Journal de Théorie des Nombres de Bordeaux
anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 34, nº 1 (2022), p. 1-39.
https://doi.org/10.5802/jtnb.1192

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Journal de Théorie des Nombres de Bordeaux **34** (2022), 1–39

Energy Minimization Principle for non-archimedean curves

par Veronika WANNER

Résumé. Baker et Rumely ont défini la notion de fonction d'Arakelov–Green sur la droite projective analytifiée au sens de Berkovich et ont établi un principe de minimisation de l'énergie pour ces fonctions. Nous étendons leur définition et démontrons leur principe de minimisation de l'énergie pour les courbes projectives lisses générales. Comme application, nous obtenons une généralisation et une nouvelle démonstration d'un résultat d'équidistribution de Baker et Petsche.

Abstract. Baker and Rumely defined a notion of Arakelov–Green's functions on the Berkovich analytification of the projective line and established an Energy Minimization Principle. We extend their definition and show their Energy Minimization Principle for general smooth projective curves. As an application we get a generalization and a different proof of an equidistribution result by Baker and Petsche.

1. Introduction

Potential theory is a very old area of mathematics and has been extended to non-archimedean analytic geometry by many different authors. In the one-dimensional case this is for example done by Favre and Jonsson in [8] for the Berkovich projective line (indeed for any metric $\mathbb{R}$-tree), by Thuillier in [13] for general analytic curves and by Baker and Rumely in [5] also for the Berkovich projective line. One important theorem in potential theory is the so called Energy Minimization Principle. There are independent approaches of a non-archimedean version of this principle in the case of the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$, one by Favre and Rivera-Letelier in [9] and one by Baker and Rumely established in [5]. Both results are respectively used in [9] and in [4] as key tools for non-archimedean equidistribution results.

In this paper, we generalize Baker and Rumely's approach and extend all of their needed notions to the Berkovich analytification X^{an} of a smooth

Manuscrit reçu le 8 octobre 2019, révisé le 6 juin 2021, accepté le 1er octobre 2021.

Mathematics Subject Classification. 32P05, 14G22, 14T05, 32U05, 32U40.

Mots-clefs. Potential theory, Berkovich spaces, Equidistribution.

The author was supported by the collaborative research center SFB 1085 'Higher Invariants' funded by the Deutsche Forschungsgemeinschaft.

projective curve X over an algebraically closed non-archimedean field K. As an application we get a generalization and a different proof of an equidistribution result by Baker and Petsche in [3]. This work is part of the author's thesis [15]. As Baker and Rumely's non-archimedean potential theory is only established for the Berkovich analytification $\mathbb{P}^{1,\mathrm{an}}$ of the projective line, we work in Thuillier's general theory which he developed in his thesis [13]. Most important to us are his class of smooth functions A^0 with its corresponding measure valued Laplacian dd^c and his class of subharmonic functions.

For the Energy Minimization Principle, we need Arakelov–Green's functions g_μ defined on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ for given probability measures μ on X^{an} with continuous potentials analogous to the complex geometrical setting. For the definition of having continuous potentials we refer to Definition 7.3. This condition assures g_μ is well-defined and is lower semicontinuous on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. Complex Arakelov–Green's functions are characterized by a special list of properties. We extend the construction of Arakelov–Green's function from [5, §8.10] to our general smooth projective curve X such that the following analogous list is satisfied:

Theorem 1.1. *For a probability measure μ on X^{an} with continuous potentials, there exists a unique symmetric function $g_\mu \colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty, \infty]$ such that the following holds.*

(1) *(Semicontinuity) The function g_μ is finite and continuous off the diagonal and strongly lower semicontinuous on the diagonal in the sense that*

$$g_\mu(x_0, x_0) = \liminf_{(x,y)\to(x_0,x_0),\, x\neq y} g_\mu(x, y).$$

(2) *(Differential equation) For each fixed $y \in X^{\mathrm{an}}$ the function $g_\mu(\,\cdot\,, y)$ satisfies*

$$\mathrm{dd}^c g_\mu(\,\cdot\,, y) = \mu - \delta_y,$$

i.e. $\int g_\mu(x, y)\,(\mathrm{dd}^c f)(x) = \int f\,\mathrm{d}(\mu - \delta_y)(x)$ for all $f \in A^0_c(X^{\mathrm{an}})$.

(3) *(Normalization)*

$$\iint g_\mu(x, y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0.$$

The function g_μ is called the *Arakelov–Green's function* corresponding to μ. With the help of g_μ, we can define the *μ-energy integral* of an arbitrary probability measure ν on X^{an} as

$$I_\mu(\nu) := \iint g_\mu(x, y)\,\mathrm{d}\nu(y)\mathrm{d}\nu(x).$$

In Theorem 8.2 we formulate and prove the following Energy Minimization Principle analogous to the one in complex potential theory and [5, §8.10]:

Theorem 1.2 (Energy Minimization Principle)**.** *Let μ be a probability measure on X^{an} with continuous potentials. Then*

(1) $I_\mu(\nu) \geq 0$ *for each probability measure ν on X^{an}, and*
(2) $I_\mu(\nu) = 0$ *if and only if $\nu = \mu$.*

As a direct application of the Energy Minimization Principle, we can give a generalization and a different proof of the non-archimedean local discrepancy result from [3] for an elliptic curve E over K. Note that in [3] everything was worked out for K coming from a number field. For our general K, we define the *local discrepancy* of a subset $Z_n \subset E(K)$ consisting of n distinct points as

$$D(Z_n) := \frac{1}{n^2}\left(\sum_{P \neq Q \in Z_n} g_{\mu_E}(P,Q) + \frac{n}{12}\log^+|j_E|\right),$$

where μ_E is the canonical measure and j_E is the j-invariant of E (see Section 9 for definitions). Note that this definition is consistent with the definition of local discrepancy from [3] and [11]. We show in Corollary 9.4 the following generalization of [3, Corollary 5.6] using the Energy Minimization Principle:

Corollary 1.3. *For each $n \in \mathbb{N}$, let $Z_n \subset E(K)$ be a set consisting of n distinct points and let δ_n be the probability measure on E^{an} that is equidistributed on Z_n. If $\lim_{n\to\infty} D(Z_n) = 0$, then δ_n converges weakly to μ_E on E^{an}.*

Terminology. In this paper, let K be an algebraically closed field endowed with a complete, non-archimedean, non-trivial absolute value $|\cdot|$. A variety over K is an irreducible separated reduced scheme of finite type over K and a curve is a 1-dimensional variety over K.

Acknowledgments. The author would like to thank Matt Baker for the opportunity to visit him for a two months research stay and for suggesting the generalization of the Energy Minimization Principle to the author. The author is also grateful to Walter Gubler for very carefully reading drafts of this work and for the helpful discussions. Furthermore the author would like to thank the anonymous referee for the detailed report, which helped to improve the presentation of this paper.

2. Non-archimedean curves and their skeleta

Let X be an algebraic smooth projective curve X over K. Then by X^{an} we always denote the Berkovich analytification of X. We briefly recall the construction of this analytification.

Definition 2.1. For an open affine subset $U = \mathrm{Spec}(A)$ of X, the analytification U^{an} is the set of all multiplicative seminorms on A extending the given absolute value $|\cdot|$ on K. We endow the set U^{an} with the coarsest topology such that $U^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(a)$ is continuous for every element $a \in A$. By gluing, we get a topological space X^{an}, which is connected, compact and Hausdorff. We call the space X^{an} the *(Berkovich) analytification* of X which is a K-analytic space in the sense of [6, §3.1].

Remark 2.2. Note that the space X^{an} is in fact path-connected. Here a *path* from x to y is a continuous injective map $\gamma\colon [a,b] \to X^{\mathrm{an}}$ with $\gamma(a) = x$ and $\gamma(b) = y$. If there is a unique path between two points $x, y \in X^{\mathrm{an}}$, we write $[x,y]$ for this path. We often use the notations $(x,y) := [x,y]\backslash\{x,y\}$, $(x,y] := [x,y]\backslash\{x\}$ and $[x,y) := [x,y]\backslash\{y\}$.

Remark 2.3. The points of X^{an} can be classified in four different types following [6, §1.4], [13, §2.1] and [2, §3.5]. The points of type I can be identified with the rational points $X(K)$. By $I(X^{\mathrm{an}})$ we denote the subset of points of type II or III, and by $\mathbb{H}(X^{\mathrm{an}})$ the subset of points of type II, III and IV, i.e. $\mathbb{H}(X^{\mathrm{an}}) = X^{\mathrm{an}}\backslash X(K)$. For any subset S of X^{an}, we write $I(S) := S \cap I(X^{\mathrm{an}})$ and $\mathbb{H}(S) := S \cap \mathbb{H}(X^{\mathrm{an}})$. The sets $I(X^{\mathrm{an}})$ and $X(K)$ are dense in X^{an}.

Remark 2.4. When we talk about the boundary of a subset W of X^{an}, we always mean (if nothing is stated otherwise) the Berkovich boundary of W, which is the topological boundary in X^{an}. For an affinoid domain the Berkovich boundary coincides with Shilov boundary and the limit boundary, and it is always a finite set of points of type II or III in X^{an} (see [13, Proposition 2.1.12] for definitions and a proof). If the affinoid domain is strictly affinoid, all boundary points are of type II.

Due to the nice properties of the topological space X^{an}, finite signed Borel measures are automatically regular.

Proposition 2.5. *Every finite signed Borel measure on X^{an} is a signed Radon measure. In particular, every net $\langle \nu_\alpha \rangle_\alpha$ of probability measures ν_α on X^{an} has a subnet that converges weakly to a probability measure ν on X^{an}.*

Proof. The first assertion follows by [10, Theorem 7.8], as every open subset of the locally compact Hausdorff space X^{an} is the countable union of compact sets by [7, (2.1.5)]. Since every probability measure is so a Radon measure, the second assertion follows by the Prohorov's theorem for nets (see for example [5, Theorem A.11]). □

Remark 2.6. Another important property of the analytification X^{an} of a smooth projective curve X over K is the existence of so called *skeleta.*

Skeleta are deformation retracts of X^{an} and they have the structure of a metric graph. We refer to [1] for their definition via semistable vertex sets. Without loss of generality all of our considered skeleta do not have any loop edges (cf. [1, Corollary 3.14]). Note that their definition of skeleta is consistent with Thuillier's notion. For a skeleton Γ of X^{an} we write Γ_0 for its vertex set and τ_Γ for its retraction map.

Proposition 2.7. *As sets we have*

$$I(X^{\mathrm{an}}) = \bigcup_{\Gamma \text{ skeleton of } X^{\mathrm{an}}} \Gamma.$$

Proof. See [1, Corollary 5.1]. □

Proposition 2.8. *Let Γ be a skeleton of X^{an}, then the following are true:*

(1) *Γ is a connected, compact subset of points of type II and III and has the structure of a metric graph.*
(2) *For a finite subset $S \subset I(X^{\mathrm{an}})$, there is a skeleton Γ' of X^{an} such that Γ' contains Γ as a finite metric subgraph and $S \subset \Gamma'$.*
(3) *For a finite subset S of type II points in Γ, there is a skeleton of X^{an} such that Γ' contains Γ as a finite metric subgraph with $\Gamma_0 \cup S = \Gamma'_0$, i.e. Γ' and Γ are equal as sets.*

Proof. See [1, Lemma 3.4], [1, Lemma 3.13] and use the last proposition. □

Definition 2.9. With the help of the shortest-path metric on every skeleton and the fact that $I(X^{\mathrm{an}})$ can be exhausted by skeleta, one can define a metric ρ on $\mathbb{H}(X^{\mathrm{an}})$ (cf. [1, §5]), which is called the *skeletal metric*.

Definition 2.10. Let Γ be a skeleton of X^{an}. Then a subset Ω of Γ is a *star-shaped open subset* of Γ if Ω is a simply-connected open subset of Γ and there is a point $x_0 \in \Omega$ such that $\Omega \backslash \{x_0\}$ is a disjoint union of open intervals. We call x_0 the *center* of Ω.

Theorem 2.11. *Let $x_0 \in X^{\mathrm{an}}$. There is a fundamental system of open neighborhoods $\{V_\alpha\}$ of x_0 of the following form:*

(1) *If x_0 is of type I or type IV, then the V_α are open balls.*
(2) *If x_0 is of type III, then the V_α are open annuli with x_0 contained in the skeleton of the annulus V_α (cf.* [1, §2]*).*
(3) *If x_0 is of type II, then $V_\alpha = \tau_\Gamma^{-1}(\Omega_\alpha)$ for a skeleton Γ of X^{an} and a star-shaped open subset Ω_α of Γ. Hence each $V_\alpha \backslash \{x_0\}$ is a disjoint union of open balls and open annuli.*

Proof. See [1, Corollary 4.27]. □

Definition 2.12. An open subset of the described form in Theorem 2.11 is called *simple open*.

Remark 2.13. Theorem 2.11 implies directly that X^{an} is locally path-connected.

3. Subharmonic functions on non-archimedean curves

Thuillier developed in [13] a potential theory on non-archimedean curves, which is based on skeleta. In this section, we introduce his subharmonic functions on X^{an} via his class of smooth functions with their corresponding Laplacian.

Definition 3.1. Let Γ be a skeleton of X^{an}.

(1) A *piecewise affine function* on Γ is a continuous function $F\colon \Gamma \to \mathbb{R}$ such that $F|_e \circ \alpha_e$ is piecewise affine for every edge e of Γ, where α_e is an identification of e with a real closed interval.

(2) We define the *outgoing slope* of a piecewise affine function F on Γ at a point $x \in \Gamma$ along a tangent direction v_e at x corresponding to an adjacent edge e as

$$d_{v_e}F(x) := \lim_{\varepsilon \to 0} (F|_e \circ \alpha_e)'(\alpha_e^{-1}(x) + \varepsilon).$$

One obtains a finite measure on X^{an} by putting

$$\mathrm{dd}^c F := \sum_{x \in \Gamma} \left(\sum_{v_e} d_{v_e}F(x) \right) \delta_x,$$

where e is running over all edges in Γ at x. Since F is piecewise affine, we have $\sum_{v_e} d_{v_e}F(x) \neq 0$ for only finitely many points in Γ.

Definition 3.2. Let $W \subset X^{\mathrm{an}}$ be open. A continuous function $f\colon W \to \mathbb{R}$ is called *smooth* if for every point $x \in W$ there is a neighborhood V of x in W, a skeleton Γ of X^{an} and a piecewise affine function F on Γ such that

$$f = F \circ \tau_\Gamma$$

on V. We denote by $A^0(W)$ the vector space of smooth functions on W, and by $A^0_c(W)$ the subspace of smooth functions on W with compact support in W.

Remark 3.3. One should note that these smooth functions are not necessarily smooth in the sense of Chambert-Loir and Ducros from [7]. In [14] and [15] we work with both notions and so smooth functions in the sense of Thuillier from Definition 3.2 are called there lisse to distinguish them form those defined by Chambert-Loir and Ducros.

Definition 3.4. We write $A^1(W)$ for the set of real measures on W with discrete support in $I(W)$, and use $A^1_c(W)$ for those with compact support in W. Then for every smooth function $f \in A^0(W)$, there is a unique real measure $\mathrm{dd}^c f$ in $A^1(W)$ such that

$$\mathrm{dd}^c f = \mathrm{dd}^c F$$

whenever $f = F \circ \tau_\Gamma$ for a skeleton Γ of X^{an} (cf. [13, Théorème 3.2.10]). We call this linear operator $\mathrm{dd}^c\colon A^0(W) \to A^1(W)$ the *Laplacian.* Note that $A^0_c(W)$ is mapped to $A^1_c(W)$ under dd^c [13, Corollaire 3.2.11].

Proposition 3.5. *For any two points $x, y \in I(X^{\mathrm{an}})$ there is a unique smooth function $g_{x,y} \in A^0(X^{\mathrm{an}})$ such that*
(1) $\mathrm{dd}^c g_{x,y} = \delta_x - \delta_y$, *and*
(2) $g_{x,y}(x) = 0$.

Proof. See [13, Proposition 3.3.7]. □

Definition 3.6. Let W be an open subset of X^{an}. We denote by $D^0(W)$ (resp. $D^1(W)$) the dual of $A^1_c(W)$ (resp. $A^0_c(W)$).

Proposition 3.7. *The map*

$$\begin{aligned} D^0(W) &\longrightarrow \mathrm{Hom}(I(W), \mathbb{R}), \\ T &\longmapsto (x \mapsto \langle T, \delta_x \rangle) \end{aligned}$$

is an isomorphism of vector spaces.

Proof. See [13, Proposition 3.3.3]. □

In the following, we always use this identification.

Remark 3.8. The Laplacian $\mathrm{dd}^c\colon A^0_c(W) \to A^1_c(W)$ on an open subset $W \subset X^{\mathrm{an}}$ leads naturally by duality to an $\mathbb{R}$-linear operator

$$\begin{aligned} \mathrm{dd}^c\colon D^0(W) &\longrightarrow D^1(W), \\ T &\longmapsto (g \mapsto \langle \mathrm{dd}^c T, g \rangle := \langle T, \mathrm{dd}^c g \rangle) \end{aligned}$$

such that the following diagram commutes

$$\begin{array}{ccc} A^0(W) & \xrightarrow{\mathrm{dd}^c} & A^1(W) \\ \downarrow & & \downarrow \\ D^0(W) & \xrightarrow{\mathrm{dd}^c} & D^1(W). \end{array}$$

Note that the vertical maps are the natural inclusion maps.

Definition 3.9. We say that a current $T \in D^1(W)$ on an open subset W of X^{an} is *positive* if $\langle T, g \rangle \geq 0$ for every non-negative smooth function $g \in A^0_c(W)$.

Before introducing subharmonic functions we recall upper respectively lower semicontinuity.

Definition 3.10. A function $f\colon W \to [-\infty, \infty)$ on an open subset W of a topological space is *upper semicontinuous* in a point x_0 of W if

$$\limsup_{x \to x_0} f(x) \leq f(x_0),$$

where the limit superior in this context is defined as

$$\limsup_{x\to x_0} f(x) := \sup_{U\in\mathcal{U}(x_0)} \inf_{x\in U\setminus\{x_0\}} f(x),$$

where $\mathcal{U}(x_0)$ is any basis of open neighborhoods of x_0. We say that f is *upper semicontinuous on* W if it is upper semicontinuous in all points of W.

Analogously, a function $f\colon W \to (-\infty,\infty]$ on an open subset W of a topological space is *lower semicontinuous* in a point x_0 of W if

$$\liminf_{x\to x_0} f(x) \geq f(x_0),$$

where the limit inferior in this context is defined as

$$\liminf_{x\to x_0} f(x) := \inf_{U\in\mathcal{U}(x_0)} \sup_{x\in U\setminus\{x_0\}} f(x),$$

where $\mathcal{U}(x_0)$ is any basis of open neighborhoods of x_0. We say that f is *lower semicontinuous on* W if it is lower semicontinuous in all points of W.

Definition 3.11. Let W be an open subset of X^{an}. An upper semicontinuous function $f\colon W \to [-\infty,\infty)$ is called *subharmonic* if and only if $f \in D^0(W)$ and $\mathrm{dd}^c f \geq 0$.

A continuous function $h\colon W \to \mathbb{R}$ is called *harmonic* if h and $-h$ are subharmonic, i.e. $\mathrm{dd}^c h = 0$.

Remark 3.12. Note that this is not Thuillier's original definition of subharmonic functions, but it is equivalent by [13, Théorème 3.4.12]. Baker and Rumely independently introduced subharmonic functions on $\mathbb{P}^{1,\mathrm{an}}$. However, their class equals Thuillier's class of subharmonic functions in this special case.

If f is smooth, then f is subharmonic if and only if $\mathrm{dd}^c f$ is a positive measure [13, Proposition 3.4.4]. Moreover, note that a harmonic function h is automatically smooth i.e. $h \in A^0(W)$ by [13, Corollaire 3.2.11].

Proposition 3.13. *Let W be an open subset of X^{an}. Then a subharmonic function $f\colon W \to [-\infty,\infty)$ admits a local maximum in a point x_0 of W if and only if it is locally constant at x_0.*

Proof. See [13, Proposition 3.1.11]. □

Proposition 3.14. *The subharmonic functions form a sheaf on X^{an}.*

Proof. See [13, Corollaire 3.1.13]. □

Remark 3.15. Let $f\colon W \to [-\infty,\infty)$ be a subharmonic function on an open subset W of X^{an} and let $[x_0,y_0]$ be an interval (i.e. a segment of an edge) in a skeleton Γ of X^{an} such that $\tau_\Gamma^{-1}((x_0,y_0)) \subset W$. Then one can show that f is convex restricted to the relative interior of $I=[x_0,y_0]$ (see for example [15, Remark 3.1.32]).

4. Potential kernel

On the way to define Arakelov–Green's functions and prove an Energy Minimization Principle, we have to introduce a lot of other things first. Our most fundamental tool is the potential kernel that is a function $g_\zeta(\,\cdot\,,y)$ for fixed ζ and y that inverts the Laplacian in the sense that $\mathrm{dd}^c g_\zeta(\,\cdot\,,y) = \delta_\zeta - \delta_y$. A function with this property was already seen in Proposition 3.5 for $\zeta, y \in I(X^{\mathrm{an}})$.

Definition 4.1. Let Γ be a metric graph. For fixed points $\zeta, y \in \Gamma$, let $g_\zeta(\,\cdot\,,y)_\Gamma\colon \Gamma \to \mathbb{R}_{\geq 0}$ be the unique piecewise affine function on Γ such that

(1) $\mathrm{dd}^c g_\zeta(\,\cdot\,,y)_\Gamma = \delta_\zeta - \delta_y$, and
(2) $g_\zeta(\zeta,y)_\Gamma = 0$.

We call $g_\zeta(x,y)_\Gamma$ the *potential kernel* on Γ. The Laplacian dd^c is defined as in Definition 3.1. More details in the context of general metric graphs can be found in [5, §3]. Note that the function $g_\zeta(x,y)_\Gamma$ is unique by [5, Proposition 3.2] and the normalization condition (2).

Lemma 4.2. *Let Γ be a metric graph, then the potential kernel $g_\zeta(x,y)_\Gamma$ on Γ is non-negative, bounded, symmetric in x and y, and jointly continuous in x, y, ζ. For every $\zeta' \in \Gamma$, we have*

$$g_\zeta(x,y)_\Gamma = g_{\zeta'}(x,y)_\Gamma - g_{\zeta'}(x,\zeta)_\Gamma - g_{\zeta'}(y,\zeta)_\Gamma + g_{\zeta'}(\zeta,\zeta)_\Gamma.$$

Proof. Follows by [5, Proposition 3.3]. □

Since every skeleton of X^{an} has the structure of a metric graph, we can define a potential kernel on every skeleton. Using the skeletal metric $\rho\colon \mathbb{H}(X^{\mathrm{an}}) \times \mathbb{H}(X^{\mathrm{an}}) \to \mathbb{R}_{\geq 0}$ from Definition 2.9, we can extend the potential kernel to all of X^{an}.

Remark 4.3. Let V be a uniquely path-connected subset of X^{an} and let ζ be a point in V. For two points $x, y \in V$, we denote by $w_\zeta(x,y)$ the unique point in V where the paths $[x,\zeta]$ and $[y,\zeta]$ first meet. For example, for a skeleton Γ of X^{an} and a point $x_0 \in \Gamma$, the subset $\tau_\Gamma^{-1}(x_0)$ is uniquely path-connected. We therefore can define for two points $x, y \in \tau_\Gamma^{-1}(x_0)$ the point $w_\Gamma(x,y) := w_{x_0}(x,y)$.

Definition 4.4. Let $\zeta \in I(X^{\mathrm{an}})$. We define the *potential kernel* $g_\zeta\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty, \infty]$ corresponding to ζ by

$$g_\zeta(x,y) := \begin{cases} \infty & \text{if } (x,y) \in \mathrm{Diag}(X(K)), \\ g_\zeta(\tau_\Gamma(x), \tau_\Gamma(y))_\Gamma & \text{if } \tau_\Gamma(x) \neq \tau_\Gamma(y), \\ g_\zeta(\tau_\Gamma(y), \tau_\Gamma(x))_\Gamma + \rho(w_\Gamma(x,y), \tau_\Gamma(y)) & \text{else} \end{cases}$$

for a skeleton Γ of X^{an} containing ζ and the skeletal metric $\rho\colon \mathbb{H}(X^{\mathrm{an}}) \times \mathbb{H}(X^{\mathrm{an}}) \to \mathbb{R}_{\geq 0}$. The different types and subsets of points in X^{an} were defined in Remark 2.3. Note that we have to exclude K-rational points in the diagonal first, otherwise the last alternative does not make sense as we need that $w_\Gamma(x,y)$ is in $\mathbb{H}(X^{\mathrm{an}})$. It is not necessary to exclude all points from the diagonal as the remaining points are covered by the last alternative.

Proposition 4.5. *The function g_ζ is well-defined for every $\zeta \in I(X^{\mathrm{an}})$.*

Proof. We have to show that g_ζ is independent of the skeleton Γ. Thus we consider $(x,y) \notin \mathrm{Diag}(X(K))$. Let Γ_1 and Γ_2 be two skeleta containing ζ, and we may assume that $\Gamma_1 \subset \Gamma_2$. Since Γ_1 is already a skeleton of X^{an}, Γ_2 arises by just adding additional edges and vertices to Γ_1 without getting new loops or cycles. Working inductively, we may assume that Γ_2 equals to the graph Γ_1 and one new edge e attached to a vertex z in Γ_1.

Note that for every $w \in \Gamma_1$, due to uniqueness of the potential kernel and because its Laplacian is supported on $\{\zeta, w\}$, we have

$$g_\zeta(\,\cdot\,,w)_{\Gamma_1} \equiv g_\zeta(\,\cdot\,,w)_{\Gamma_2} \text{ on } \Gamma_1, \text{ and } g_\zeta(\,\cdot\,,w)_{\Gamma_2} \equiv g_\zeta(z,w)_{\Gamma_2} \text{ on } e. \tag{4.1}$$

Hence $g_\zeta(v,w)_{\Gamma_1} = g_\zeta(v,w)_{\Gamma_2}$ for every pair $(v,w) \in \Gamma_1 \times \Gamma_1$.

First, we consider (x,y) with $\tau_{\Gamma_1}(x) \neq \tau_{\Gamma_1}(y)$. Then automatically $\tau_{\Gamma_2}(x) \neq \tau_{\Gamma_2}(y)$ and $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(y) = \tau_{\Gamma_2}(y)$. Without loss of generality $\tau_{\Gamma_1}(y) = \tau_{\Gamma_2}(y)$. As argued in (4.1) and using symmetry, we get

$$\begin{aligned} g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_1} &= g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_2}(y))_{\Gamma_2}. \end{aligned}$$

Note for the second equation that either $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(x) = z$ and $\tau_{\Gamma_2}(x) \in e$.

Now consider the case $\tau_{\Gamma_1}(x) = \tau_{\Gamma_1}(y)$, and we set $w := w_{\Gamma_1}(x,y)$ (cf. Remark 4.3). Then (4.1) implies

$$g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_1} = g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} = g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(y))_{\Gamma_2}. \tag{4.2}$$

Note again that either $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(x) = z$ and $\tau_{\Gamma_2}(x) \in e$. In the case $\tau_{\Gamma_2}(x) = \tau_{\Gamma_2}(y) = \tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x)$, then the line above implies the claim.

If $\tau_{\Gamma_2}(x) = \tau_{\Gamma_2}(y) \neq \tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x)$, then $\tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x) = z$ and we have

$$\rho(w, \tau_{\Gamma_1}(x)) = \rho(w, \tau_{\Gamma_2}(x)) + \rho(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(x)).$$

Identity (4.2) implies

$$\begin{aligned} g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(x))_{\Gamma_1} &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(x))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_2}(x))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_2}(x))_{\Gamma_2} - \rho(\tau_{\Gamma_1}(x), \tau_{\Gamma_2}(x)), \end{aligned}$$

where we use for the last equation that $g_\zeta(\,\cdot\,,\tau_{\Gamma_2}(x))_{\Gamma_2}$ restricted to the path $[z,\tau_{\Gamma_2}(x)]$ is affine with slope 1. Adding these two equations up, we get

$$g_\zeta(\tau_{\Gamma_1}(x),\tau_{\Gamma_1}(x))_{\Gamma_1}+\rho(w,\tau_{\Gamma_1}(x))=g_\zeta(\tau_{\Gamma_2}(x),\tau_{\Gamma_2}(x))_{\Gamma_2}+\rho(w,\tau_{\Gamma_2}(x))$$

as we desired.

If $\tau_{\Gamma_2}(x)\neq\tau_{\Gamma_2}(y)$, then $z=\tau_{\Gamma_1}(x)=\tau_{\Gamma_1}(y)$ and $w=\tau_{\Gamma_2}(x)$ or $w=\tau_{\Gamma_2}(y)$. Without loss of generality, $w=\tau_{\Gamma_2}(y)$. The potential kernel $g_\zeta(\,\cdot\,,\tau_{\Gamma_2}(x))_{\Gamma_2}$ restricted to the path $[z,\tau_{\Gamma_2}(x)]$ (which contains $w=\tau_{\Gamma_2}(y)$) is affine with slope 1. Hence (4.2) and symmetry yield

$$\begin{aligned}g_\zeta(\tau_{\Gamma_1}(x),\tau_{\Gamma_1}(y))_{\Gamma_1}&=g_\zeta(\tau_{\Gamma_1}(y),\tau_{\Gamma_2}(x))_{\Gamma_2}\\&=g_\zeta(\tau_{\Gamma_2}(y),\tau_{\Gamma_2}(x))_{\Gamma_2}-\rho(\tau_{\Gamma_1}(y),w).\end{aligned}$$

Consequently, $g_\zeta(x,y)$ is well-defined. □

Remark 4.6. If $X=\mathbb{P}^1$, it is easy to see that the function g_ζ coincides with the potential kernel j_ζ from [5, §4.2] for every $\zeta\in I(X^{\mathrm{an}})$. One should also mention that the potential kernel $j_\zeta(x,y)$ on a finite $\mathbb{R}$-tree coincides with the Gromov product $(x|y)_\zeta$ which plays an important role in [9].

Lemma 4.7. *Fix $\zeta\in I(X^{\mathrm{an}})$. As a function of two variables $g_\zeta(x,y)$ satisfies the following properties:*

(1) *It is non-negative and $g_\zeta(\zeta,y)=0$.*
(2) $g_\zeta(x,y)=g_\zeta(y,x)$.
(3) *For every $\zeta'\in I(X^{\mathrm{an}})$, we have*

$$g_\zeta(x,y)=g_{\zeta'}(x,y)-g_{\zeta'}(x,\zeta)-g_{\zeta'}(y,\zeta)+g_{\zeta'}(\zeta,\zeta). \tag{4.3}$$

(4) *It is finitely valued and continuous off the diagonal and it is lower semicontinuous on $X^{\mathrm{an}}\times X^{\mathrm{an}}$ (where we understand $X^{\mathrm{an}}\times X^{\mathrm{an}}$ set theoretically and endowed with the product topology).*

Proof. All properties follow by construction and the properties of the potential kernel on a metric graph from Lemma 4.2. Note for the third assertion that we choose a skeleton such that $\zeta,\zeta'\in\Gamma$. A detailed proof of property (4) can be found in [15]. □

Proposition 4.8. *For fixed points $\zeta\in I(X^{\mathrm{an}})$ and $y\in X^{\mathrm{an}}$, we consider the function $G_{\zeta,y}:=g_\zeta(\,\cdot\,,y)\colon X^{\mathrm{an}}\to(-\infty,\infty]$. Then $G_{\zeta,y}$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$\mathrm{dd}^cG_{\zeta,y}=\delta_\zeta-\delta_y.$$

Moreover, the following hold:

(1) *If y is of type II or III, then $G_{\zeta,y}\in A^0(X^{\mathrm{an}})$ and coincides with $g_{y,\zeta}$ from Proposition 3.5.*
(2) *If y is of type IV, the function $G_{\zeta,y}$ is finitely valued and continuous on X^{an}.*

(3) *If y is of type I, then $G_{\zeta,y}$ is finitely valued on $X^{\mathrm{an}}\backslash\{y\}$ and continuous on X^{an} when we endow $(-\infty,\infty]$ with the topology of a half-open interval.*

Hence $G_{\zeta,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$ for every fixed $y \in X^{\mathrm{an}}$.

Proof. First, note that by construction $G_{\zeta,y}(x) = \infty$ if and only if $x = y \in X(K)$. Thus the restriction of $G_{\zeta,y}$ to $I(X^{\mathrm{an}})$ is always finite, and so $G_{\zeta,y}$ defines a current in $D^0(X^{\mathrm{an}})$. Here, one should have in mind that the vector space $D^0(X^{\mathrm{an}})$ is isomorphic to the vector space $\mathrm{Hom}(I(X^{\mathrm{an}}),\mathbb{R})$ endowed with the topology of pointwise convergence (see Proposition 3.7). We always use this identification.

Let Γ always be a skeleton that contains ζ. To calculate the Laplacian, we first consider a point $y \in I(X^{\mathrm{an}})$. We may extend Γ such that $y \in \Gamma$. Then $G_{\zeta,y} = g_\zeta(\,\cdot\,,y)_\Gamma \circ \tau_\Gamma$ on X^{an} since

$$\rho(w_\Gamma(x,y),y) = \rho(\tau_\Gamma(x),\tau_\Gamma(y)) = 0$$

if $\tau_\Gamma(x) = \tau_\Gamma(y) = y$. Since the potential kernel $g_\zeta(\,\cdot\,,y)_\Gamma$ is the unique piecewise affine function on the metric graph Γ such that $\mathrm{dd}^c g_\zeta(\,\cdot\,,y)_\Gamma = \delta_\zeta - \delta_y$ and $g_\zeta(\zeta,y)_\Gamma = 0$, we have $G_{\zeta,y} = g_\zeta(\,\cdot\,,y) \in A^0(X^{\mathrm{an}})$ and $\mathrm{dd}^c G_{\zeta,y} = \delta_\zeta - \delta_y$ on X^{an} by the construction of the Laplacian. In particular, the function $G_{\zeta,y}$ is continuous on X^{an}. Uniqueness in Proposition 3.5 implies that $G_{\zeta,y}$ coincides with $g_{y,\zeta}$.

Now consider an arbitrary $y \in X^{\mathrm{an}}\backslash I(X^{\mathrm{an}})$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points $y_n \in I(X^{\mathrm{an}})$ converging to y. Then $g_\zeta(\,\cdot\,,y_n)$ converges to $g_\zeta(\,\cdot\,,y)$ in the topological vector space $\mathrm{Hom}(I(X^{\mathrm{an}}),\mathbb{R}) \simeq D^0(X^{\mathrm{an}})$, i.e. for every fixed point $x \in I(X^{\mathrm{an}})$ we have $g_\zeta(x,y_n) = g_\zeta(y_n,x)$ converges to $g_\zeta(x,y) = g_\zeta(y,x)$ for $n \to \infty$ since $g_\zeta(\,\cdot\,,x)$ is smooth, and so continuous. The differential operator $\mathrm{dd}^c\colon D^0(X^{\mathrm{an}}) \to D^1(X^{\mathrm{an}})$ is continuous by [13, Proposition 3.3.4], and hence $\mathrm{dd}^c g_\zeta(\,\cdot\,,y) = \delta_\zeta - \delta_y$.

If y is a point of type I or IV, the connected component U of $X^{\mathrm{an}}\backslash\Gamma$ containing y is an open ball. For a type IV point y, we have

$$\begin{aligned} G_{\zeta,y}(x) &= g_\zeta(\tau_\Gamma(y),\tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(x,y),\tau_\Gamma(x)) \\ &= g_\zeta(\tau_\Gamma(y),\tau_\Gamma(y))_\Gamma + \rho(w_{\tau_\Gamma(y)}(x,y),\tau_\Gamma(x)) \end{aligned}$$

for every $x \in U$. Note that $\tau_\Gamma(x) = \tau_\Gamma(y)$ and $\zeta \in \Gamma$. If y is of type I, we have this identity on $U\backslash\{y\}$. Since the path distance metric ρ is continuous on U, it follows that $G_{\zeta,y}$ is continuous on U in both cases with $\lim_{x\to y} G_{\zeta,y}(x) = G_{\zeta,y}(y) = \infty$ if y is of type I.

In particular, $G_{\zeta,y}$ is upper semicontinuous on $X^{\mathrm{an}}\backslash\{y\}$ with $\mathrm{dd}^c G_{\zeta,y} = \delta_\zeta - \delta_y$ for every fixed $y \in X^{\mathrm{an}}$. Hence $G_{\zeta,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$. □

To introduce a capacity theory and define potential functions on X^{an} in the following sections, we define a potential kernel $g_\zeta(x,y)$ for every point

$\zeta \in X^{\mathrm{an}}$ (cf. [5, §4.4]). In [5], this is done with the help of the Gauss point. In our case, we have to fix a base point for the definition.

Definition 4.9. Fix $\zeta_0 \in I(X^{\mathrm{an}})$. We define $g_{\zeta_0}\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \times X^{\mathrm{an}} \to [-\infty, \infty]$ as $g_{\zeta_0}(\zeta, x, y) = \infty$ if $x = y = \zeta \in X(K)$ and else as

$$g_{\zeta_0}(\zeta, x, y) := g_{\zeta_0}(x, y) - g_{\zeta_0}(x, \zeta) - g_{\zeta_0}(y, \zeta).$$

Note that $g_{\zeta_0}(x, y) = \infty$ if and only if $x = y \in X(K)$ and so $g_{\zeta_0}(\zeta, x, y)$ is well-defined.

Corollary 4.10. *For fixed points $\zeta_0 \in I(X^{\mathrm{an}})$ and $\zeta, y \in X^{\mathrm{an}}$, the potential kernel $g_{\zeta_0}(\zeta, \cdot\,, y)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$dd^c g_{\zeta_0}(\zeta, \cdot\,, y) = \delta_\zeta - \delta_y,$$

and it extends $g_\zeta(x, y)$ in the sense that

$$g_\zeta(\zeta, x, y) = g_\zeta(x, y)$$

if $\zeta \in I(X^{\mathrm{an}})$.

Proof. Follows directly by construction, Proposition 4.8 and linearity of dd^c. □

5. Capacity theory

The main goal of this paper is to prove an analogue of the Energy Minimization Principle. In order to do this, we need to prove some partial results as for example Frostman's theorem. One of the tools for showing Frostman's theorem is the notion of capacity. We therefore introduce capacity analogously as in [5, §6.1], show all needed properties, and compare our notion with Thuillier's capacity in [13, §3.6.1].

Definition 5.1. Let $\zeta_0 \in I(X^{\mathrm{an}})$ be a fixed base point. Then for a point $\zeta \in X^{\mathrm{an}}$ and for a probability measure ν on X^{an} with $\mathrm{supp}(\nu) \subset X^{\mathrm{an}} \backslash \{\zeta\}$, we define the *energy integral* as

$$I_{\zeta_0,\zeta}(\nu) := \iint g_{\zeta_0}(\zeta, x, y)\, \mathrm{d}\nu(x) \mathrm{d}\nu(y).$$

Recall from Definition 4.9 the extended potential kernel $g_{\zeta_0}(\zeta, \cdot\,, \cdot\,)$, which is lower semicontinuous on $X^{\mathrm{an}} \backslash \{\zeta\} \times X^{\mathrm{an}} \backslash \{\zeta\}$ by Lemma 4.7 and Proposition 4.8. Hence the Lebesgue integral with respect to ν is well-defined.

With the help of the energy integral, one can introduce *capacity* of a proper subset E of X^{an} with respect to $\zeta \in X^{\mathrm{an}} \backslash E$ as

$$\gamma_{\zeta_0,\zeta}(E) := e^{-\inf_\nu I_{\zeta_0,\zeta}(\nu)}$$

where ν varies over all probability measures supported on E. We say that E *has positive capacity* if there is a $\zeta_0 \in I(X^{\mathrm{an}})$ and a point $\zeta \in X^{\mathrm{an}} \backslash E$ such that $\gamma_{\zeta_0,\zeta}(E) > 0$, i.e. there exists a probability measure ν supported on E with $I_{\zeta_0,\zeta}(\nu) < \infty$. Otherwise, we say that E *has capacity zero*.

Remark 5.2. It follows from the definition of the capacity of E with respect to $\zeta \in X^{\mathrm{an}} \backslash E$ that

$$\gamma_{\zeta_0,\zeta}(E) = \sup_{E' \subset E,\, E' \text{compact}} \gamma_{\zeta_0,\zeta}(E').$$

Lemma 5.3. *Positive capacity is independent of the choice of the chosen base point ζ_0.*

Proof. Consider $\zeta_0, \zeta_0' \in I(X^{\mathrm{an}})$, a proper subset E of X^{an}, a point $\zeta \in X^{\mathrm{an}} \backslash E$ and a probability measure ν supported on E. We show that $I_{\zeta_0,\zeta}(\nu)$ is finite if and only if $I_{\zeta_0',\zeta}(\nu)$ is finite. Using Definition 4.9 and Lemma 4.7, we obtain for every $x, y \in E$ (note that $\zeta \notin E$) $g_{\zeta_0}(\zeta, x, y) = g_{\zeta_0'}(\zeta, x, y) + 2g_{\zeta_0'}(\zeta, \zeta_0) - g_{\zeta_0'}(\zeta_0, \zeta_0)$ where the last two terms are finite for all $\zeta \in X^{\mathrm{an}} \backslash E$. Considering the energy integrals, we get

$$I_{\zeta_0,\zeta}(\nu) = I_{\zeta_0',\zeta}(\nu) + 2g_{\zeta_0'}(\zeta, \zeta_0) - g_{\zeta_0'}(\zeta_0, \zeta_0).$$

Hence they differ by a finite constant. □

For the rest of the section, we therefore just fix a base point $\zeta_0 \in I(X^{\mathrm{an}})$.

Remark 5.4. Let E be a proper subset of X^{an} and let ν be a probability measure supported on E. Then for every $\zeta \in X^{\mathrm{an}} \backslash E$

$$\begin{aligned} I_{\zeta_0,\zeta}(\nu) &= \iint g_{\zeta_0}(\zeta, x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y) \\ &= \iint g_{\zeta_0}(x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y) - 2\int g_{\zeta_0}(x, \zeta)\, \mathrm{d}\nu(x), \end{aligned}$$

where the last term of the right hand side is finite since $g_{\zeta_0}(\,\cdot\,, \zeta)$ is continuous on the compact set $\mathrm{supp}(\nu)$ by Proposition 4.8. Thus $I_{\zeta_0,\zeta}(\nu)$ is finite if and only if $I_{\zeta_0,\xi}(\nu)$ is finite for every point $\xi \in X^{\mathrm{an}} \backslash E$.

Lemma 5.5. *If E is a proper subset of X^{an} containing a point of $\mathbb{H}(X^{\mathrm{an}})$, then E has positive capacity.*

Proof. Choose a point $\zeta \in X^{\mathrm{an}} \backslash E$ and assume there is a point $z \in \mathbb{H}(X^{\mathrm{an}}) \cap E$. Then the Dirac measure $\nu := \delta_z$ is a probability measure supported on E and

$$I_{\zeta_0,\zeta}(\nu) = \iint g_{\zeta_0}(\zeta, x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y) = g_{\zeta_0}(z, z) - 2g_{\zeta_0}(\zeta, z) < \infty$$

since $z \in \mathbb{H}(X^{\mathrm{an}})$. □

Note that $I_{\zeta_0,\zeta_0}(\nu)$ is also well-defined for a probability measure ν supported on X^{an} with $\zeta_0 \in \mathrm{supp}(\nu)$ as

$$I_{\zeta_0,\zeta_0}(\nu) = \iint g_{\zeta_0}(\zeta_0, x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y) = \iint g_{\zeta_0}(x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y)$$

by Corollary 4.10 and g_{ζ_0} is lower semicontinuous on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Lemma 4.7.

Lemma 5.6. *Let ζ be a point in X^{an}, let E be a subset of $X^{\mathrm{an}}\backslash\{\zeta\}$ that has capacity zero and let ν be a probability measure on X^{an}. If*

(1) $\mathrm{supp}(\nu) \subset X^{\mathrm{an}}\backslash\{\zeta\}$ *with* $I_{\zeta_0,\zeta}(\nu) < \infty$ *for some base point* $\zeta_0 \in I(X^{\mathrm{an}})$*, or*
(2) $\zeta \in I(X^{\mathrm{an}})$ *with* $I_{\zeta,\zeta}(\nu) < \infty$*,*

then $\nu(E) = 0$.

Proof. The proof is analogous to [5, Lemma 6.16]. Note that $g_{\zeta_0}(\,\cdot\,,\zeta)$ is continuous on the compact set $\mathrm{supp}(\nu)$ and $g_{\zeta_0}(x,y)$ as a function of two variables is lower semicontinuous on $\mathrm{supp}(\nu)\times\mathrm{supp}(\nu)$ (Proposition 4.8 and Lemma 4.7). Hence the extended potential kernel $g_{\zeta_0}(\zeta,x,y) = g_{\zeta_0}(x,y) - g_{\zeta_0}(x,\zeta) - g_{\zeta_0}(y,\zeta)$ is bounded from below on $\mathrm{supp}(\nu)\times\mathrm{supp}(\nu)$ by a constant if (1) is satisfied. If $\zeta \in I(X^{\mathrm{an}})$, then the function $g_\zeta(\zeta,x,y) = g_\zeta(x,y)$ (cf. Corollary 4.10) is lower semicontinuous on $X^{\mathrm{an}}\times X^{\mathrm{an}}$ by Lemma 4.7, and so also bounded from below on $\mathrm{supp}(\nu)$. In both cases let C be this constant. If $\nu(E) > 0$, then there is a compact subset e of E such that $\nu(e) > 0$. Consider the probability measure $\omega := (1/\nu(e))\cdot\nu|_e$ on e. Then

$$\begin{aligned} I_{\zeta_0,\zeta}(\omega) &= \iint g_{\zeta_0}(\zeta,x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) \\ &= \iint (g_{\zeta_0}(\zeta,x,y) - C)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) + \iint C\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) \\ &\leq \frac{1}{\nu(e)^2}\cdot\iint (g_{\zeta_0}(\zeta,x,y) - C)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y) + C \\ &= \frac{1}{\nu(e)^2}\cdot I_{\zeta_0,\zeta}(\nu) - \frac{\nu(E)^2}{\nu(e)^2}\cdot C + C < \infty \end{aligned}$$

contradicting that E has capacity zero. Note that in case (2) we have $\zeta_0 = \zeta$ in the calculation. ☐

Corollary 5.7. *Let ζ be a point in X^{an} and let E_n be a countable collection of Borel sets in $X^{\mathrm{an}}\backslash\{\zeta\}$ such that E_n has capacity zero for every $n \in \mathbb{N}$. Then the set $E := \bigcup_{n\in\mathbb{N}} E_n$ has capacity zero.*

Proof. Assume E has positive capacity, i.e. there is a $\zeta \in X^{\mathrm{an}}\backslash E$ and a probability measure ν supported on E such that $I_{\zeta_0,\zeta}(\nu) < \infty$. The set E is measurable since all E_n are, and $\sum_{n\in\mathbb{N}}\nu(E_n) \geq \nu(E) = 1$. Thus there has to be an E_n such that $\nu(E_n) > 0$ contradicting Lemma 5.6. ☐

Remark 5.8. Thuillier introduced in [13, §3.6.1] relative capacity in an open subset Ω of X^{an} with a non-empty boundary $\partial\Omega \subset I(X^{\mathrm{an}})$. The capacity of a compact subset E of Ω is then defined as

$$C(E,\Omega)^{-1} := \left(\inf_\nu \int_E\int_E -g_x(y)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y)\right) \in [0,\infty]$$

where ν runs over all probability measures supported on E. Here $g_x\colon \Omega \to [-\infty, 0)$ for $x \in \Omega$ is the unique subharmonic function on Ω such that

(1) $\mathrm{dd}^c g_x = \delta_x$, and
(2) $\lim_{y\in\Omega,\ y\to\zeta} g_x(y) = 0$

for every $\zeta \in \partial\Omega$ (see [13, Lemma 3.4.14]). This notion of relative capacity can be extended canonically to all subsets of Ω by

$$C(E,\Omega) := \sup_{E'\subset E \text{ compact}} C(E',\Omega).$$

One can show that when $\partial\Omega = \{\zeta\} \subset I(X^{\mathrm{an}})$, a subset E of Ω has positive capacity (as defined in Definition 5.1) if and only if $C(E,\Omega) > 0$ (cf. [15, Proposition 3.2.19]).

6. Potential functions

With the help of the potential kernel from Section 4, one can introduce potential functions on X^{an} attached to a finite signed Borel measure. Baker and Rumely defined these functions on the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$ in [5, §6.3]. For the generalization to X^{an}, we have to fix a type II or III point ζ_0 serving as a base point as the Gauss point does for $\mathbb{P}^{1,\mathrm{an}}$. We define potential functions with respect to this base point and use them to define Arakelov–Green's functions in Section 7. Later in Lemma 7.8, we see that the definition of the Arakelov–Green's functions is independent of this choice.

Definition 6.1. Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, we define the corresponding *potential function* as

$$u_{\zeta_0,\nu}(x,\zeta) := \int_{X^{\mathrm{an}}} g_{\zeta_0}(\zeta,x,y)\,\mathrm{d}\nu(y)$$

for every $x \in X^{\mathrm{an}}$. Here $g_{\zeta_0}(\zeta,x,y)$ is the potential kernel defined in Definition 4.9.

Lemma 6.2. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the function $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is well-defined on X^{an} with values in $\mathbb{R}\cup\{\pm\infty\}$ and we can write*

$$u_{\zeta_0,\nu}(\,\cdot\,,\zeta) = \int g_{\zeta_0}(\,\cdot\,,y)\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(\,\cdot\,,\zeta) + C_{\zeta_0,\zeta} \tag{6.1}$$

on X^{an} for a finite constant $C_{\zeta_0,\zeta}$.

Proof. By the definition of the potential kernel $g_{\zeta_0}(\zeta,x,y)$, we get for every $x \in X^{\mathrm{an}}$

$$u_{\zeta_0,\nu}(x,\zeta) = \int g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(x,\zeta) - \int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\nu(y).$$

Since $g_{\zeta_0}(\,\cdot\,,\zeta)$ is continuous on the compact subset $\mathrm{supp}(\nu)$ if $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$ (cf. Proposition 4.8), the last term is always a finite constant, and so we get the description in (6.1) with $C_{\zeta_0,\zeta} := -\int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\nu(y)$.

To prove that $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is well-defined, we have to show that $\infty - \infty$ or $-\infty + \infty$ cannot occur.

If $\zeta \in I(X^{\mathrm{an}})$, then $g_{\zeta_0}(x,\zeta)$ is finite for every $x \in X^{\mathrm{an}}$, and so $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is well-defined.

Next, we consider $\zeta \notin \mathrm{supp}(\nu)$. For every $x \neq \zeta$, we know that $g_{\zeta_0}(x,\zeta)$ is finite as well, and so $\infty - \infty$ or $-\infty + \infty$ cannot occur. It remains to show that the function is well-defined in $x = \zeta \notin \mathrm{supp}(\nu)$. Since $g_{\zeta_0}(x,\cdot)$ is continuous on the compact subset $\mathrm{supp}(\nu)$ as $x \notin \mathrm{supp}(\nu)$, the first term $\int g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y)$ is finite, and so $u_{\zeta_0,\nu}(x,\zeta)$ is well-defined if $x = \zeta$. □

Remark 6.3. Let ζ_0' be another chosen base point in $I(X^{\mathrm{an}})$. Then Lemma 4.7 implies that for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$ and $x \in X^{\mathrm{an}}$ we have

$$u_{\zeta_0,\nu}(x,\zeta) = u_{\zeta_0',\nu}(x,\zeta) + 2\nu(X^{\mathrm{an}})g_{\zeta_0'}(\zeta,\zeta_0) - \nu(X^{\mathrm{an}})g_{\zeta_0'}(\zeta_0,\zeta_0),$$

i.e. the corresponding potential function differ by a constant depending on ζ_0', ζ_0 and ζ.

Lemma 6.4. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every skeleton Γ of X^{an} or every path $\Gamma = [z,\omega] \subset \mathbb{H}(X^{\mathrm{an}})$, and for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the restriction of $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ to Γ is finite and continuous.*

Proof. First, we consider a skeleton Γ of X^{an}. We may assume $\zeta_0 \in \Gamma$ by Remark 6.3. Note that the potential kernel satisfies by construction a retraction formula as in [5, Proposition 4.5], i.e.

$$g_{\zeta_0}(x,y) = g_{\zeta_0}(x,\tau_\Gamma(y))_\Gamma = g_{\zeta_0}(x,\tau_\Gamma(y)) \tag{6.2}$$

for every $x \in \Gamma$ and $y \in X^{\mathrm{an}}$. Furthermore, recall the description of $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ in (6.1).

Then for every $x \in \Gamma$

$$\begin{aligned} u_{\zeta_0,\nu}(x,\zeta) &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(x,\zeta) + C_{\zeta_0,\zeta} \\ &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,\tau_\Gamma(y))_\Gamma\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(x,\tau_\Gamma(\zeta))_\Gamma + C_{\zeta_0,\zeta} \\ &= \int_{\Gamma} g_{\zeta_0}(x,t)_\Gamma\,\mathrm{d}((\tau_\Gamma)_*\nu)(t) - \nu(X^{\mathrm{an}})g_{\zeta_0}(x,\tau_\Gamma(\zeta))_\Gamma + C_{\zeta_0,\zeta}. \end{aligned}$$

The first term is finite and continuous by Lemma 4.2 and the second term is as well by Lemma 4.8. Hence $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is finite and continuous on Γ.

In the following, we consider a path $\Sigma := [z,\omega]$. Recall that $\mathbb{H}(X^{\mathrm{an}})$ is the set of points of type II, III and IV, and every point of type IV has only one tangent direction in X^{an} [1, Lemma 5.12]. We already know that $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ restricted to every skeleton is finite and continuous. Moreover, every path $[z,\omega]$ for $z,\omega\in I(X^{\mathrm{an}})$ lies in some skeleton. Thus it remains to consider paths of the form $[z,\tau_\Gamma(z)]$ for a type IV point z and an arbitrary large skeleton Γ of X^{an}. From now on let Σ be the considered path $[z,\omega]$ with $\omega:=\tau_\Gamma(z)$.

Let ζ_0 be some base point in $I(X^{\mathrm{an}})\cap\Gamma$, which we may choose that way by Remark 6.3. Again, we consider each term of

$$u_{\zeta_0,\nu}(x,\zeta)=\int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y)-\nu(X^{\mathrm{an}})g_{\zeta_0}(x,\zeta)+C_{\zeta_0,\zeta}$$

for $x\in\Sigma$ separately. The second term is finite and continuous in x by Proposition 4.8 (note that $\Sigma\cap X(K)=\emptyset$).

It remains to consider the first term. Let V be the connected component of $X^{\mathrm{an}}\backslash\Gamma$ containing z, which is an open ball with unique boundary point $\omega=\tau_\Gamma(z)$. We can consider the canonical retraction map $\tau_\Sigma\colon\overline{V}\to[z,\omega]$, where a point $x\in\overline{V}$ is retracted to $w_\Gamma(x,z)$ (cf. Remark 4.3). Note that for $x\in\Sigma$, we have

$$g_{\zeta_0}(x,y)=\begin{cases} g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma & \text{if } y\notin V,\\ g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma+\rho(w_\Gamma(x,y),\omega) & \text{if } y\in V=\tau_\Sigma^{-1}([z,\omega)).\end{cases}$$

Hence for $x\in\Sigma$ the following is true

$$\begin{aligned}\int_{X^{\mathrm{an}}} & g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y)\\ &=\int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma\,\mathrm{d}\nu(y)+\int_{\tau_\Sigma^{-1}((\omega,z])}\rho(w_\Gamma(x,y),\omega)\,\mathrm{d}\nu(y)\\ &=\int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma\,\mathrm{d}\nu(y)+\int_{\tau_\Sigma^{-1}((\omega,z])}\rho(w_\Gamma(x,\tau_\Sigma(y)),\omega)\,\mathrm{d}\nu(y)\\ &=\int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma\,\mathrm{d}\nu(y)+\int_{\Sigma}\rho(w_\Gamma(x,t),\omega)\,\mathrm{d}((\tau_\Sigma)_*\nu)(t)\\ &=\int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega,\tau_\Gamma(y))_\Gamma\,\mathrm{d}\nu(y)+\int_{\Sigma} g_\omega(x,t)_\Sigma\,\mathrm{d}((\tau_\Sigma)_*\nu)(t).\end{aligned}$$

Note that our path $\Sigma=[z,\omega]\subset\mathbb{H}(X^{\mathrm{an}})$ is a metric graph, and so we can consider the potential kernel $g_\omega(x,t)_\Sigma$ on Σ from Definition 4.1. For the last identity we used $\rho(w_\Gamma(x,t),\omega)=\rho(w_\omega(x,t),\omega)=g_\omega(x,t)_\Sigma$, which follows by Remark 4.6 and [5, §4.2 p. 77]. Then Lemma 4.2 tells us again that the second term is finite and continuous. As $g_{\zeta_0}(\omega,\tau_\Gamma(\,\cdot\,))_\Gamma=g_{\zeta_0}(\omega,\,\cdot\,)$ (see (6.2)) is finitely valued and continuous on the compact set $\mathrm{supp}(\nu)$ by Proposition 4.8, the first one is a finite constant, and hence the claim follows. $\square$

Proposition 6.5. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be a finite positive Borel measure on X^{an}. Then for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \operatorname{supp}(\nu)$ the following are true:*

1. *If $\zeta \notin X(K)$, then $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is finitely valued and continuous on $X^{\mathrm{an}}\backslash \operatorname{supp}(\nu)$ and it is lower semicontinuous on X^{an}.*
2. *If $\zeta \in X(K)$, then $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is continuous on $X^{\mathrm{an}}\backslash(\operatorname{supp}(\nu) \cup \{\zeta\})$ with $u_{\zeta_0,\nu}(x,\zeta) = \infty$ if and only if $x = \zeta$, and it is lower semicontinuous on $X^{\mathrm{an}}\backslash\{\zeta\}$.*
3. *For each $z \in X^{\mathrm{an}}$ and each path $[z,\omega]$, we have*

$$\begin{aligned}\liminf_{t\to z} u_{\zeta_0,\nu}(t,\zeta) &= \liminf_{\substack{t\to z,\\ t\in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta)\\ &= \lim_{\substack{t\to z,\\ t\in[\omega,z)}} u_{\zeta_0,\nu}(t,\zeta)\\ &= u_{\zeta_0,\nu}(z,\zeta). \end{aligned} \tag{6.3}$$

Proof. Recall from (6.1) that we can write

$$u_{\zeta_0,\nu}(\,\cdot\,,\zeta) = \int g_{\zeta_0}(\,\cdot\,,y)\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(\,\cdot\,,\zeta) + C_{\zeta_0,\zeta}.$$

Since $g_{\zeta_0}(\,\cdot\,,\zeta)$ is finitely valued and continuous on X^{an} if $\zeta \notin X(K)$ and $g_{\zeta_0}(\,\cdot\,,\zeta)$ is finitely valued and continuous on $X^{\mathrm{an}}\backslash\{\zeta\}$ if $\zeta \in X(K)$ by Proposition 4.8, it remains to show the assertions (1) and (2) for the function $f(x) := \int g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y)$ on X^{an}. As g_{ζ_0} is finitely valued and continuous off the diagonal by Lemma 4.7 and $\operatorname{supp}(\nu)$ is a compact subset, it follows that f is finitely valued and continuous on $X^{\mathrm{an}}\backslash \operatorname{supp}(\nu)$. For the lower semicontinuity of f we use techniques from the proof of [5, Proposition 6.12]. By Lemma 4.7, g_{ζ_0} is lower semicontinuous on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$, and so it is bounded from below by a constant M. Note that the finite positive Borel measure ν is a positive Radon measure on X^{an} by Proposition 2.5. Using [5, Proposition A.3], we get the identity

$$f(x) = \sup\left\{ \int_{X^{\mathrm{an}}} g(x,y)\,\mathrm{d}\nu(y) \,\middle|\, g \in \mathcal{C}(X^{\mathrm{an}} \times X^{\mathrm{an}}),\ M \le g \le g_{\zeta_0} \right\}$$

on X^{an}. Due to the compactness of X^{an}, the integral function $x \mapsto \int_{X^{\mathrm{an}}} g(x,y)\,\mathrm{d}\nu(y)$ is continuous on X^{an} for every $g \in \mathcal{C}(X^{\mathrm{an}}\times X^{\mathrm{an}})$. Then [5, Lemma A.2] tells us that f has to be lower semicontinuous on X^{an}.

Thus it remains to prove identity (6.3). First, we show the last equation

$$\lim_{t\to z, t\in[\omega,z)} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta).$$

If $z \notin X(K)$, then by shrinking our path we may assume $[z,\omega] \subset \mathbb{H}(X^{\mathrm{an}})$, and so the restriction of $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ to $[z,\omega]$ is continuous by Lemma 6.4 and the equation is true. If $z \in X(K)$, we may assume that $[z,\omega]$ lies in a connected component of $X^{\mathrm{an}}\backslash\Gamma$ for a skeleton Γ of X^{an} with $\zeta_0 \in \Gamma$. Then $\tau_\Gamma(t) = \tau_\Gamma(z)$ for every $t \in (z, \omega]$, and so for every $y \in X^{\mathrm{an}}$ and $t \in (z, \omega]$

$$g_{\zeta_0}(t,y) = \begin{cases} g_{\zeta_0}(\tau_\Gamma(z), \tau_\Gamma(y))_\Gamma & \text{if } \tau_\Gamma(z) \neq \tau_\Gamma(y), \\ g_{\zeta_0}(\tau_\Gamma(z), \tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(t,y), \tau_\Gamma(z)) & \text{if } \tau_\Gamma(z) = \tau_\Gamma(y). \end{cases}$$

Since $\rho(w_\Gamma(t,y), \tau_\Gamma(z))$ increases monotonically as t tends to z along $(z,\omega]$ for every $y \in X^{\mathrm{an}}$, the Monotone Convergence Theorem implies as in the proof of [5, Proposition 6.12] that the integral function $\int g_{\zeta_0}(t,y)\,\mathrm{d}\nu(y)$ converges to $\int g_{\zeta_0}(z,y)\,\mathrm{d}\nu(y)$ as t tends to z along $(z,\omega]$. Furthermore, $g_{\zeta_0}(t,\zeta)$ converges to $g_{\zeta_0}(z,\zeta)$ as t tends to z along $(z,\omega]$ by Proposition 4.8. At most one of the terms $\int g_{\zeta_0}(z,y)\,\mathrm{d}\nu(y)$ and $g_{\zeta_0}(z,\zeta)$ is infinite (due to $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$), so the description stated at the beginning of the proof (or see (6.1)) implies

$$\lim_{t\to z, t\in[\omega,z)} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta).$$

Now, we deduce the rest of (6.3) from that. When $\zeta = z \in X(K)$, we have

$$\liminf_{t\to z} u_{\zeta_0,\nu}(t,\zeta) \leq \liminf_{\substack{t\to z,\\ t\in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta) \leq \lim_{\substack{t\to z,\\ t\in[\omega,z)}} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta) = -\infty,$$

and so clearly (6.3) is true. When $\zeta \notin X(K)$ or $\zeta \neq z$, then $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ is lower semicontinuous at z by (1) and (2), and so we get

$$\begin{aligned} u_{\zeta_0,\nu}(z,\zeta) \leq \liminf_{t\to z} u_{\zeta_0,\nu}(t,\zeta) &\leq \liminf_{\substack{t\to z,\\ t\in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta) \leq \lim_{\substack{t\to z,\\ t\in[\omega,z)}} u_{\zeta_0,\nu}(t,\zeta) \\ &= u_{\zeta_0,\nu}(z,\zeta). \end{aligned}$$

Hence we also have equality. □

Proposition 6.6. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. Then for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the potential function $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$\mathrm{dd}^c u_{\zeta_0,\nu}(\,\cdot\,,\zeta) = \nu(X^{\mathrm{an}})\delta_\zeta - \nu.$$

Proof. A function on X^{an} defines a current in $D^0(X^{\mathrm{an}})$ if and only if its restriction to $I(X^{\mathrm{an}})$ is finite (cf. Proposition 3.7). Recall from (6.1) that for every $x \in X^{\mathrm{an}}$

$$u_{\zeta_0,\nu}(\,\cdot\,,\zeta) = \int g_{\zeta_0}(\,\cdot\,,y)\,\mathrm{d}\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(\,\cdot\,,\zeta) + C_{\zeta_0,\zeta}.$$

If we fix $x \in I(X^{\mathrm{an}})$, the function $g_{\zeta_0}(x,\cdot) = g_{\zeta_0}(\,\cdot\,,x)$ (symmetry follows by Lemma 4.7) is a finitely valued continuous function on X^{an} by Proposition 4.8 (1). Hence all terms define currents in $D^0(X^{\mathrm{an}})$, and so does $u_{\zeta_0,\nu}(\,\cdot\,,\zeta)$. For the first term we also use that $\mathrm{supp}(\nu)$ is compact.

Furthermore, we know by Proposition 4.8 that for any fixed y we have $\mathrm{dd}^c g_{\zeta_0}(\,\cdot\,,y) = \delta_{\zeta_0} - \delta_y$. Due to the calculation

$$\begin{aligned}\langle \mathrm{dd}^c \left(\int g_{\zeta_0}(\,\cdot\,,y)\,\mathrm{d}\nu(y)\right), \varphi\rangle &= \int \langle \mathrm{dd}^c g_{\zeta_0}(\,\cdot\,,y), \varphi\rangle \,\mathrm{d}\nu(y) \\ &= \int \left(\int \varphi \,\mathrm{d}(\delta_{\zeta_0} - \delta_y)(x)\right) \mathrm{d}\nu(y) \\ &= \int \varphi \,\mathrm{d}(\nu(X^{\mathrm{an}})\delta_{\zeta_0} - \nu)(y)\end{aligned}$$

for every $\varphi \in A_c^0(X^{\mathrm{an}})$, we obtain

$$\mathrm{dd}^c \left(\int g_{\zeta_0}(\,\cdot\,,y)\,\mathrm{d}\nu(y)\right) = \nu(X^{\mathrm{an}})\delta_{\zeta_0} - \nu.$$

Hence

$$\mathrm{dd}^c u_{\zeta_0,\nu}(\,\cdot\,,\zeta) = \nu(X^{\mathrm{an}})\delta_{\zeta_0} - \nu - \nu(X^{\mathrm{an}})(\delta_{\zeta_0} - \delta_\zeta) = \nu(X^{\mathrm{an}})\delta_\zeta - \nu. \quad \square$$

7. Arakelov–Green's functions

Baker and Rumely developed a theory of Arakelov–Green's functions on $\mathbb{P}^{1,\mathrm{an}}$ in [5, §8.10]. This class of functions arise naturally in the study of dynamics and can be seen as a generalization of the potential kernel from Section 4. Arakelov–Green's functions are characterized by a list of properties which can be found in Definition 7.1. We generalize Baker and Rumely's definition of an Arakelov–Green's function from $\mathbb{P}^{1,\mathrm{an}}$ to X^{an}, and show that the characteristic properties are still satisfied.

Definition 7.1. A symmetric function g on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ that satisfies the following list of properties for a probability measure μ on X^{an} is called a *normalized Arakelov–Green's function* on X^{an}.

(1) (Semicontinuity) The function g is finite and continuous off the diagonal and strongly lower semicontinuous on the diagonal in the sense that

$$g(x_0,x_0) = \liminf_{(x,y)\to(x_0,x_0),x\neq y} g(x,y).$$

(2) (Differential equation) For each fixed $y \in X^{\mathrm{an}}$ the function $g(\,\cdot\,,y)$ is an element of $D^0(X^{\mathrm{an}})$ and

$$\mathrm{dd}^c g(\,\cdot\,,y) = \mu - \delta_y.$$

(3) (Normalization)

$$\iint g(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0.$$

The list of properties is an analogue of the one in the complex case and can for example also be found in [4, §3.5 (B1)–(B3)].

Remark 7.2. As in the complex case, the list of properties in Definition 7.1 for a probability measure μ on X^{an} determines a normalized Arakelov–Green's function on X^{an} uniquely. If $\widetilde{g}$ is another symmetric function on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ satisfying (1)–(3), then for a fixed $y \in X^{\mathrm{an}}$

$$g(\,\cdot\,, y) - \widetilde{g}(\,\cdot\,, y) = h_y$$

on $I(X^{\mathrm{an}})$ for a harmonic function h_y on X^{an} by property (2) and [13, Lemme 3.3.12]. This harmonic function h_y has to be constant on X^{an} by the Maximum Principle (Proposition 3.13). Since $I(X^{\mathrm{an}})$ is dense in X^{an}, the identity holds on all of X^{an} by property (1).

Thanks to the symmetry of g and $\widetilde{g}$, the constant function h_y is independent of y. The last property (3), implies that this constant has to be zero, i.e. $g = \widetilde{g}$ on $X^{\mathrm{an}} \times X^{\mathrm{an}}$.

Definition 7.3. A probability measure μ on X^{an} *has continuous potentials* if each $\zeta \in I(X^{\mathrm{an}})$ defines a continuous function

$$\begin{aligned} X^{\mathrm{an}} &\longrightarrow \mathbb{R}, \\ x &\longmapsto \int_{X^{\mathrm{an}}} g_\zeta(x, y)\,\mathrm{d}\mu(y). \end{aligned}$$

These functions are bounded as X^{an} is compact.

Remark 7.4. Let μ be a probability measure on X^{an}. If there exists a point $\zeta_0 \in I(X^{\mathrm{an}})$ such that $X^{\mathrm{an}} \to \mathbb{R},\ x \mapsto \int_{X^{\mathrm{an}}} g_{\zeta_0}(x, y)\,\mathrm{d}\mu(y)$ defines a continuous function, thenμ has continuous potentials.

Example 7.5. Let μ be a probability measure supported on a skeleton Γ of X^{an} (e.g. $\mu = \delta_z$ for some $z \in I(X^{\mathrm{an}})$), then μ has continuous potentials (using the last remark and Lemma 4.2).

Definition 7.6. For every probability measure μ on X^{an} with continuous potentials and a fixed base point $\zeta_0 \in I(X^{\mathrm{an}})$, we define $g_{\zeta_0,\mu} \colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty, \infty]$ by

$$g_{\zeta_0,\mu}(x, y) := g_{\zeta_0}(x, y) - \int_{X^{\mathrm{an}}} g_{\zeta_0}(x, \zeta)\,\mathrm{d}\mu(\zeta) - \int_{X^{\mathrm{an}}} g_{\zeta_0}(y, \zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0},$$

where C_{ζ_0} is a constant chosen such that

$$\iint g_{\zeta_0,\mu}(x, y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0.$$

Remark 7.7. Recall that $g_{\zeta_0}(\zeta_0, x, y) = g_{\zeta_0}(x, y)$ (see Definition 4.9) by Corollary 4.10, and so the potential function from Section 6 can be written as

$$u_{\zeta_0,\mu}(\,\cdot\,, \zeta_0) = \int g_{\zeta_0}(\zeta_0, \cdot, \zeta)\,\mathrm{d}\mu(\zeta) = \int g_{\zeta_0}(\,\cdot\,, \zeta)\,\mathrm{d}\mu(\zeta).$$

Hence we have the description

$$g_{\zeta_0,\mu}(x,y) = g_{\zeta_0}(x,y) - u_{\zeta_0,\mu}(x,\zeta_0) - u_{\zeta_0,\mu}(y,\zeta_0) + C_{\zeta_0} \tag{7.1}$$

on $X^{\mathrm{an}} \times X^{\mathrm{an}}$.

In the following lemma, we see that this function is independent of the chosen base point, and hence we just write g_μ.

Lemma 7.8. *For every probability measure μ on X^{an} with continuous potentials, the function $g_{\zeta_0,\mu}$ is independent of the chosen base point ζ_0.*

Proof. First, we determine C_{ζ_0}:

$$\begin{aligned} 0 = \iint g_{\zeta_0,\mu}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) &= \iint g_{\zeta_0}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &\quad - \iint g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\mu(x) \\ &\quad - \iint g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\mu(y) + C_{\zeta_0} \\ &= -\iint g_{\zeta_0}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) + C_{\zeta_0}. \end{aligned}$$

Hence $C_{\zeta_0} = \iint g_{\zeta_0}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)$. Now let $\zeta_0' \in I(X^{\mathrm{an}})$. Applying Lemma 4.7 to C_{ζ_0}, we get

$$\begin{aligned} C_{\zeta_0} &= \iint g_{\zeta_0}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &= \iint \Big(g_{\zeta_0'}(x,y) - g_{\zeta_0'}(x,\zeta_0) - g_{\zeta_0'}(y,\zeta_0) + g_{\zeta_0'}(\zeta_0,\zeta_0)\Big)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &= C_{\zeta_0'} - 2\int g_{\zeta_0'}(x,\zeta_0)\,\mathrm{d}\mu(x) + g_{\zeta_0'}(\zeta_0,\zeta_0), \end{aligned}$$

where $-2\int g_{\zeta_0'}(\zeta,\zeta_0)\,\mathrm{d}\mu(\zeta) + g_{\zeta_0'}(\zeta_0,\zeta_0)$ is a finite constant as μ has continuous potentials.

Using Lemma 4.7 also for the other terms of $g_{\zeta_0,\mu}$, i.e. for $g_{\zeta_0}(x,y)$, $g_{\zeta_0}(x,\zeta)$ and $g_{\zeta_0}(y,\zeta)$, and plugging in the identity from above, we get $g_{\zeta_0,\mu}(x,y) = g_{\zeta_0',\mu}(x,y)$. □

Proposition 7.9. *Let μ be a probability measure on X^{an} with continuous potentials. Then as a function of two variables $g_\mu\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty,\infty]$ is symmetric, finite and continuous off the diagonal, and strongly lower semicontinuous on the diagonal in the sense that*

$$g_\mu(x_0,x_0) = \liminf_{(x,y)\to(x_0,x_0),x\neq y} g_\mu(x,y),$$

where we understand $X^{\mathrm{an}} \times X^{\mathrm{an}}$ set theoretically and endowed with the product topology.

Proof. As

$$g_\mu(x,y) = g_{\zeta_0}(x,y) - \int g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta) - \int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0}$$

for some base point $\zeta_0 \in I(X^{\mathrm{an}})$ and as we required μ to has continuous potentials, Lemma 4.7 implies that $g_\mu\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty,\infty]$ is symmetric, finite and continuous off the diagonal and lower semicontinuous on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. Thus we only need to prove

$$\begin{aligned} g_\mu(x_0,x_0) &\geq \liminf_{(x,y)\to(x_0,x_0),x\neq y} g_\mu(x,y) \\ &= \sup_{U\in\mathcal{U}((x_0,x_0))} \inf_{(x,y)\in U\setminus(x_0,x_0)} g_\mu(x,y). \end{aligned} \tag{7.2}$$

Here $\mathcal{U}((x_0,x_0))$ is any basis of open neighborhoods of (x_0,x_0) in $X^{\mathrm{an}}\times X^{\mathrm{an}}$ endowed with the product topology.

In the following, let Γ be any skeleton of X^{an} with $\zeta_0 \in \Gamma$. If x_0 is of type I, we have $g_\mu(x_0,x_0) = g_{\zeta_0}(x_0,x_0) = \infty$ by the definition of the potential kernel, and so (7.2) is obviously true.

If x_0 is of type II or III, we may choose $\zeta_0 = x_0$ by Lemma 7.8, and so

$$g_\mu(x_0,x_0) = g_{x_0}(x_0,x_0) - \int g_{x_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta) - \int g_{x_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0} = C_{\zeta_0}$$

as $g_{x_0}(x_0,\zeta) = 0$ for every $\zeta \in X^{\mathrm{an}}$ by Lemma 4.7. On the other hand, every U in $\mathcal{U}((x_0,x_0))$ contains an element of the form (x_0,y) with $y \in X^{\mathrm{an}}\setminus\{x_0\}$, and

$$\begin{aligned} g_\mu(x_0,y) &= g_{x_0}(x_0,y) - \int g_{x_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta) - \int g_{x_0}(y,\zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0} \\ &= -\int g_{x_0}(y,\zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0} \leq C_{\zeta_0} \end{aligned}$$

since μ and $g_{x_0}(y,\cdot)$ are non-negative (see Lemma 4.7 (1)). Thus (7.2) has to be true.

For the rest of the proof let x_0 be of type IV. There is a basis of open neighborhoods of x_0 that is contained in the connected component V of $X^{\mathrm{an}}\setminus\Gamma$ that contains x_0 (cf. Theorem 2.11). Consider the corresponding basis of open neighborhoods $\mathcal{U}((x_0,x_0))$ of (x_0,x_0) in $X^{\mathrm{an}} \times X^{\mathrm{an}}$ endowed with the product topology. In every $U \in \mathcal{U}((x_0,x_0))$ we consider tuples of the form (x_0,y) where y lies in the interior of the unique path $[x_0,\tau_\Gamma(x_0)]$ (such tuples always exist). Then $\tau_\Gamma(y) = \tau_\Gamma(x_0)$ and $w_\Gamma(x_0,y) = y$ (recall its definition from Remark 4.3), and so

$$\begin{aligned} g_{\zeta_0}(x_0,x_0)-g_{\zeta_0}(x_0,y) &= g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(x_0))_\Gamma+\rho(w_\Gamma(x_0,x_0),\tau_\Gamma(x_0)) \\ &\quad -(g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(y))_\Gamma+\rho(w_\Gamma(x_0,y),\tau_\Gamma(y))) \\ &= \rho(w_\Gamma(x_0,x_0),\tau_\Gamma(x_0))-\rho(w_\Gamma(x_0,y),\tau_\Gamma(y)) \\ &= \rho(x_0,\tau_\Gamma(x_0))-\rho(y,\tau_\Gamma(x_0)) \\ &= \rho(x_0,y). \end{aligned}$$

Consequently, we get

$$\begin{aligned} g_\mu(x_0,x_0)-g_\mu(x_0,y) &= g_{\zeta_0}(x_0,x_0)-2\int g_{\zeta_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta)+C_{\zeta_0} \\ &\quad -(g_{\zeta_0}(x_0,y)-\int g_{\zeta_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta) \\ &\quad -\int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta)+C_{\zeta_0}) \\ &= g_{\zeta_0}(x_0,x_0)-g_{\zeta_0}(x_0,y) \\ &\quad -\int g_{\zeta_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta)+\int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta) \\ &= \rho(x_0,y)+\int g_{\zeta_0}(y,\zeta)-g_{\zeta_0}(x_0,\zeta)\,\mathrm{d}\mu(\zeta). \end{aligned} \tag{7.3}$$

To prove (7.2), we need to show that (7.3) is non-negative.

Recall that y lies in the interior of the unique path $[x_0,\tau_\Gamma(x_0)]$. We denote by V_0 the connected component of $V\backslash\{y\}$ that contains x_0 (note that V_0 is an open ball as x_0 is of type IV and V is an open ball). We will see that $g_{\zeta_0}(y,\zeta)-g_{\zeta_0}(x_0,\zeta)$ in (7.3) is zero for every $\zeta\in X^{\mathrm{an}}\backslash V_0$. Recall that V is the connected component of $X^{\mathrm{an}}\backslash\Gamma$ that contains x_0. Hence V is an open ball with $\partial V=\{\tau_\Gamma(x_0)\}$ and $V_0\subset V$. Furthermore, one should have in mind that $\tau_\Gamma(y)=\tau_\Gamma(x_0)$ as $y\in[x_0,\tau_\Gamma(x_0)]$.

If $\zeta\in X^{\mathrm{an}}\backslash V$, then by the definition of the potential kernel

$$g_{\zeta_0}(y,\zeta)-g_{\zeta_0}(x_0,\zeta)=g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma-g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma=0.$$

If $\zeta\in V\backslash V_0$, then $\tau_\Gamma(\zeta)=\tau_\Gamma(x_0)=\tau_\Gamma(y)$ and $w_\Gamma(x_0,\zeta)=w_\Gamma(y,\zeta)$, and hence

$$\begin{aligned} g_{\zeta_0}(y,\zeta)-g_{\zeta_0}(x_0,\zeta) &= g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma+\rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) \\ &\quad -(g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma+\rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0))) \\ &= 0. \end{aligned}$$

Thus $g_{\zeta_0}(y,\zeta)-g_{\zeta_0}(x_0,\zeta)=0$ for every $\zeta\in X^{\mathrm{an}}\backslash V_0$.

For every $\zeta \in V_0$ we have $\tau_\Gamma(\zeta) = \tau_\Gamma(x_0) = \tau_\Gamma(y)$, $w_\Gamma(y,\zeta) = y$, and $w_\Gamma(x_0,\zeta) \in [x_0, y]$. Hence

$$\begin{aligned} g_{\zeta_0}(y,\zeta) - g_{\zeta_0}(x_0,\zeta) &= g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) \\ &\qquad - (g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0))) \\ &= \rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) - \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0)) \\ &= \rho(y,\tau_\Gamma(x_0)) - \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0)) \\ &= -\rho(w_\Gamma(x_0,\zeta),y) \end{aligned}$$

for every $\zeta \in V_0$. Plugging everything in (7.3), we get

$$\begin{aligned} g_\mu(x_0,x_0) - g_\mu(x_0,y) &= \rho(x_0,y) + \int_{V_0} -\rho(w_\Gamma(x_0,\zeta),y)\,\mathrm{d}\mu(\zeta) \\ &\geq \int_{V_0} \rho(x_0,y) - \rho(w_\Gamma(x_0,\zeta),y)\,\mathrm{d}\mu(\zeta) \\ &\geq 0 \end{aligned}$$

as $\rho(x_0,y) \geq \rho(w_\Gamma(x_0,\zeta),y)$ on V_0 and μ is a non-negative measure.

Consequently, (7.2) has to be also true for x_0 of type IV. ☐

Proposition 7.10. *For every probability measure μ on X^{an} with continuous potentials and for every fixed $y \in X^{\mathrm{an}}$, $G_{\mu,y} := g_\mu(\,\cdot\,,y)\colon X^{\mathrm{an}} \to (-\infty,\infty]$ defines a current in $D^0(X^{\mathrm{an}})$ and satisfies*

$$\mathrm{dd}^c G_{\mu,y} = \mu - \delta_y.$$

Moreover, $G_{\mu,y}$ is continuous on X^{an} with $G_{\mu,y}(x) = \infty$ if and only if $x = y \in X(K)$. In particular, $G_{\mu,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$.

Proof. By the definition of the Arakelov–Green's function, we have

$$G_{\mu,y}(x) = g_{\zeta_0}(x,y) - \int g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta) - \int g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta) + C_{\zeta_0}$$

for every $x \in X^{\mathrm{an}}$. Due to Proposition 4.8, the first term is continuous on X^{an} and attains values in $\mathbb{R} \cup \{\infty\}$ with $g_{\zeta_0}(x,y) = \infty$ if and only if $x = y \in X(K)$. In particular, the first term is finitely valued on $I(X^{\mathrm{an}})$. Since μ has continuous potentials, the other two terms are finitely valued and continuous on X^{an}. Hence $G_{\mu,y}\colon X^{\mathrm{an}} \to (-\infty,\infty]$ is continuous on X^{an} with $G_{\mu,y}(x) = \infty$ if and only if $x = y \in X(K)$. In particular, $G_{\mu,y}$ is finitely valued on $I(X^{\mathrm{an}})$, and so defines a current in $D^0(X^{\mathrm{an}})$ by Proposition 3.7. It remains to calculate the Laplacian of $G_{\mu,y}$. By (7.1), we have

$$G_{\mu,y} = g_{\zeta_0}(\,\cdot\,,y) - u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) - u_{\zeta_0,\mu}(y,\zeta_0) + C_{\zeta_0}.$$

Proposition 4.8 and Proposition 6.6 imply $\mathrm{dd}^c G_{\mu,y} = \mu - \delta_y$. Hence $G_{\mu,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$. ☐

Using the previous propositions, g_μ is a normalized Arakelov–Green's function as defined in Definition 7.1 for every probability measure μ on X^{an} with continuous potentials.

Corollary 7.11. *Let μ be a probability measure on X^{an} with continuous potentials. Then the function g_μ is a normalized Arakelov–Green's function on X^{an}.*

Proof. We need to know that all properties of the list in Definition 7.1 hold. Property (1) and symmetry are true due to Proposition 7.9, (2) was shown in Proposition 7.10, and (3) follows by construction. □

Remark 7.12. For a probability measure μ on X^{an} and for a point $y \in I(X^{\mathrm{an}})$, Thuillier constructs in his thesis [13, §3.4.3] a unique function $g_{y,\mu} \colon X^{\mathrm{an}} \to [-\infty, \infty)$ such that $\mathrm{dd}^c g_{y,\mu} = \mu - \delta_y$, $g_{y,\mu}(y) = 0$ and its restriction to $X^{\mathrm{an}} \backslash \{y\}$ is subharmonic. His construction uses [13, Théorème 3.3.13 & 3.4.12]. If μ has continuous potentials, then $g_{y,\mu}$ and $G_{\mu,y}$ define two currents in $D^0(X^{\mathrm{an}})$ (cf. Proposition 3.11) having the same Laplacian $\mu - \delta_y$. [13, Lemma 3.3.12] implies that $g_{y,\mu}$ and $G_{\mu,y}$ differ only by a harmonic function on X^{an}, which has to be constant by the Maximum Principle 3.13.

8. Energy Minimization Principle

The Energy Minimization Principle is a very important theorem in dynamics and has many applications. The goal is to translate this principle into our non-archimedean setting. For $X = \mathbb{P}^1$ this was already done in [5, §8.10], and Matt Baker suggested to generalize their definition of Arakelov–Green's functions and their result to the author. In the following section, we give a proof of the Energy Minimization Principle for a smooth projective curve X over our non-archimedean field K using the techniques from [5, §8.10].

Definition 8.1. Let μ be a probability measure on X^{an} with continuous potentials. Then for every probability measure ν on X^{an}, we define the corresponding *μ-energy integral* as

$$I_\mu(\nu) := \iint g_\mu(x, y)\, \mathrm{d}\nu(y)\mathrm{d}\nu(x).$$

Note that the integral is well-defined since g_μ is lower semicontinuous on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Proposition 7.9, and hence Borel measurable and bounded from below.

Theorem 8.2 (Energy Minimization Principle)**.** *Let μ be a probability measure on X^{an} with continuous potentials. Then*

(1) *$I_\mu(\nu) \geq 0$ for each probability measure ν on X^{an}, and*
(2) *$I_\mu(\nu) = 0$ if and only if $\nu = \mu$.*

We show the principle in several steps. At first, we prove analogues of Maria's theorem (Theorem 8.8) and Frostman's theorem (Theorem 8.11). In Maria's theorem we study the boundedness of the generalized potential function that is defined in the subsequent definition.

Definition 8.3. Let μ be a probability measure on X^{an} with continuous potentials. Then for every probability measure ν on X^{an}, we define the corresponding *generalized potential function* by

$$u_\nu(\,\cdot\,,\mu) := \int g_\mu(\,\cdot\,,y)\,\mathrm{d}\nu(y).$$

Lemma 8.4. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then for every $\zeta_0 \in I(X^{\mathrm{an}})$ we can write*

$$u_\nu(\,\cdot\,,\mu) = u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0) - u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) + C \tag{8.1}$$

on X^{an} for a finite constant C.

Proof. Let ζ_0 be a point in $I(X^{\mathrm{an}})$. Then by Corollary 4.10

$$u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0) = \int g_{\zeta_0}(\zeta_0,\cdot,\zeta)\,\mathrm{d}\nu(\zeta) = \int g_{\zeta_0}(\,\cdot\,,\zeta)\,\mathrm{d}\nu(\zeta).$$

The same identity is true for μ, i.e. $u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) = \int g_{\zeta_0}(\,\cdot\,,\zeta)\,\mathrm{d}\mu(\zeta)$, which is a finitely valued continuous function on X^{an} as μ has continuous potentials. Thus we can write using the definition of the Arakelov–Green's function (Definition 7.6)

$$\begin{aligned} u_\nu(x,\mu) &= \int g_\mu(x,y)\,\mathrm{d}\nu(y) \\ &= \int g_{\zeta_0}(x,y)\,\mathrm{d}\nu(y) - \int g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta) \\ &\qquad - \iint g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\nu(y) + C_{\zeta_0} \\ &= u_{\zeta_0,\nu}(x,\zeta_0) - u_{\zeta_0,\mu}(x,\zeta_0) - \int u_{\zeta_0,\mu}(y,\zeta_0)\,\mathrm{d}\nu(y) + C_{\zeta_0} \end{aligned}$$

for every $x \in X^{\mathrm{an}}$. Since $u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0)$ is bounded and continuous on X^{an}, we get

$$u_\nu(\,\cdot\,,\mu) = u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0) - u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) + C$$

on X^{an} for a finite constant C. □

Proposition 8.5. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then $u_\nu(\,\cdot\,,\mu)\colon X^{\mathrm{an}} \to (-\infty,\infty]$ is continuous on $X^{\mathrm{an}} \backslash \operatorname{supp}(\nu)$ and lower semicontinuous on X^{an}. Moreover, the restriction of $u_\nu(\,\cdot\,,\mu)$ to every skeleton Γ of X^{an} and to every path $[y,z]$ is finite and continuous.*

Proof. Let ζ_0 be some point in $I(X^{\mathrm{an}})$, then

$$u_\nu(\,\cdot\,,\mu) = u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0) - u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) + C$$

on X^{an} for a finite constant C by Lemma 8.4. Since μ has continuous potentials, $u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0)$ is a finitely valued continuous function on X^{an}. Thus it remains to prove the continuity assertions for $u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0)$. But these were all already shown in Lemma 6.4 and Proposition 6.5. □

Proposition 8.6. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then $u_\nu(\,\cdot\,,\mu)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$\mathrm{dd}^c u_\nu(\,\cdot\,,\mu) = \mu - \nu.$$

In particular, $u_\nu(\,\cdot\,,\mu)$ is subharmonic on $X^{\mathrm{an}}\backslash \operatorname{supp}(\nu)$.

Proof. Let ζ_0 be a point in $I(X^{\mathrm{an}})$, then

$$u_\nu(\,\cdot\,,\mu) = u_{\zeta_0,\nu}(\,\cdot\,,\zeta_0) - u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0) + C$$

on X^{an} for a finite constant C by Lemma 8.4. By Proposition 6.6 and linearity, the function $u_{\zeta_0,\mu}(\,\cdot\,,\zeta_0)$ belongs to $D^0(X^{\mathrm{an}})$ with $\mathrm{dd}^c u_\nu(\,\cdot\,,\mu) = \mu - \nu$. Then the generalized potential function $u_\nu(\,\cdot\,,\mu)$ is therefore subharmonic on $X^{\mathrm{an}}\backslash \operatorname{supp}(\nu)$ as it is upper semicontinuous by Proposition 8.5. □

The key tool of the proof of Maria's theorem in [5] is [5, Proposition 8.16], which we can translate to our situation in the following form.

Lemma 8.7. *Let W be an open ball or an open annulus in X^{an} and let f be a subharmonic function on a connected open subset V of W with $\overline{V} \subset W$. For every $x \in \mathbb{H}(V)$, there is a path Λ from x to a boundary point $y \in \partial V$ such that f is non-decreasing along Λ.*

Proof. Since $\overline{V}$ is contained in an open ball or in an open annulus, we can view it is a subset of $\mathbb{P}^{1,\mathrm{an}}$. Then [5, Proposition 8.16] and Remark 3.12 yield the claim. □

With the help of Proposition 8.5 and Lemma 8.7, we can prove Maria's theorem.

Theorem 8.8 (Maria)**.** *Let μ be a probability measure on X^{an} with continuous potentials and let ν be an arbitrary probability measure on X^{an}. If there is a constant $M < \infty$ such that $u_\nu(\,\cdot\,,\mu) \leq M$ on $\operatorname{supp}(\nu)$, then $u_\nu(\,\cdot\,,\mu) \leq M$ on X^{an}.*

Proof. Let V be a connected component of $X^{\mathrm{an}}\backslash \operatorname{supp}(\nu)$ and assume there is a point $x_0 \in V$ such that $u_\nu(x_0,\mu) > M$. Note that V is path-connected since X^{an} is locally path-connected. If B is an open ball in X^{an}, then between two points $x, y \in B$ there is only one path in X^{an} by the structure of X^{an}. Thus $V \cap B$ is uniquely path-connected for every open ball B in

X^{an}. We have seen in Proposition 8.5 that the generalized potential function $u_\nu(\,\cdot\,,\mu)$ is continuous on $V \subset X^{\mathrm{an}} \backslash \operatorname{supp}(\nu)$. Hence we may assume x_0 to be contained in the dense subset $I(V)$ of V, and so we can choose a skeleton Γ of X^{an} containing x_0 by Proposition 2.8.

Let $(Y_\alpha)_\alpha$ be the directed system of connected strictly affinoid domains contained in V and containing x_0. Note that the union of two connected strictly affinoid domains Y_1, Y_2 in X^{an} both containing x_0 with $Y_1 \cup Y_2 \neq X^{\mathrm{an}}$ is again a connected strictly affinoid domain in X^{an} by [13, Corollaire 2.1.17]. Then $u_\nu(\,\cdot\,,\mu)$ is continuous on Y_α and subharmonic on the relative interior Y_α° by Proposition 8.5 and Proposition 8.6. Hence $u_\nu(\,\cdot\,,\mu)$ attains a maximum on Y_α in a point $z_\alpha \in \partial Y_\alpha$ (see Maximum Principle 3.13), i.e.

$$u_\nu(z_\alpha, \mu) = \max_{x \in Y_\alpha} u_\nu(x, \mu) \geq \max_{x \in Y_\alpha^\circ} u_\nu(x, \mu) \geq u_\nu(x_0, \mu) > M \tag{8.2}$$

for every α. Then $\langle z_\alpha \rangle_\alpha$ defines a net of type II points in V. As $\overline{V}$ is compact, we may assume by passing to a subnet that $\langle z_\alpha \rangle_\alpha$ converges to a point $z \in \overline{V}$. Due to $V = \bigcup_\alpha Y_\alpha$ and $z_\alpha \in \partial Y_\alpha$, the point z has to lie in $\partial V \subset \operatorname{supp}(\nu)$. In the following, we use this net to get a contradiction to $u_\nu(\,\cdot\,,\mu) \leq M$ on ∂V. Recall that Γ is a skeleton of X^{an} containing x_0.

If $z \in \partial V \backslash \Gamma$, there exists an open ball B_z in $X^{\mathrm{an}} \backslash \Gamma$ containing z. We can find B_z such that $\overline{B_z} = B_z \cup \{\zeta_z\} \subset X^{\mathrm{an}} \backslash \Gamma$. We may assume $\langle z_\alpha \rangle_\alpha$ to lie in B_z. Then every path from a z_α to x_0, or more generally to the skeleton, goes by construction through ζ_z. Hence for every α the path $[z_\alpha, \zeta_z]$ lies inside Y_α as z_α and x_0 do, and so $u_\nu(z_\alpha, \mu) \geq u_\nu(\,\cdot\,,\mu)$ on $[z_\alpha, \zeta_z]$ by (8.2). Assume we have equality for every α, then

$$u_\nu(\,\cdot\,,\mu) \equiv u_\nu(z_\alpha, \mu) \geq u_\nu(x_0, \mu)$$

on $(z, \zeta_z]$ since we can write $(z, \zeta_z] \subset \bigcup_\alpha [z_\alpha, \zeta_z]$ as z_α converges to z. Proposition 8.5 implies

$$u_\nu(z, \mu) = \lim_{x \in [\zeta_z, z),\ x \to z} u_\nu(x, \mu) = u_\nu(\zeta_z, \mu) \geq u_\nu(x_0, \mu) > M$$

contradicting $u_\nu(\,\cdot\,,\mu) \leq M$ on $\operatorname{supp}(\nu)$. Consequently, we may assume that there is a z_α and a point $y_\alpha \in (z_\alpha, \zeta_z]$ such that $u_\nu(z_\alpha, \mu) > u_\nu(y_\alpha, \mu)$. Our function $u_\nu(\,\cdot\,,\mu)$ is subharmonic on the connected open subset $V \cap B_z$ and $z_\alpha \in I(V \cap B_z)$, and so there exists a path Λ from z_α to a boundary point of $V \cap B_z$ by Lemma 8.7 such that $u_\nu(\,\cdot\,,\mu)$ is non-decreasing along Λ. The boundary points of $V \cap B_z$ consist of points in ∂V and ζ_z. Since we have already seen that there is a point $y_\alpha \in (z_\alpha, \zeta_z]$ such that $u_\nu(z_\alpha, \mu) > u_\nu(y_\alpha, \mu)$, Λ cannot be the path $[z_\alpha, \zeta_z]$. Hence Λ is a path to a boundary point $z' \in \partial V$ and we get the contradiction

$$u_\nu(z', \mu) = \lim_{x \in \Lambda^\circ,\ x \to z'} u_\nu(x, \mu) \geq u_\nu(z_\alpha, \mu) > u_\nu(x_0, \mu) > M,$$

where $u_\nu(\,\cdot\,,\mu)$ restricted to Λ is continuous by Proposition 8.5.

If $z \in \partial V \cap \Gamma$, we show that $\langle \tau_\Gamma(z_\alpha)\rangle_\alpha$ defines a net in $V \cap \Gamma$ converging to z with $u_\nu(\tau_\Gamma(z_\alpha),\mu) \geq u_\nu(x_0,\mu) > M$ for every α. Then we use again Proposition 8.5. Since τ_Γ is continuous, the net $\langle \tau_\Gamma(z_\alpha)\rangle_\alpha$ converges to $\tau_\Gamma(z) = z$. Clearly, $\langle \tau_\Gamma(z_\alpha)\rangle_\alpha$ lies in Γ. The open set V is path-connected, and so there exists a path between z_α and x_0 in V. By the construction of the retraction map and due to $x_0 \in \Gamma$, $\tau_\Gamma(z_\alpha)$ lies inside this path, and hence it lies in V. We continue with $u_\nu(\tau_\Gamma(z_\alpha),\mu) \geq u_\nu(x_0,\mu)$ for every z_α. Assume that $z_\alpha \neq \tau_\Gamma(z_\alpha)$ because otherwise we are done by (8.2). Denote by B_α the connected component of $X^{\mathrm{an}}\backslash\Gamma$ containing z_α, and choose a sequence of type II points $\zeta_n \in [z_\alpha, \tau_\Gamma(z_\alpha)]^\circ$ converging to $\tau_\Gamma(z_\alpha)$. Note that there is only one path from z_α to $\tau_\Gamma(z_\alpha)$ in X^{an}, and this path lies in V because $z_\alpha, \tau_\Gamma(z_\alpha) \in V$ and V is path-connected. Thus each ζ_n lies in V as well. Let $B_{\alpha,n}$ be the open ball containing z_α and having ζ_n as unique boundary point. Since $u_\nu(\,\cdot\,,\mu)$ is subharmonic on $V \cap B_{\alpha,n}$ for every $n \in \mathbb{N}$, there is a path Λ_n from z_α to a boundary point z'_n in $\partial(V \cap B_{\alpha,n}) \subset \partial V \cup \{\zeta_n\}$ such that $u_\nu(\,\cdot\,,\mu)$ is non-decreasing along Λ_n by Lemma 8.7. If there exists an $n \in \mathbb{N}$ with $z'_n \in \partial V$, then Proposition 8.5 and (8.2) imply

$$u_\nu(z'_n,\mu) = \lim_{x\in\Lambda_n^\circ,\ x\to z'_n} u_\nu(x,\mu) \geq u_\nu(z_\alpha,\mu) \geq u_\nu(x_0,\mu) > M$$

contradicting $u_\nu(\,\cdot\,,\mu) \leq M$ on $\mathrm{supp}(\nu)$. Hence $\Lambda_n = [z_\alpha,\zeta_n]$ for all $n \in \mathbb{N}$. Recall that $(\zeta_n)_n$ is a sequence in V converging to $\tau_\Gamma(z_\alpha) \in V$. Since $u_\nu(\,\cdot\,,\mu)$ is continuous on V and $u_\nu(\,\cdot\,,\mu)$ is non-decreasing along $\Lambda_n = [z_\alpha,\zeta_n]$, Proposition 8.5 yields

$$u_\nu(\tau_\Gamma(z_\alpha),\mu) = \lim_{n\to\infty} u_\nu(\zeta_n,\mu) \geq u_\nu(z_\alpha,\mu). \tag{8.3}$$

Altogether, we have a net $\langle \tau_\Gamma(z_\alpha)\rangle_\alpha$ in $V \cap \Gamma$ converging to z such that

$$u_\nu(\tau_\Gamma(z_\alpha),\mu) \geq u_\nu(x_0,\mu) > M$$

for every α. Proposition 8.5 tells us that $u_\nu(\,\cdot\,,\mu)$ restricted to Γ is continuous, and hence using (8.2) and (8.3) we get

$$u_\nu(z,\mu) = \lim_\alpha u_\nu(\tau_\Gamma(z_\alpha),\mu) \geq u_\nu(x_0,\mu) > M$$

contradicting $u_\nu(\,\cdot\,,\mu) \leq M$ on $\mathrm{supp}(\nu)$.

Hence there cannot exist a point x_0 in V with $u_\nu(x_0,\mu) > M$. □

Definition 8.9. Let μ be a probability measure with continuous potentials, then we define the *μ-Robin constant* as

$$V(\mu) := \inf_\nu I_\mu(\nu),$$

where ν runs over all probability measures supported on X^{an}.

Lemma 8.10. *We have $V(\mu) \in \mathbb{R}_{\leq 0}$ and there exists a probability measure ω on X^{an} such that $I_\mu(\omega) = V(\mu)$.*

Proof. First, we explain why $V(\mu)$ is a non-positive real number. The normalized Arakelov–Green's function g_μ is bounded from below as a lower semicontinuous function on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Proposition 7.9, and hence we have

$$V(\mu) = \iint g_\mu(x, y)\, \mathrm{d}\nu(x)\mathrm{d}\nu(y) > -\infty.$$

On the other hand,

$$V(\mu) \leq I_\mu(\mu) = \iint g_\mu(x, y)\, \mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0$$

by the normalization of g_μ. Thus $V(\mu) \in \mathbb{R}_{\leq 0}$.

We show the second part of the assertion applying the same argument used to prove the existence of an equilibrium measure in [5, Proposition 6.6]. Let ω_i be a sequence of probability measures such that $\lim_{i\to\infty} I_\mu(\omega_i) = V(\mu)$. By Proposition 2.5, we can pass to a subsequence converging weakly to a probability measure ω on X^{an}. Due to $I_\mu(\omega) \geq V(\mu)$ by the definition of the Robin constant, it remains to show the inequality $I_\mu(\omega) \leq V(\mu)$. By Proposition 7.9, the normalized Arakelov–Green's function g_μ is lower semicontinuous on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$, and so it is bounded from below by some constant $M \in \mathbb{R}$. Proposition 2.5 tells us that ω is a Radon measure, and so [5, Proposition A.3] yields the following description

$$I_\mu(\omega) = \iint g_\mu(x, y)\, \mathrm{d}\omega(x)\mathrm{d}\omega(y) = \sup_{\substack{g \in \mathcal{C}(X^{\mathrm{an}} \times X^{\mathrm{an}}), \\ M \leq g \leq g_\mu}} \iint g(x, y)\, \mathrm{d}\omega(x)\mathrm{d}\omega(y),$$

for the space $\mathcal{C}(X^{\mathrm{an}} \times X^{\mathrm{an}})$ of real-valued continuous functions on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. For every $g \in \mathcal{C}(X^{\mathrm{an}} \times X^{\mathrm{an}})$ satisfying $M \leq g \leq g_\mu$, we have

$$\begin{aligned} \iint g(x, y)\, \mathrm{d}\omega(x)\mathrm{d}\omega(y) &= \lim_{i\to\infty} \iint g(x, y)\, \mathrm{d}\omega_i(x)\mathrm{d}\omega_i(y) \\ &\leq \lim_{i\to\infty} \iint g_\mu(x, y)\, \mathrm{d}\omega_i(x)\mathrm{d}\omega_i(y) \\ &= \lim_{i\to\infty} I_\mu(\omega_i) = V(\mu), \end{aligned}$$

where the first identity is proven for example in [5, Lemma 6.5] and the inequality holds as every ω_i is positive. Hence $I_\mu(\omega) \leq V(\mu)$. □

Theorem 8.11 (Frostman). *Let μ be a probability measure on X^{an} with continuous potentials and let ω be a probability measure on X^{an} such that $I_\mu(\omega) = V(\mu)$. Then we have on X^{an}*

$$u_\omega(\,\cdot\,, \mu) \equiv V(\mu).$$

Proof. The strategy is as in the proof of [5, Proposition 8.55] with using analogous capacity results from Section 5.

Step 1: Show that $E := \{x \in X^{\mathrm{an}} \mid u_\omega(x,\mu) < V(\mu)\} \subset X(K)$. By Lemma 5.5, it remains to show that E is a proper subset of X^{an} of capacity zero. Assume that $E \subset \mathrm{supp}(\omega)$, then we get the contradiction

$$V(\mu) = I_\mu(\omega) = \iint g_\mu(x,y)\,\mathrm{d}\omega(y)\mathrm{d}\omega(x) = \int u_\omega(x,\mu)\,\mathrm{d}\omega(x) < V(\mu).$$

Thus there has to be a point $\xi \in \mathrm{supp}(\omega)\backslash E$, and so E is indeed a proper subset of X^{an}. To show that it has capacity zero, we consider

$$E_n := \{x \in X^{\mathrm{an}} \mid u_\omega(x,\mu) \le V(\mu) - 1/n\}$$

for every $n \in \mathbb{N}_{\ge 1}$. Clearly, $\xi \notin E_n$ for every $n \in \mathbb{N}_{\ge 1}$. Since $u_\omega(\,\cdot\,,\mu)$ is lower semicontinuous on X^{an} by Proposition 8.5, each E_n is closed and so compact as a closed subset of a compact space. If every E_n has capacity zero, then $E = \bigcup_{n\in\mathbb{N}_{\ge 1}} E_n$ has capacity zero as well by Corollary 5.7.

We therefore assume that there is an E_n with positive capacity, i.e. there exist a probability measure ν supported on E_n, a base point $\zeta_0 \in I(X^{\mathrm{an}})$ and $\zeta \in X^{\mathrm{an}}\backslash E_n$ such that $I_{\zeta_0,\zeta}(\nu) < \infty$. Since E_n is closed and $I(X^{\mathrm{an}})$ is a dense subset of X^{an}, we may choose $\zeta_0 = \zeta \in I(X^{\mathrm{an}})\backslash E_n$ by Remark 5.4. Then

$$\begin{aligned} I_{\zeta_0,\zeta_0}(\nu) &= \iint g_{\zeta_0}(\zeta_0,x,y)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y) \\ &= \iint g_{\zeta_0}(x,y)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y) < \infty, \end{aligned} \tag{8.4}$$

where we used $g_{\zeta_0}(\zeta_0,x,y) = g_{\zeta_0}(x,y)$ from Corollary 4.10. We can write by the definition of the Arakelov–Green's function g_μ

$$I_\mu(\nu) = I_{\zeta_0,\zeta_0}(\nu) - 2\iint g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\nu(x) + C_{\zeta_0}.$$

Since μ has continuous potentials, the term $2\iint g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\nu(x)$ is finite. Hence $I_{\zeta_0,\zeta_0}(\nu) < \infty$ implies $I_\mu(\nu) < \infty$.

Recall that ξ is a point in $\mathrm{supp}(\omega)\backslash E_n$ and $u_\omega(\xi,\mu) \ge V(\mu)$. Since $u_\omega(\,\cdot\,,\mu)$ is lower semicontinuous on X^{an} by Proposition 8.5, we can find an open neighborhood U of ξ such that $u_\omega(\,\cdot\,,\mu) > V(\mu) - 1/(2n)$ on $\overline{U}$. Then $\overline{U} \cap E_n = \emptyset$ and $M := \omega(\overline{U}) > 0$ using that ω is a positive measure and $\xi \in \overline{U} \cap \mathrm{supp}(\omega)$. We define the following measure on X^{an}

$$\sigma := \begin{cases} M\cdot\nu & \text{on } E_n, \\ -\omega & \text{on } \overline{U}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\sigma(X^{\mathrm{an}}) = M \cdot \nu(E_n) - \omega(\overline{U}) = 0$ as ν is a probability measure supported on E_n. Moreover, we can consider

$$\begin{aligned} I_\mu(\sigma) &:= \iint g_\mu(x,y)\ d\sigma(x)d\sigma(y) \\ &= M^2 \cdot \int_{E_n}\int_{E_n} g_\mu(x,y)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y) \\ &\quad - 2M \cdot \int_{E_n}\int_{\overline{U}} g_\mu(x,y)\,\mathrm{d}\nu(x)\mathrm{d}\omega(y) + \int_{\overline{U}}\int_{\overline{U}} g_\mu(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y). \end{aligned}$$

We will explain why $I_\mu(\sigma)$ is finite. Note that g_μ is lower semicontinuous on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$ (cf. Proposition 7.9), and so bounded from below. The first term is equal to $M^2 \cdot I_\mu(\nu)$, and we have already seen that $I_\mu(\nu) < \infty$. Since g_μ is bounded from below and ν is a positive measure, the first term is finite. The second term is finite because $\overline{U}$ and E_n are compact disjoint sets and g_μ is continuous off the diagonal (see Proposition 7.9). The third term has to be finite as well as g_μ is bounded from below, ω is a positive measure, and we have

$$\iint g_\mu(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) = I_\mu(\omega) = V(\mu) \in \mathbb{R}$$

by Lemma 8.10. Consequently, $I_\mu(\sigma)$ is finite.

For every $t \in [0,1]$, we define the probability measure $\omega_t := \omega + t\sigma$ on X^{an}. Then

$$\begin{aligned} I_\mu(\omega_t) - I_\mu(\omega) &= \iint g_\mu(x,y)\,\mathrm{d}\omega_t(x)\mathrm{d}\omega_t(y) - \iint g_\mu(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) \\ &= \iint g_\mu(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) + 2\iint g_\mu(x,y)\,\mathrm{d}\omega(x)d(t\sigma)(y) \\ &\quad + \iint g_\mu(x,y)\,\mathrm{d}(t\sigma)(x)d(t\sigma)(y) - \iint g_\mu(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y) \\ &= 2t \cdot \int u_\omega(y,\mu)\ d\sigma(y) + t^2 \cdot I_\mu(\sigma). \end{aligned}$$

Inserting the definition of the measure σ, we obtain

$$\begin{aligned} &I_\mu(\omega_t) - I_\mu(\omega) \\ &\qquad = 2t \cdot \left(M \cdot \int_{E_n} u_\omega(y,\mu)\,\mathrm{d}\nu(y) - \int_{\overline{U}} u_\omega(y,\mu)\,\mathrm{d}\omega(y) \right) + t^2 \cdot I_\mu(\sigma). \end{aligned}$$

Since $u_\omega(\,\cdot\,,\mu) \le V(\mu) - 1/n$ on E_n and $\mathrm{supp}(\nu) \subset E_n$, $u_\omega(\,\cdot\,,\mu) > V(\mu) - 1/(2n)$ on $\overline{U}$ and $M = \omega(\overline{U}) > 0$, we get

$$\begin{aligned} I_\mu(\omega_t) - I_\mu(\omega) &\le 2t \cdot (M \cdot (V(\mu) - 1/n) - M \cdot (V(\mu) - 1/(2n))) + t^2 \cdot I_\mu(\sigma) \\ &= (-M/n) \cdot t + t^2 \cdot I_\mu(\sigma). \end{aligned}$$

The right hand side is negative for sufficiently small $t > 0$ as $I_\mu(\sigma)$ is finite, and so this contradicts $I_\mu(\omega) = V(\mu)$. Hence each E_n has capacity zero, and so does E. By Lemma 5.5, we get $E \cap \mathbb{H}(X^{\mathrm{an}}) = \emptyset$.

Step 2: Show that $\omega(E) = 0$. Pick a base point $\zeta_0 \in I(X^{\mathrm{an}})$. We have seen in Step 1 that $E \subset X(K)$, so ζ_0 cannot by contained in E. Because of $I_{\zeta_0,\zeta_0}(\omega) = \iint g_{\zeta_0}(x,y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y)$ by Corollary 4.10, we have

$$\begin{aligned} I_\mu(\omega) &= \iint g_\mu(x,y)\,\mathrm{d}\omega(y)\mathrm{d}\omega(x) \\ &= \iint g_{\zeta_0}(x,y)\,\mathrm{d}\omega(y)\mathrm{d}\omega(x) - \iint g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\omega(x) \\ &\qquad - \iint g_{\zeta_0}(y,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\omega(y) + C_{\zeta_0} \\ &= I_{\zeta_0,\zeta_0}(\omega) - 2\iint g_{\zeta_0}(x,\zeta)\,\mathrm{d}\mu(\zeta)\mathrm{d}\omega(x) + C_{\zeta_0}, \end{aligned}$$

where the double integral is finite since μ has continuous potentials. As $I_\mu(\omega) = V(\mu)$ is finite by Lemma 8.10, it follows directly from the calculation that $I_{\zeta_0,\zeta_0}(\omega) < \infty$. Moreover, we have seen in the proof of Step 1 that E has capacity zero and we also know that $\zeta_0 \notin E$. Lemma 5.6 yields $\omega(E) = 0$.

Step 3: Show that $u_\omega(\,\cdot\,,\mu) \leq V(\mu)$ *on* X^{an}. Using Maria's theorem 8.8, it remains to prove $u_\omega(\,\cdot\,,\mu) \leq V(\mu)$ on $\mathrm{supp}(\omega)$. Assume there is a point $z \in \mathrm{supp}(\omega)$ such that $u_\omega(z,\mu) > V(\mu)$. Choose $\varepsilon > 0$ such that $u_\omega(z,\mu) > V(\mu)+\varepsilon$. Since $u_\omega(\,\cdot\,,\mu)$ is lower semicontinuous on X^{an} by Proposition 8.5, there is an open neighborhood U_z of z with $u_\omega(\,\cdot\,,\mu) > V(\mu) + \varepsilon$ on U_z. Then $\omega(U_z) > 0$ as $z \in \mathrm{supp}(\omega)$. By the construction of E, we have $u_\omega(\,\cdot\,,\mu) < V(\mu)$ on E. Hence E and U_z are disjoint and we get the following decomposition of $V(\mu) = I_\mu(\omega)$

$$\begin{aligned} V(\mu) &= \int_{X^{\mathrm{an}}} u_\omega(x,\mu)\,\mathrm{d}\omega(x) \\ &= \int_{U_z} u_\omega(x,\mu)\,\mathrm{d}\omega(x) + \int_{X^{\mathrm{an}}\setminus(U_z\cup E)} u_\omega(x,\mu)\,\mathrm{d}\omega(x). \end{aligned}$$

Note that we also use that the integral of $u_\omega(\,\cdot\,,\mu)$ over E has to be zero as $\omega(E) = 0$ by Step 2. For the first term we know that $u_\omega(\,\cdot\,,\mu) > V(\mu) + \varepsilon$ on U_z and $\omega(U_z) > 0$. Thus

$$\int_{X^{\mathrm{an}}} u_\omega(x,\mu)\,\mathrm{d}\omega(x) \geq \omega(U_z)\cdot(V(\mu)+\varepsilon). \tag{8.5}$$

We have $u_\omega(\,\cdot\,,\mu) \geq V(\mu)$ on $X^{\mathrm{an}}\setminus E$ by the definition of E, and so

$$\int_{X^{\mathrm{an}}\setminus(U_z\cup E)} u_\omega(x,\mu)\,\mathrm{d}\omega(x) \geq (1-\omega(U_z)-\omega(E))\cdot V(\mu). \tag{8.6}$$

Putting (8.5), (8.6) and $\omega(E) = 0$ together, we get the contradiction

$$\begin{aligned}V(\mu) &\geq \omega(U_z) \cdot (V(\mu) + \varepsilon) + (1 - \omega(U_z) - \omega(E)) \cdot V(\mu) \\ &= \omega(U_z) \cdot (V(\mu) + \varepsilon) + (1 - \omega(U_z)) \cdot V(\mu) \\ &= V(\mu) + \omega(U_z)\varepsilon > V(\mu).\end{aligned}$$

Hence $u_\omega(\,\cdot\,, \mu) \leq V(\mu)$ on $\mathrm{supp}(\omega)$. Maria's theorem 8.8, implies that $u_\omega(\,\cdot\,, \mu) \leq V(\mu)$ on X^{an}. This shows the third step.

By the first step we know that $u_\omega(\,\cdot\,, \mu) \geq V(\mu)$ on $X^{\mathrm{an}} \backslash X(K)$. For every point $y \in X(K)$, we can find a path $[z, y]$ from a point $z \in I(X^{\mathrm{an}})$ to y such that $[z, y)$ is contained in $I(X^{\mathrm{an}}) \subset X^{\mathrm{an}} \backslash X(K)$. Then Proposition 8.5 implies

$$u_\omega(y, \mu) = \lim_{x \in [z,y)} u_\omega(x, \mu) \geq V(\mu).$$

Hence $E = \{x \in X^{\mathrm{an}} \mid u_\omega(x, \mu) < V(\mu)\}$ is empty, and so $u_\omega(\,\cdot\,, \mu) \geq V(\mu)$ on X^{an}. Step 3 implies $u_\omega(\,\cdot\,, \mu) \equiv V(\mu)$ on X^{an}. □

Proof of Theorem 8.2. Let ω be a probability measure on X^{an} that minimizes the energy integral, i.e. $I_\mu(\omega) = V(\mu)$. Such a measure always exists by Lemma 8.10. By Frostman's theorem 8.11, $u_\omega(\,\cdot\,, \mu)$ is constant on X^{an}, and hence

$$0 = \mathrm{dd}^c u_\omega(\,\cdot\,, \mu) = \mu - \omega$$

by Proposition 8.5. Thus ω minimizes the energy integral if and only if $\omega = \mu$. Since $I_\mu(\mu) = \iint g_\mu(x, y)\, \mathrm{d}\mu(y)\mathrm{d}\mu(x) = 0$ by the normalization of the Arakelov–Green's function g_μ, it follows that $I_\mu(\nu) \geq 0$ for every probability measure ν on X^{an}. □

Corollary 8.12. *Let $\zeta \in I(X^{\mathrm{an}})$ and μ be a probability measure on X^{an} with continuous potentials. Then $g_\mu(\zeta, \zeta) \geq 0$, and $g_\mu(\zeta, \zeta) = 0$ if and only if $\mu = \delta_\zeta$.*

Proof. Since

$$g_\mu(\zeta, \zeta) = \iint g_\mu(x, y)\, \mathrm{d}\delta_\zeta(x) d\delta_\zeta(y) = I_\mu(\delta_\zeta),$$

the Energy Minimization Principle (Theorem 8.2) gives the assertion immediately. □

9. Local discrepancy

Let E be an elliptic curve over K with j-invariant j_E. In this section, we give a different proof of the local discrepancy result from [3, Corollary 5.6] using our Energy Minimization Principle (Theorem 8.2).

Remark 9.1. In the following, let Γ_E be the minimal skeleton of E^{an}. Then Γ_E is a single point ζ_0 when E has good reduction and Γ_E corresponds to the circle $\mathbb{R}/\mathbb{Z}$ when it has multiplicative reduction. One has a canonical probability measure μ_E supported on Γ_E, where

(1) μ_E is the dirac measure in ζ_0 if E has good reduction, and
(2) μ_E is the uniform probability measure (i.e. Haar measure) supported on the circle $\Gamma_E \simeq \mathbb{R}/\mathbb{Z}$ if E has multiplicative reduction.

Then μ_E has in particular continuous potentials by Example 7.5. Hence we can consider its corresponding Arakelov–Green's function g_{μ_E} on $E^{\mathrm{an}} \times E^{\mathrm{an}}$.

Definition 9.2. Let $Z = \{P_1, \ldots, P_N\}$ be a set of N distinct points in $E(K)$. Then the *local discrepancy* of Z is defined as

$$D(Z) := \frac{1}{N^2}\left(\sum_{i \neq j} g_{\mu_E}(P_i, P_j) + \frac{N}{12}\log^+ |j_E|\right).$$

Remark 9.3. Baker and Petsche defined the local discrepancy in [3, §3.4] and [11, §2.2] of a set $Z = \{P_1, \ldots, P_N\}$ of N distinct points in $E(K)$ as

$$\frac{1}{N^2}\left(\sum_{i \neq j} \lambda(P_i - P_j) + \frac{N}{12}\log^+ |j_E|_v\right)$$

for the Néron function $\lambda\colon E(K)\backslash\{O\} \to \mathbb{R}$ (cf. [12, §VI.1]).

Note that our definition is consistent with theirs. As it is also mentioned in [3, Remark 5.3], the Néron function can be extend to an Arakelov–Green's function corresponding to the canonical measure μ on E^{an}. By the uniqueness of the Arakelov–Green's function (see Remark 7.2), we have $g_{\mu_E}(P, Q) = \lambda(P - Q)$ for $P \neq Q \in E(K)$.

Baker and Petsche showed in [3, Corollary 5.6] the following result for the local discrepancy when $K = \mathbb{C}_v$. Here, v is a non-archimedean place of a number field k and $\mathbb{C}_v$ is the completion of the algebraic closure of the completion of k with respect to v. We can prove this statement for our general K using our characterization of the local discrepancy and the Energy Minimization Principle (Theorem 8.2).

Corollary 9.4. *For each $n \in \mathbb{N}$, let $Z_n \subset E(K)$ be a set consisting of n distinct points and let δ_n be the probability measure on E^{an} that is equidistributed on Z_n. If $\lim_{n\to\infty} D(Z_n) = 0$, then δ_n converges weakly to μ_E on E^{an}.*

Proof. By passing to a subsequence we may assume that δ_n converges weakly to a probability measure ν on E^{an} (see Proposition 2.5). We show that $I_{\mu_E}(\nu)$ is zero and we then use the Energy Minimization Principle 8.2. We have seen in the Energy Minimization Principle that $I_{\mu_E}(\nu) \geq 0$. Thus it remains to show $I_{\mu_E}(\nu) \leq 0$. Due to the definition of the μ_E-energy integral and [5, Lemma 7.54], the following inequality holds

$$\begin{aligned} I_{\mu_E}(\nu) &= \iint_{E^{\mathrm{an}} \times E^{\mathrm{an}}} g_{\mu_E}(x,y)\,\mathrm{d}\nu(x)\mathrm{d}\nu(y) \\ &\leq \liminf_{n\to\infty} \iint_{(E^{\mathrm{an}} \times E^{\mathrm{an}})\setminus\Delta} g_{\mu_E}(x,y)\,\mathrm{d}\delta_n(x)d\delta_n(y) \\ &= \liminf_{n\to\infty} \frac{1}{n^2} \sum_{P\neq Q\in Z_n} g_{\mu_E}(P,Q), \end{aligned}$$

where $\Delta := \mathrm{Diag}(E^{\mathrm{an}})$. Since the term $D(Z_n) = \frac{1}{n^2}\sum_{P\neq Q\in Z_n} g_{\mu_E}(P,Q) + \frac{1}{12n}\log^+|j_E|$ converges to zero, and $\frac{1}{12n}\log^+|j_E|$ does as well, we have

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{P\neq Q\in Z_n} g_{\mu_E}(P,Q) = 0.$$

Hence $I_{\mu_E}(\nu) \leq 0$. The Energy Minimization Principle yields $\mu_E = \nu$. We just have seen that every convergent subsequence converges weakly to same limiting measure μ_E, and so does the overall sequence. □

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