\ll\alpha^{\lambda(n-n_1)},$$

where $\lambda=\lambda(G,s)<1$ and the implied constant depends only on the linear recurrence base G and the integer s.

Proof. Let $(T_n)_{n\ge0}$ be the linear recurrent sequence given by (3.17) (with initial values satisfying (3.18)). By the definition of $b_1,\dots,b_D$, and $K_k(n_1)$

we get, using (3.19), that

$$\begin{aligned}\sum_{k=1}^{n-n_1} \sum_{(j_1,\dots,j_k)\in K_k(n_1)} \prod_{\ell=1}^{k} \left|B_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}\left(y, \frac{r}{s}\right)\right| &\le \sum_{k=1}^{n-n_1} \sum_{(j_1,\dots,j_k)\in K_k(n_1)} \prod_{\ell=1}^{k} b_{j_\ell} \\ &\ll \alpha^{-\lambda n_1} \sum_{k=1}^{n-n_1} \sum_{(j_1,\dots,j_k)\in K_k(n_1)} \left(\prod_{\ell=1}^{k} b_{j_\ell}\right) T_{n-\sum_{r=1}^{k} j_r} \\ &= \alpha^{-\lambda n_1} T_n \\ &\ll \alpha^{\lambda(n-n_1)}. \qquad \square\end{aligned}$$

We mention that, analogously to (2.8) we get the estimate

$$(3.21)\quad |S_n(y,\beta)| \le \sum_{k=1}^{n-n_0} \sum_{(j_1,\dots,j_k)\in K_k(n_0)} \prod_{\ell=1}^{k} \left|B_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right| \cdot \left|S_{n-\sum_{r=1}^{k} j_r}(y,\beta)\right|$$

for each $n_0 \in \{D, \dots, n-1\}$.

3.3. Upper bounds for m_G. Let G be a linear recurrence base as in Definition 1.1 and let α be the dominant root of the characteristic polynomial $X^d - a_1X^{d-1} - \dots - a_{d-1}X - a_d$ of G. According to Proposition 3.3 the 1-norm of $S_n(\,\cdot\,,\beta)$ can be easily bounded by $\alpha^{n/2}$ by using Cauchy's inequality followed by Parseval's identity. However, often Proposition 3.3 is of use only if this bound can be sharpened (and the same holds for Proposition 3.4). In particular, in view of (1.7) it will turn out that it is desirable to get $m_G + 3 \le \alpha^{0.4886061}$, where the quantity m_G is defined in (3.7). Such a sharpened bound is needed for instance in the proof of Corollary 1.4. Unfortunately, we are not able to achieve such an improvement for all G satisfying the conditions of Definition 1.1, however, we can achieve it if the coefficient a_1 is large enough. To get the threshold value for a_1 as low as possible we will now study m_G in some detail. Since the dependence of m_G on the linear recurrence base G will be crucial we keep the index G in m_G as well as in $m_G(j)$ (defined in (3.6)) throughout this section.

We start with the following estimate which is related to estimates established in [12, Section VI].

Lemma 3.7. *Let G be a linear recurrence base as in Definition 1.1 and let α be the dominant root of the characteristic polynomial $X^d - a_1X^{d-1} - \dots - a_{d-1}X - a_d$ of G. Let $m_G = \max_{j\in\mathcal{I}} m_G(j)$ with $m_G(j)$ as in* (3.6).

Then for $a_1 \geq 3$ we have

$$m_G \leq 2 + \frac{2}{a_1 \sin \frac{\pi}{a_1}} - \frac{2}{\pi} \log \tan \frac{\pi}{2a_1}. \tag{3.22}$$

This implies that $m_G \ll \log a_1 \ll \log \alpha$ for large a_1.

Proof. For convenience we set $a = a_1$. Fix G in a way that $a \geq 3$ and set $I(b) = (\frac{b}{a}, \frac{b+1}{a})$ for $b \in \mathbb{Z}$. First observe that, since $a_j \leq a$ for $j \in \mathcal{I}$,

$$m_G(j) \leq 2 + \frac{1}{a} \sum_{b=1}^{a-2} \sup_{y \in I(b)} \frac{1}{\sin \pi y}.$$

If $a \equiv 1 \pmod 2$ we obtain

$$\begin{aligned} m_G(j) &\leq 2 + \frac{2}{a} \sum_{b=1}^{\frac{a-3}{2}} \sup_{y \in I(b)} \frac{1}{\sin \pi y} + \frac{1}{a} \sup_{y \in I((a-1)/2)} \frac{1}{\sin \pi y} \\ &\leq 2 + \frac{2}{a} \sum_{b=1}^{\frac{a-1}{2}} \sup_{y \in I(b)} \frac{1}{\sin \pi y} \end{aligned} \tag{3.23}$$

for each interval $I(b)$ in the rightmost sum the supremum of $\frac{1}{\sin \pi y}$ is located on the left end point of $I(b)$. Thus

$$\begin{aligned} \sum_{b=1}^{\frac{a-1}{2}} \sup_{y \in I(b)} \frac{1}{\sin \pi y} &= \sum_{b=1}^{\frac{a-1}{2}} \frac{1}{\sin \frac{\pi b}{a}} \\ &\leq \frac{1}{\sin \frac{\pi}{a}} + \int_1^{\frac{a-1}{2}} \frac{\mathrm{d}x}{\sin \frac{\pi x}{a}} \\ &= \frac{1}{\sin \frac{\pi}{a}} + \frac{a}{\pi} \log \frac{\tan(\frac{\pi}{4} - \frac{\pi}{4a})}{\tan \frac{\pi}{2a}} \\ &\leq \frac{1}{\sin \frac{\pi}{a}} - \frac{a}{\pi} \log \tan \frac{\pi}{2a}. \end{aligned}$$

Inserting this in (3.23) we arrive at

$$m_G(j) \leq 2 + \frac{2}{a \sin \frac{\pi}{a}} - \frac{2}{\pi} \log \tan \frac{\pi}{2a}. \tag{3.24}$$

If $a \equiv 0 \pmod 2$ we obtain

$$m_G(j) \leq 2 + \frac{2}{a} \sum_{b=1}^{\frac{a}{2}-1} \sup_{y \in I(b)} \frac{1}{\sin \pi y}. \tag{3.25}$$

Similar to the case of odd a we now gain

$$\begin{aligned}\sum_{b=1}^{\frac{a}{2}-1}\sup_{y\in I(b)}\frac{1}{\sin\pi y}&=\sum_{b=1}^{\frac{a}{2}-1}\frac{1}{\sin\frac{\pi b}{a}}\\&\le\frac{1}{\sin\frac{\pi}{a}}+\int_1^{\frac{a}{2}-1}\frac{\mathrm{d}x}{\sin\frac{\pi x}{a}}\\&=\frac{1}{\sin\frac{\pi}{a}}+\frac{a}{\pi}\log\frac{\tan(\frac{\pi}{4}-\frac{\pi}{2a})}{\tan\frac{\pi}{2a}}\\&\le\frac{1}{\sin\frac{\pi}{a}}-\frac{a}{\pi}\log\tan\frac{\pi}{2a}.\end{aligned}$$

Inserting this in (3.25) we get (3.24) also in this case. The estimate (3.22) now follows from the definition of m_G. The asymptotic result is an immediate consequence of (3.22) since $\tan x\sim x$ for $x\to 0$, and $a\le\alpha$ holds by (1.3). □

The above result immediately implies that $m_G+3<\alpha^{0.4886061}$ holds for all $a_1\ge 72$. By calculating m_G directly (the suprema have to be approximated numerically which has been done using `Mathematica`) we get that this even holds for $a_1\ge 59$. Thus Proposition 3.3 and Proposition 3.4 immediately imply the following lemma.

Lemma 3.8. *Let $G=(G_j)_{j\ge0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 with characteristic polynomial $X^d-a_1X^{d-1}-\cdots-a_{d-1}X-a_d$ having dominant root α. If $a_1\ge 59$ then*

$$\int_0^1|S_n(y,\beta)|\,\mathrm{d}y\ll\alpha^{n\eta}\quad\text{and}\quad\int_0^1\left|\frac{\partial S_n}{\partial y}(y,\beta)\right|\mathrm{d}y\ll\alpha^{n(1+\eta)}$$

hold for some explicitly computable $\eta<0.4886061$.

For the special family $(G_j)_{j\ge0}$ with $G_{j+2}=a_1G_{j+1}+G_j$ we use Remark 3.5 to get this result for even smaller values of a_1. Indeed, if $a_1\ge 40$ we may choose $r=2$ in this remark and, again using `Mathematica`, we can calculate the quantity $m_G^{(2)}$ defined in (3.13) for $40\le a_1\le 58$. Since $m_G^{(2)}+2<\alpha^{0.4886061}$ holds for all $a_1\ge 40$, the estimate in (3.15) yields the following result.

Lemma 3.9. *Let $(G_j)_{j\ge0}$ be a linear recurrence base whose characteristic polynomial is given by X^2-a_1X-1 and has dominant root α. If $a_1\ge 40$ then*

$$\int_0^1|S_n(y,\beta)|\,\mathrm{d}y\ll\alpha^{n\eta}\quad\text{and}\quad\int_0^1\left|\frac{\partial S_n}{\partial y}(y,\beta)\right|\mathrm{d}y\ll\alpha^{n(1+\eta)}$$

hold for some explicitly computable $\eta<0.4886061$.

4. Estimates of the 1-norm for smaller values of a_1

4.1. Blocking. As mentioned at the beginning of Section 3.3, in order to derive results on almost primes we need to get good bounds for the 1-norm of $S_n(\,\cdot\,,\beta)$ and of its derivative. To obtain such good estimates also for smaller coefficients a_1, instead of taking suprema after each step of the recurrence, we deal with "blocks" or "windows" of "width" w and take the suprema after each w-th iteration. To keep things as simple as possible we only do this for recurrences having characteristic polynomial $X^2 - a_1X - 1$ for some $a_1 \geq 1$ (it should then be clear how to treat the general case). Thus in the present section $G = (G_n)$ is defined by

$$G_{n+2} = a_1G_{n+1} + G_n \qquad (n \geq 0) \tag{4.1}$$

with $G_0 = 1$ and $G_1 \geq a_1 + 1$. In this case (2.5) becomes

$$S_n(y,\beta) = A_{n,1}(y,\beta)S_{n-1}(y,\beta) + A_{n,2}(y,\beta)S_{n-2}(y,\beta) \qquad (n \geq 2). \tag{4.2}$$

Now set $A^{(1)}_{n,j}(y,\beta) = A_{n,j}(y,\beta)$ for $j \in \{1,2\}$ and recursively define

$$\begin{aligned} A^{(\ell)}_{n,\ell}(y,\beta) &= A^{(\ell-1)}_{n,\ell-1}(y,\beta)\cdot A_{n-\ell+1,1}(y,\beta) + A^{(\ell-1)}_{n,\ell}(y,\beta) \quad \text{and} \\ A^{(\ell)}_{n,\ell+1}(y,\beta) &= A^{(\ell-1)}_{n,\ell-1}(y,\beta)\cdot A_{n-\ell+1,2}(y,\beta). \end{aligned}$$

If we iterate (4.2) appropriately we obtain

$$S_n(y,\beta) = A^{(w)}_{n,w}(y,\beta)S_{n-w}(y,\beta) + A^{(w)}_{n,w+1}(y,\beta)S_{n-w-1}(y,\beta). \tag{4.3}$$

Setting

$$J^{(w)}_k = \left\{(j_1,\dots,j_k) \in \{w,w+1\}^k : n - \sum_{\ell=1}^{k-1} j_\ell > w+1 \geq n - \sum_{\ell=1}^{k} j_\ell\right\} \tag{4.4}$$

for $1 \leq k \leq \lfloor n/w \rfloor$ and iterating (4.3) we find in the same way as in the proof of Proposition 3.3 that

$$|S_n(y,\beta)| \ll \sum_{k=1}^{\lfloor n/w \rfloor} \sum_{(j_1,\dots,j_k)\in J^{(w)}_k} \prod_{\ell=1}^{k} \left| A^{(w)}_{n-\sum_{r=1}^{\ell-1} j_r,\, j_\ell}(y,\beta)\right|.$$

The functions $A^{(w)}_{n,j}(y,\beta)$ are exponential sums containing linear combinations of $G_{n-1},\dots,G_{n-w-2}$ in the exponents. Moreover, their definition implies that $|A^{(w)}_{n,j}(y,\beta)| \leq \alpha^j$ for $j \in \{w,w+1\}$, where α is the dominant root of the characteristic polynomial of G. Thus as in the proof of Proposition 3.4

we get

$$\left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \ll \sum_{k=1}^{\lfloor n/w\rfloor} \sum_{(j_1,\ldots,j_k)\in J_k^{(w)}} \sum_{1\le i\le k} G_{n-\sum_{r=1}^{i} j_r} \prod_{\substack{\ell=1\\ \ell\neq i}}^{k} \left|A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right|$$

$$\ll \alpha^n \sum_{k=1}^{\lfloor n/w\rfloor} \sum_{(j_1,\ldots,j_k)\in J_k^{(w)}} \sum_{1\le i\le k} \prod_{\ell=i+1}^{k} \left|A_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right|.$$

For the recurrences in (4.1) the asymptotic estimate (1.4) can clearly be strengthened to $G_\ell = c\alpha^\ell + \mathcal{O}(\alpha^{-\ell})$ for some $c>0$. Thus if we replace G_ℓ by $c\alpha^\ell$ in $A_{n,j}^{(w)}(y,\beta)$ and call the resulting expression $\widetilde{A}_{n,j}^{(w)}(y,\beta)$, we have $|\widetilde{A}_{k,j}^{(w)}(y,\beta) - A_{k,j}^{(w)}(y,\beta)| \ll \alpha^{-k}$ for $j\in\{w,w+1\}$. This entails that for each $\delta>0$ we have

$$|S_n(y,\beta)| \ll \sum_{k=1}^{\lfloor n/w\rfloor} \sum_{(j_1,\ldots,j_k)\in J_k^{(w)}} \prod_{\ell=1}^{k} \left(\left|\widetilde{A}^{(w)}_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y,\beta)\right| + \delta\right),$$

where the implied constant depends on δ. Obviously, an analogous estimate holds *mutatis mutandis* for $\left|\frac{\partial S_n}{\partial y}(y,\beta)\right|$. Instead of the intervals $I_k(b)$ defined in (3.2) we now use the intervals (as before we set $a=a_1$ for convenience)

$$I'_k(b) = \left[\frac{b-a\beta}{ac\alpha^k}, \frac{b+1-a\beta}{ac\alpha^k}\right) \qquad (0\le b<\lfloor ac\alpha^k\rfloor).$$

(For large k, the intervals $I_k(b)$ and $I'_k(b)$ are almost the same.) Now we define

$$M_w(j,b) = \sup_{y\in I'_{n-1}(b)} \left|\widetilde{A}_{n,j}^{(w)}(y,\beta)\right| + \delta \qquad (j\in\{w,w+1\},\ b\in\mathbb{Z}) \tag{4.5}$$

and note that $M_w(j,b)$ does not depend on n. Indeed, by (2.6) and the definition of $\widetilde{A}_{n,j}^{(w)}(y,\beta)$ the variable n in $\widetilde{A}_{n,j}^{(w)}(y,\beta)$ occurs only in linear combinations of terms of the form $\alpha^{n-k}y$ for some k depending only on a and w. Thus n cancels out if we insert the bounds of the interval $I'_{n-1}(b)$ for y. However, contrary to $m(j,b)$, the function $M_w(j,b)$ is in general *not* periodic in b (also note that, contrary to the definition of $m(j,b)$ we use $n-1$ instead of $n-j$ as index of $I'_{n-1}(b)$; this is because we want to split in *finer* subintervals in each step than we did in Section 3.1). Setting

$$F_{n,j}^{(w)}(y) = M_w(j,b) \quad \text{for} \quad y\in I'_{n-1}(b)$$

and integrating we gain

$$\int_0^1 |S_n(y,\beta)|\mathrm{d}y \ll \sum_{k=1}^{\lfloor n/w\rfloor} \sum_{(j_1,\ldots,j_k)\in J_k^{(w)}} \int_0^1 \prod_{\ell=1}^{k} F^{(w)}_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y)\mathrm{d}y \tag{4.6}$$

and

$$\text{(4.7)}\quad \int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \ll \alpha^n \sum_{k=1}^{\lfloor n/w\rfloor} \sum_{(j_1,\dots,j_k)\in J_k^{(w)}} \sum_{1\le i\le k} \int_0^1 \prod_{\ell=i+1}^{k} F^{(w)}_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}(y)\mathrm{d}y.$$

Writing

$$\text{(4.8)}\quad \begin{aligned} n_\ell &= n - \sum_{r=1}^{k-\ell} j_r \quad \text{for } 0\le \ell \le k \quad \text{and}\\ i_\ell &= n_\ell - n_{\ell-1} = j_{k+1-\ell} \quad \text{for } 1\le \ell\le k \end{aligned}$$

we now consider the integrals $\int_0^1 \prod_{\ell=1}^k F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y$ in (4.6) and $\int_0^1 \prod_{\ell=1}^{k-i} F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y$ in (4.7). From the definition of the intervals $I'_k(b)$ it is clear that each interval of the form $I'_{n_{\ell-1}}(b)$ can be covered by $\lfloor \alpha^{i_\ell}\rfloor + 2$ adjacent intervals of the form $I_{n_\ell}(b')$. To be more precise, there is $c\in\mathbb{N}$ such that

$$\text{(4.9)}\quad I'_{n_{\ell-1}}(b) \subset I'_{n_\ell}(c) \cup I'_{n_\ell}(c+1) \cup \cdots \cup I'_{n_\ell}(c+\lfloor\alpha^{i_\ell}\rfloor+1).$$

We can now argue in a similar way as in (3.10) to gain (the functions $c_1,\dots,c_{k-1}$ are chosen in accordance with (4.9))

$$\text{(4.10)}\quad \begin{aligned} \int_0^1 \prod_{\ell=1}^k F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y &= \int_0^1 F^{(w)}_{n_1,i_1}(y) \prod_{\ell=2}^k F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y\\ &\ll \sum_{b_0=0}^{\lfloor ac\alpha^{n_1-1}\rfloor} M_w(i_1,b_0) \int_{I'_{n_1-1}(b_0)} \prod_{\ell=2}^k F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y\\ &\ll \sum_{b_0=0}^{\lfloor ac\alpha^{n_1-1}\rfloor} M_w(i_1,b_0) \sum_{b_1=0}^{\lfloor\alpha^{i_2}\rfloor+1} M_w(i_2,c_1(b_0)+b_1)\\ &\qquad \cdot \int_{I'_{n_2-1}(c_1(b_0)+b_1)} \prod_{\ell=3}^k F^{(w)}_{n_\ell,i_\ell}(y)\mathrm{d}y\\ &\ll \frac{1}{ac\alpha^{n-1}} \sum_{b_0=0}^{\lfloor ac\alpha^{n_1-1}\rfloor} M_w(i_1,b_0) \sum_{b_1=0}^{\lfloor\alpha^{i_2}\rfloor+1} M_w(i_2,c_1(b_0)+b_1)\cdots\\ &\qquad \cdots \sum_{b_{k-1}=0}^{\lfloor\alpha^{i_k}\rfloor+1} M_w(i_k, c_{k-1}(b_0,\dots,b_{k-2})+b_{k-1}). \end{aligned}$$

Let

$$M_w(r) = \sup_{q\in\mathbb{Z}} \sum_{b=0}^{\lfloor \alpha^r \rfloor + 1} M_w(r, b+q) \qquad (r \in \{w, w+1\}). \tag{4.11}$$

According to (4.8) we have $i_\ell = n_\ell - n_{\ell-1} \in \{w, w+1\}$. Thus, if

$$s = s(j_1, \dots, j_k) = \#\{1 \le \ell \le k \,:\, i_\ell = w\} = \#\{1 \le \ell \le k \,:\, j_\ell = w\},$$

we have

$$\int_0^1 \prod_{\ell=1}^{k} F_{n_\ell, i_\ell}^{(w)}(y)\mathrm{d}y \ll \frac{1}{\alpha^n} M_w(w)^s M_w(w+1)^{k-s}. \tag{4.12}$$

If we iterate only $k-i-1$ times (instead of $k-1$ times) in (4.10) we get

$$\begin{aligned} \int_0^1 \prod_{\ell=1}^{k-i} F_{n_\ell, i_\ell}^{(w)}(y)\mathrm{d}y &\ll \frac{1}{\alpha^n} M_w(w)^{s(i)} M_w(w+1)^{k-i-s} \\ &\ll \frac{1}{\alpha^n} \max\{M_w(w), 1\}^s \max\{M_w(w+1), 1\}^{k-s} \end{aligned} \tag{4.13}$$

with $s(i) = \#\{1 \le \ell \le k-i \,:\, i_\ell = w\}$.

By the definition of $J_k^{(w)}$ in (4.4) the sum $ws(j_1, \dots, j_k) + (w+1) \times (k - s(j_1, \dots, j_k))$ must be close to n to make sure that $(j_1, \dots, j_k) \in J_k^{(w)}$ holds. Using this fact and inserting (4.12) in (4.6) we finally gain

$$\begin{aligned} &\int_0^1 |S_n(y, \beta)|\mathrm{d}y \\ &\quad \ll \frac{1}{\alpha^n} \sum_{k=1}^{\lfloor n/w \rfloor} \sum_{(j_1,\dots,j_k)\in J_k^{(w)}} M_w(w)^{s(j_1,\dots,j_k)} M_w(w+1)^{k-s(j_1,\dots,j_k)} \\ &\quad \ll \frac{1}{\alpha^n} \sum_{s=1}^{\lfloor n/w \rfloor} \binom{\lfloor n/w \rfloor}{s} M_w(w)^s M_w(w+1)^{(n-ws)/(w+1)} \\ &\quad = \frac{1}{\alpha^n} \sum_{s=1}^{\lfloor n/w \rfloor} \binom{\lfloor n/w \rfloor}{s} M_w(w)^s (M_w(w+1)^{w/(w+1)})^{n/w-s} \\ &\quad \ll \frac{1}{\alpha^n} \big(M_w(w) + M_w(w+1)^{w/(w+1)}\big)^{n/w}. \end{aligned} \tag{4.14}$$

Since $k \le n$ inserting (4.13) in (4.7) in an analogous way we derive

$$\begin{aligned} &\int_0^1 \left|\frac{\partial S_n}{\partial y}(y, \beta)\right| \mathrm{d}y \\ &\qquad \ll n\big(\max\{M_w(w), 1\} + \max\{M_w(w+1), 1\}^{w/(w+1)}\big)^{n/w}. \end{aligned} \tag{4.15}$$

As mentioned in Section 3.3 we want to get

$$\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y \ll \alpha^{n\eta} \quad \text{and} \quad \int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right| \mathrm{d}y \ll \alpha^{n(1+\eta)} \tag{4.16}$$

for some $\eta \leq 0.4886061$. Thus in view of (4.14) and (4.15) we are left with finding bounds for the suprema $M_w(w)$ and $M_w(w+1)$ that imply

$$\max\{M_w(w),1\} + \max\{M_w(w+1),1\}^{w/(w+1)} < \alpha^{1.4886061\cdot w}. \tag{4.17}$$

4.2. Blocks of width two. In this section we derive the estimate (4.16) for the recurrences (4.1) with $15 \leq a_1 \leq 39$ by setting $w = 2$ for the width of the blocks. Indeed, if we take $w = 2$ we get from (4.3) that

$$S_n(y,\beta) = A^{(2)}_{n,2}(y,\beta)S_{n-2}(y,\beta) + A^{(2)}_{n,3}(y,\beta)S_{n-3}(y,\beta)$$

with

$$\begin{aligned} A^{(2)}_{n,2}(y,\beta) &= A_{n,1}(y,\beta)A_{n-1,1}(y,\beta) + A_{n,2}(y,\beta) \quad \text{and} \\ A^{(2)}_{n,3}(y,\beta) &= A_{n,1}(y,\beta)A_{n-1,2}(y,\beta). \end{aligned}$$

Inserting $w = 2$ in (4.5) yields therefore (recall that $a = a_1$)

$$\begin{aligned} M_2(2,b) - \delta &= \sup_{y\in I'_{n-1}(b)} |\widetilde{A}_{n,1}(y,\beta)\widetilde{A}_{n-1,1}(y,\beta) + \widetilde{A}_{n,2}(y,\beta)| \\ &\leq \sup_{y\in I'_{n-1}(b)} \left|\frac{\sin \pi a(\beta + c\alpha^{n-1}y)}{\sin \pi(\beta + c\alpha^{n-1}y)} \frac{\sin \pi a(\beta + c\alpha^{n-2}y)}{\sin \pi(\beta + c\alpha^{n-2}y)}\right| + 1 \\ &= \sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)} \left|\frac{\sin \pi a y}{\sin \pi y} \frac{\sin \pi a(\beta(1-\alpha^{-1}) + \alpha^{-1}y)}{\sin \pi(\beta(1-\alpha^{-1}) + \alpha^{-1}y)}\right| + 1 \end{aligned}$$

and

$$M_2(3,b) - \delta = \sup_{y\in I_{n-1}(b)} |\widetilde{A}_{n,1}(y,\beta)\widetilde{A}_{n-1,2}(y,\beta)| = \sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)} \left|\frac{\sin \pi a y}{\sin \pi y}\right|.$$

Thus, setting $\delta' = (\lfloor\alpha^2\rfloor + 2)\delta$ we obtain from (4.11) that

$$\begin{aligned} &M_2(2) - \delta' \\ &= \sup_{q\in\mathbb{Z}} \sum_{b=0}^{\lfloor\alpha^2\rfloor+1} (M_2(2,b+q) - \delta) \\ &\leq \sup_{q\in\mathbb{Z}} \sum_{b=0}^{\lfloor\alpha^2\rfloor+1} \left(\sup_{y\in\left(\frac{b+q}{a},\frac{b+q+1}{a}\right)} \left|\frac{\sin \pi a y}{\sin \pi y} \frac{\sin \pi a(\beta(1-\alpha^{-1}) + \alpha^{-1}y)}{\sin \pi(\beta(1-\alpha^{-1}) + \alpha^{-1}y)}\right| + 1 \right). \end{aligned}$$

Since $\{1,\alpha^{-1}\}$ are rationally independent over $\mathbb{Q}$ we can omit the offset in the arguments of the sine functions in the last quotient without changing the supremum over $\mathbb{Z}$. This yields that

$$\begin{aligned}
&M_2(2)-\delta'\\
&\quad\le \sup_{q\in\mathbb{Z}}\sum_{b=0}^{\lfloor\alpha^2\rfloor+1}\left(\sup_{y\in\left(\frac{b+q}{a},\frac{b+q+1}{a}\right)}\left|\frac{\sin\pi ay}{\sin\pi y}\frac{\sin\pi a\alpha^{-1}y}{\sin\pi\alpha^{-1}y}\right|+1\right)\\
&\quad= \sup_{q\in\mathbb{Z}}\sum_{b=0}^{\lfloor\alpha^2\rfloor+1}\left(\sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)}\left|\frac{\sin\pi ay}{\sin\pi(y+\frac{q}{a})}\frac{\sin\pi a(\alpha^{-1}y+\frac{q\alpha^{-1}}{a})}{\sin\pi(\alpha^{-1}y+\frac{q\alpha^{-1}}{a})}\right|+1\right),
\end{aligned}$$

which is an estimate that is uniform in $\beta\in[0,1)$. Now we again use the rational independence of $\{1,\alpha^{-1}\}$ and the fact that $|\sin\pi(y+\frac{q}{a})|$ is periodic in $q\in\mathbb{Z}$ with period a to gain (setting $\gamma=\{q\alpha^{-1}/a\}$) that

$$\begin{aligned}
M_2(2)-\delta'\le{}&\lfloor\alpha^2\rfloor+1\\
&+\max_{q\in\{0,\dots,a-1\}}\sup_{\gamma\in[0,1)}\sum_{b=0}^{\lfloor\alpha^2\rfloor+1}\sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)}\left|\frac{\sin\pi ay}{\sin\pi(y+\frac{q}{a})}\frac{\sin\pi a(\alpha^{-1}y+\gamma)}{\sin\pi(\alpha^{-1}y+\gamma)}\right|.
\end{aligned}\tag{4.18}$$

We have to derive upper bounds for the right hand side. Set

$$g(x)=\frac{\sin\pi ax}{\sin\pi x}\qquad\text{and}\qquad h(y,\gamma,q)=g\Big(y+\frac{q}{a}\Big)g(\alpha^{-1}y+\gamma).$$

Then, Taylor expansion yields

$$\begin{aligned}
&|h(y,\gamma,q)|\\
&\le|h(y_0,\gamma_0,q)|+\frac{\varepsilon}{2}\max_{(y',\gamma')\in J(\varepsilon,\eta)}|h_y(y',\gamma',q)|+\frac{\eta}{2}\max_{(y',\gamma')\in J(\varepsilon,\eta)}|h_\gamma(y',\gamma',q)|
\end{aligned}$$

for $(y,\gamma)\in J(\varepsilon,\eta):=(y_0-\varepsilon/2,y_0+\varepsilon/2)\times(\gamma_0-\eta/2,\gamma_0+\eta/2)$; note that these intervals depend on y_0 and γ_0. We now want to estimate the derivatives. By the product rule,

$$\begin{aligned}
|h_y(y,\gamma,q)|&\le\Big|g'\Big(y+\frac{q}{a}\Big)g(\alpha^{-1}y+\gamma)\Big|+\alpha^{-1}\Big|g\Big(y+\frac{q}{a}\Big)g'(\alpha^{-1}y+\gamma)\Big|,\\
|h_\gamma(y,\gamma,q)|&=\Big|g\Big(y+\frac{q}{a}\Big)g'(\alpha^{-1}y+\gamma)\Big|.
\end{aligned}$$

Now, $|g(x)|\le a$ and by expanding g in an exponential series we get

$$|g'(x)|=\left|\sum_{j=0}^{a-1}2\pi\sqrt{-1}je(jx)\right|\le\pi a(a-1).$$

Inserting this in (4.18) yields that for each $\varepsilon, \eta > 0$

$$\begin{aligned} M_2(2) - \delta' &\le \lfloor \alpha^2 \rfloor + 1 \\ &+ \max_{q\in\{0,\dots,a-1\}} \max_{\substack{\gamma_0\in\{\ell\eta:\ell\in\mathbb{N}\}\\ \cap[0,1+\frac{\eta}{2})}} \sum_{b=0}^{\lfloor \alpha^2\rfloor+1} \max_{\substack{y_0\in\{\ell\varepsilon:\ell\in\mathbb{N}\}\\ \cap\left(\frac{b}{a}-\frac{\varepsilon}{2},\frac{b+1}{a}+\frac{\varepsilon}{2}\right)}} \Bigg\{ |h(y_0,\gamma_0,q)| \\ &\qquad + \frac{\varepsilon}{2} \max_{(y',\gamma')\in J(\varepsilon,\eta)} |h_y(y',\gamma',q)| + \frac{\eta}{2} \max_{(y',\gamma')\in J(\varepsilon,\eta)} |h_\gamma(y',\gamma',q)| \Bigg\}, \end{aligned}$$

and thus, again for each $\varepsilon, \eta > 0$,

$$\begin{aligned} &(4.19)\quad M_2(2) - \delta' \\ &\le \lfloor \alpha^2 \rfloor + 1 + \max_{q\in\{0,\dots,a-1\}} \max_{\substack{\gamma_0\in\{\ell\eta:\ell\in\mathbb{N}\}\\ \cap[0,1+\frac{\eta}{2})}} \sum_{b=0}^{\lfloor \alpha^2\rfloor+1} \max_{\substack{y_0\in\{\ell\varepsilon:\ell\in\mathbb{N}\}\\ \cap\left(\frac{b}{a}-\frac{\varepsilon}{2},\frac{b+1}{a}+\frac{\varepsilon}{2}\right)}} |h(y_0,\gamma_0,q)| \\ &\quad + \varepsilon a \max_{q\in\{0,\dots,a-1\}} \sum_{b=0}^{\lfloor \alpha^2\rfloor+1} \sup_{y\in\left(\frac{b}{a}-\frac{\varepsilon}{2},\frac{b+1}{a}+\frac{\varepsilon}{2}\right)} \left| g'\left(y+\frac{q}{a}\right)\right| \\ &\quad + \varepsilon\alpha^{-1}\pi a(a-1) \max_{q\in\{0,\dots,a-1\}} \sum_{b=0}^{\lfloor \alpha^2\rfloor+1} \sup_{y\in\left(\frac{b}{a}-\frac{\varepsilon}{2},\frac{b+1}{a}+\frac{\varepsilon}{2}\right)} \left| g\left(y+\frac{q}{a}\right)\right| \\ &\quad + \eta\pi a(a-1) \max_{q\in\{0,\dots,a-1\}} \sum_{b=0}^{\lfloor \alpha^2\rfloor+1} \sup_{y\in\left(\frac{b}{a}-\frac{\varepsilon}{2},\frac{b+1}{a}+\frac{\varepsilon}{2}\right)} \left| g\left(y+\frac{q}{a}\right)\right|. \end{aligned}$$

The estimation of $M_2(3)$ is much easier. By periodicity we have

$$\begin{aligned} (4.20)\quad M_2(3) - (\lfloor \alpha^3 \rfloor + 2)\delta &= \sup_{q\in\mathbb{Z}} \sum_{b=0}^{\lfloor \alpha^3\rfloor+1} (M_2(3, b+q) - \delta) \\ &\le \max_{q\in\{0,\dots,a-1\}} \sum_{b=0}^{\lfloor \alpha^3\rfloor+1} \sup_{y\in\left(\frac{b}{a},\frac{b+1}{a}\right)} \left| g\left(y+\frac{q}{a}\right)\right|. \end{aligned}$$

Treating the estimates (4.19) and (4.20) with `Mathematica` (accelerated by a `C` program for the calculation of the “main term” in the first line of (4.19)) and choosing $\delta = 10^{-10}$ led to the results displayed in Table 4.1 (recall again that $a = a_1$). This is used to prove the following lemma.

TABLE 4.1. Results of the computer calculations for the upper bound of $M_2 := \max\{M_2(2), 1\} + \max\{M_2(3), 1\}^{2/3}$ in comparison with α^3; see (4.17) with $w = 2$. The entry in the column "Power of α" is just the number κ satisfying $M_2 \leq \alpha^\kappa$ according to the estimate.

a_1	ε	η	Upper bound for M_2	Power of α	α^3
39	0.005	0.0005	46695.7	2.93416	59436
38	0.005	0.0005	43255.2	2.93405	54986
37	0.005	0.0005	39994.9	2.93398	50764
36	0.005	0.0005	36989.9	2.93458	46764
35	0.005	0.0008	39595.4	2.97694	42980
34	0.005	0.0008	36279.6	2.97656	39406
33	0.005	0.0008	33182.6	2.97641	36036
32	0.005	0.0008	30243.8	2.97603	32864
31	0.005	0.0008	27544.8	2.97627	29884
30	0.005	0.0008	24991.4	2.97630	27090
29	0.005	0.0008	22665.7	**2.97719**	24476
28	0.005	0.0007	19735.6	2.96693	22036
27	0.005	0.0007	17807.7	2.96839	19764
26	0.005	0.0007	16017.7	2.97016	17654
25	0.005	0.0007	14374.2	2.97261	15700
24	0.005	0.0007	12841.2	2.97517	13896
23	0.005	0.0006	11122.8	2.96960	12236
22	0.005	0.0006	9885.92	2.97399	10714
21	0.005	0.0005	8524.75	2.97059	9324
20	0.005	0.0005	7518.04	2.97678	8060
19	0.005	0.0004	6454.22	2.97655	6916
18	0.001	0.0004	5303.48	2.96398	5886
17	0.001	0.0004	4613.01	2.97415	4964
16	0.001	0.0001	3773.67	2.96628	4144
15	0.001	0.00003	3212.43	2.97692	3420

Lemma 4.1. *Let $(G_j)_{j\geq 0}$ be a linear recurrence base satisfying the conditions of Definition 1.1 whose characteristic polynomial is given by $X^2 - a_1X - 1$ and has dominant root α. If $15 \leq a_1 \leq 39$ then*

$$\int_0^1 |S_n(y,\beta)|\,\mathrm{d}y \ll \alpha^{n\eta} \quad \text{and} \quad \int_0^1 \left|\frac{\partial S_n}{\partial y}(y,\beta)\right|\mathrm{d}y \ll \alpha^{n(1+\eta)}$$

hold for some explicitly computable constant $\eta < 0.4886061$.

Proof. In view of (4.14), (4.15) and (4.17) we have to show that

$$M_2 := \max\{M_2(2), 1\} + \max\{M_2(3), 1\}^{2/3} < \alpha^{1.4886061\cdot 2} = \alpha^{2.9772122}. \tag{4.21}$$

This follows from the results listed in Table 4.1 (see the penultimate column whose largest value, which is typeset in boldface, is still smaller than 2.9772122). □

Using blocks of length greater than two with increasing effort it should be possible to treat even smaller values of a_1.

5. Proofs of the main results

5.1. Proof of Theorem 1.2. In order to prove Theorem 1.2 we have to show that the estimate in (2.3) holds. To this end we employ the exponential sum estimates established in Section 3. Moreover, we use the following inequality due to Sobolev and Gallagher (see [23, Lemma 1.2]).

Lemma 5.1. *Let $T_0, T \geq \delta > 0$ be real numbers and $f : [T_0, T_0 + T] \to \mathbb{C}$ a continuously differentiable function. Furthermore let $\mathcal{R} \subset [T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$ such that $|t - t'| \geq \delta$ holds for $t, t' \in \mathcal{R}$ with $t \neq t'$. Then we have the inequality*

$$\sum_{t\in\mathcal{R}} |f(t)| \leq \delta^{-1} \int_{T_0}^{T_0+T} |f(x)| \,\mathrm{d}x + \frac{1}{2} \int_{T_0}^{T_0+T} |f'(x)| \,\mathrm{d}x.$$

Proof of Theorem 1.2. We need to prove the estimate in (2.3). First we rewrite the sum on the left hand side of (2.3) to get

$$\sum_{Q<q\leq 2Q} \sum_{h=1}^{q-1} \left| S_n\left(\frac{h}{q}, \frac{r}{s}\right)\right| = \sum_{\delta=1}^{2Q} \sum_{Q\delta^{-1}<q\leq 2Q\delta^{-1}} \sum_{\substack{h=1\\(h,q)=1}}^{q-1} \left| S_n\left(\frac{h}{q}, \frac{r}{s}\right)\right|. \tag{5.1}$$

Now we concentrate on the two innermost sums and set

$$L_Q(\delta) := \sum_{Q\delta^{-1}<q\leq 2Q\delta^{-1}} \sum_{\substack{h=1\\(h,q)=1}}^{q-1} \left| S_n\left(\frac{h}{q}, \frac{r}{s}\right)\right|.$$

Let $D \in \mathbb{N}$ be the order of the recurrence in (3.16). Since the estimate in (2.3) is trivially true for $n \leq D$ we will assume that $n > D$ in the sequel. Using the product representation for S_n in (3.21) we obtain for each

$n_1 \in \{D, \dots, n-1\}$ the estimate (we use the abbreviation $\mathbf{j} = (j_1, \dots, j_k)$)

$$L_Q(\delta) \le \sum_{Q\delta^{-1} < q \le 2Q\delta^{-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \sum_{k=1}^{n-n_1} \sum_{\mathbf{j} \in K_k(n_1)} \prod_{\ell=1}^{k} \left| B_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}\left(\frac{h}{q}, \frac{r}{s}\right) \right| \cdot \left| S_{n-\sum_{r=1}^{k} j_r}\left(\frac{h}{q}, \frac{r}{s}\right) \right|.$$

Later we will choose n_1 depending on Q and δ. By the definition of $K_k(n_1)$ in (3.20) the index $n - \sum_{r=1}^{k} j_r$ always satisfies $n_1 - D < n - \sum_{r=1}^{k} j_r \le n_1$. Thus

$$\begin{aligned} L_Q(\delta) &\le \sum_{Q\delta^{-1} < q \le 2Q\delta^{-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \max_{n_1 - D < i \le n_1} \left| S_i\left(\frac{h}{q}, \frac{r}{s}\right) \right| \sum_{k=1}^{n-n_1} \sum_{\mathbf{j} \in K_k(n_1)} \prod_{\ell=1}^{k} \left| B_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}\left(\frac{h}{q}, \frac{r}{s}\right) \right| \\ &\ll \sum_{n_1 - D < i \le n_1} \sum_{Q\delta^{-1} < q \le 2Q\delta^{-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \left| S_i\left(\frac{h}{q}, \frac{r}{s}\right) \right| \sum_{k=1}^{n-n_1} \sum_{\mathbf{j} \in K_k(n_1)} \prod_{\ell=1}^{k} \left| B_{n-\sum_{r=1}^{\ell-1} j_r, j_\ell}\left(\frac{h}{q}, \frac{r}{s}\right) \right|. \end{aligned}$$

Now we apply Proposition 3.6 which yields (recall that α is the dominant root of the characteristic polynomial of G)

$$L_Q(\delta) \ll \sum_{n_1 - D < i \le n_1} \alpha^{\lambda(n-n_1)} \sum_{Q\delta^{-1} < q \le 2Q\delta^{-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \left| S_i\left(\frac{h}{q}, \frac{r}{s}\right) \right|$$

for some constant $\lambda < 1$. In this estimate λ and the implied constant depend only on G and s. In the next step we apply Lemma 5.1 together with the 1-norm estimates in Propositions 3.3 and 3.4. Setting $\eta = \log_\alpha \min\{\alpha^{\frac{1}{2}}, (m+3)\} \le \frac{1}{2}$ we get

$$\begin{aligned} L_Q(\delta) &\ll \sum_{n_1 - D < i \le n_1} \alpha^{\lambda(n-n_1)} \left(Q^2 \delta^{-2} \left\| S_i\left(\cdot, \frac{r}{s}\right) \right\|_1 + \left\| \frac{\partial S_i}{\partial y}\left(\cdot, \frac{r}{s}\right) \right\|_1 \right) \\ &\ll \alpha^{\lambda(n-n_1)} (Q^2 \delta^{-2} \alpha^{\eta n_1} + \alpha^{(1+\eta) n_1}). \end{aligned} \tag{5.2}$$

We choose n_1 by setting

$$n_1 := \min\left(\lfloor 2 \log_\alpha (Q\delta^{-1}) \rfloor + D, n-1 \right).$$

we gain (note that for $n_1 = \lfloor 2\log_\alpha(Q\delta^{-1})\rfloor + D$ both summands in (5.2) are roughly of the same size)

$$L_Q(\delta) \ll Q^2\delta^{-2}\alpha^{\eta n} + \alpha^{\lambda n}\alpha^{2(1+\eta-\lambda)\log_\alpha(Q\delta^{-1})} = Q^2\delta^{-2}\alpha^{\eta n} + \alpha^{\lambda n}(Q\delta^{-1})^{2(1+\eta-\lambda)}.$$

It suffices to prove the theorem for small ε. Thus if $\eta < \frac{1}{2}$ we may assume that $2\eta+\varepsilon < 1$. On top of this, for all $\eta \le \frac{1}{2}$ we may assume that ε is small enough and that the constant $\lambda < 1$ from Proposition 3.6 is close enough to 1 such that $\varepsilon(\frac{1}{2}-\varepsilon) < 1-\lambda < \frac{\varepsilon}{2}$ holds (note that if we increase λ, the estimate in Proposition 3.6 clearly remains valid). This yields

$$L_Q(\delta) \ll Q^2\delta^{-2}\alpha^{\eta n} + \alpha^{\lambda n}(Q\delta^{-1})^{2\eta+\varepsilon}$$

Taking into account the sum over δ in (5.1) we end up with

$$\sum_{Q<q\le 2Q}\sum_{h=1}^{q-1}\left|S_n\left(\frac{h}{q},\frac{r}{s}\right)\right| \ll \begin{cases} Q^2\alpha^{\eta n} + \alpha^{\lambda n}Q^{1+\varepsilon} & \text{if } \eta = \frac{1}{2}, \\ Q^2\alpha^{\eta n} + \alpha^{\lambda n}Q & \text{if } \eta < \frac{1}{2}. \end{cases} \tag{5.3}$$

Let $\vartheta = 1-\eta$. Then $\vartheta \ge \frac{1}{2}$ and by Lemma 3.7 we have $\vartheta \to 1$ for $a_1 \to \infty$. Also recall that $Q \le x^{\vartheta-\varepsilon}$ and $n \le \log_\alpha x + C$ for some constant C depending on G. Thus for $\eta = \frac{1}{2}$ we get

$$Q^2\alpha^{\eta n} + \alpha^{\lambda n}Q^{1+\varepsilon} \ll Qx^{\eta+\vartheta-\varepsilon} + Qx^{\lambda+\varepsilon(\vartheta-\varepsilon)} = Qx^{1-\varepsilon} + Qx^{\lambda+\varepsilon(\frac{1}{2}-\varepsilon)} \ll Qx^{\gamma} \tag{5.4}$$

for some $\gamma < 1$. For $\eta < \frac{1}{2}$ we gain

$$Q^2\alpha^{\eta n} + \alpha^{\lambda n}Q \ll Qx^{\eta+\vartheta-\varepsilon} + Qx^{\lambda} = Qx^{1-\varepsilon} + Qx^{\lambda} \ll Qx^{\lambda}. \tag{5.5}$$

Inserting (5.4) and (5.5) in (5.3) we finally see that

$$\sum_{Q<q\le 2Q}\sum_{h=1}^{q-1}\left|S_n\left(\frac{h}{q},\frac{r}{s}\right)\right| \ll Qx(\log 2x)^{-A}$$

holds for each $A > 0$ and the proof is finished. □

5.2. Improvements on the level of distribution and proof of Corollary 1.4. Let G be a linear recurrence base as in Definition 1.1 and let α be the dominant root of the characteristic polynomial $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$ of G. In the proof of Theorem 1.2 we see that the level of distribution $\vartheta(G)$ is equal to $1-\eta$ where η satisfies $\|S_n(\,\cdot\,,\beta)\|_1 \ll \alpha^{\eta n}$ and $\|\frac{\partial S_n}{\partial y}(\,\cdot\,,\beta)\|_1 \ll \alpha^{(1+\eta)n}$. Together with our estimates of these 1-norms, we gain the following result.

Lemma 5.2. *Let $G = (G_j)_{j\geq 0}$ be a linear recurrence base whose characteristic polynomial is given by $X^d - a_1X^{d-1} - \cdots - a_{d-1}X - a_d$. If $a_1 \geq 59$ then in Theorem 1.2 the level of distribution satisfies*

$$\vartheta(G) \geq 0.5113939 = 1 - 0.4886061.$$

If the characteristic polynomial of G is of the special form $X^2 - a_1X - 1$ then this estimate even holds for $a_1 \geq 15$.

Proof. From Lemma 3.8 we see that $\|S(\,\cdot\,,\beta)\| \ll \alpha^{0.4886061}$ for $a_1 \geq 59$. This proves the first assertion.

If the characteristic polynomial of G is of the special form $X^2 - a_1X - 1$ then for $a_1 \geq 40$ the result follows because Lemma 3.9 yields again $\|S(\,\cdot\,,\beta)\| \ll \alpha^{0.4886061}$. If $15 \leq a_1 \leq 39$ then the result is a consequence of Lemma 4.1. □

Along the lines indicated in Section 1.3 we can now prove Corollary 1.4.

Proof of Corollary 1.4. From Greaves [19, Proposition 1 (see also Theorem 1) of Chapter 5]) it follows that (1.6) holds provided that $\frac{1}{\vartheta(G)} < 2 - \delta_2$ for a certain constant δ_2. Since $\delta_2 = 0.044560$ is an admissible choice for this constant according to Greaves [18], we conclude that (1.6) holds if $\vartheta(G) > 0.5113938\ldots$ Since this is true in view of Lemma 5.2 whenever the conditions of the corollary are in force, the result is established. □

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