Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

Ryotaro SAKAMOTO

On the theory of Kolyvagin systems of rank 0

Tome 33, nº 3.2 (2021), p. 1077-1102.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2021__33_3.2_1077_0>

Journal de Théorie des Nombres
de Bordeaux **33** (2021), 1077–1102

On the theory of Kolyvagin systems of rank 0

par Ryotaro SAKAMOTO

Résumé. Dans cet article, nous définissons un système Kolyvagin de rang 0 et développons la théorie des systèmes Kolyvagin de rang 0. En particulier, nous prouvons que le module des systèmes Kolyvagin de rang 0 est libre de rang un sous les hypothèses standard.

Abstract. In this paper, we define a Kolyvagin system of rank 0 and develop the theory of Kolyvagin systems of rank 0. In particular, we prove that the module of Kolyvagin systems of rank 0 is free of rank one under standard assumptions.

1. Introduction

Let R be a complete noetherian local ring with finite residue field of characteristic $p > 2$ and T a free R-module with an R-linear continuous action of the absolute Galois group G_k of a number field k which is unramified outside a finite set of places of k. Let $\mathcal{F}$ be a Selmer structure on T (see Definition 2.1).

In [11], Mazur and Rubin introduced the notion of Kolyvagin systems of rank 1 for the pair $(T, \mathcal{F})$ in order to understand well the theory of Euler systems. Howard, Mazur, and Rubin proved in [11] the following

Theorem 1.1 ([11, Corollary 4.5.2 (i), Theorems 4.5.6, 5.2.10, and 5.2.12])**.** *Suppose that*

- *R is a principal ideal ring,*
- *$\mathcal{F}$ is cartesian (see Definition 2.11),*
- *T satisfies hypothesis 2.7,*
- *the core rank $\chi(\mathcal{F})$ is 1 (see Definition 2.15).*

Then the R-module $\mathrm{KS}_1(T, \mathcal{F})$ of Kolyvagin systems of rank 1 for $(T, \mathcal{F})$ is free of rank 1. Furthermore, a basis of $\mathrm{KS}_1(T, \mathcal{F})$ gives a sharp bound for the order of the dual Selmer module associated to $\mathcal{F}$.

In [6], Büyükboduk generalized Theorem 1.1 to the case that the coefficient ring R is the formal power series ring over $\mathbb{Z}_p$. Moreover, in the papers [3, 4, 5, 7], Büyükboduk considered locally restricted Euler and

---

Manuscrit reçu le 24 février 2020, révisé le 8 octobre 2020, accepté le 4 novembre 2020.
Mathematics Subject Classification. 11F80, 11R34, 11R23.
Mots-clefs. Kolyvagin system, Selmer group.
The author was supported by JSPS KAKENHI Grant Number 20J00456.

Kolyvagin systems in order to get the relation between higher rank Euler systems and Kolyvagin systems (of rank 1) and proved that, under certain assumptions, the higher rank Euler system associated with the conjectural Rubin–Stark units gives a sharp bound for the ideal class group.

To remove the assumption in Theorem 1.1 that $\chi(\mathcal{F}) = 1$, in [12], Mazur and Rubin defined higher rank Kolyvagin systems by using the exterior power. However, as they mentioned in [12, Remark 11.9], it is possible that there is a higher rank Kolyvagin system entirely unrelated to the dual Selmer module. Therefore, Mazur and Rubin considered special Kolyvagin systems that they call stub Kolyvagin systems and proved that the corresponding dual Selmer module is controlled by stub Kolyvagin systems.

After Mazur–Rubin's works, in [2], Burns and Sano modified the definition of higher rank Kolyvagin systems by using the exterior bi-dual instead of the exterior power. Furthermore, in [1], Burns, Sakamoto, and Sano proved the following theorem which is a generalization of Theorem 1.1 to the higher rank and Gorenstein coefficient case:

Theorem 1.2 ([1, Theorem 5.2]). *Suppose that*

- *R is a zero dimensional Gorenstein ring,*
- *$\mathcal{F}$ is cartesian,*
- *T satisfies hypothesis 2.7,*
- *$r := \chi(\mathcal{F}) \geq 1$.*

Then the R-module $\mathrm{KS}_r(T, \mathcal{F})$ of Kolyvagin systems of rank r for $(T, \mathcal{F})$ defined in [2] is free of rank 1. Furthermore, a basis of $\mathrm{KS}_r(T, \mathcal{F})$ controls the initial Fitting ideal of the dual Selmer module associated to $\mathcal{F}$.

In spite of the fact that there are important Selmer structures $\mathcal{F}$ on T with $\chi(\mathcal{F}) = 0$ (see Section 2.4), there is no definition of Kolyvagin systems of rank 0 because of some technical obstacles (see Remark 4.3). In this paper, we define Kolyvagin systems of rank 0 and prove the same result as Theorem 1.2, namely, the main result of the present article is the following

Theorem 1.3 (Theorems 5.5 and 5.8). *Suppose that*

- *R is a zero dimensional Gorenstein ring,*
- *$\mathcal{F}$ is cartesian,*
- *T satisfies hypothesis 2.7,*
- *$\chi(\mathcal{F}) = 0$.*

Then the R-module $\mathrm{KS}_0(T, \mathcal{F})$ of Kolyvagin systems of rank 0 for $(T, \mathcal{F})$ defined in Definition 5.1 is free of rank 1. Furthermore, a basis of $\mathrm{KS}_0(T, \mathcal{F})$ controls the initial Fitting ideal of the dual Selmer module associated to $\mathcal{F}$.

1.1. Notations. Let p be an odd prime. For a field k, we fix a separable closure $\overline{k}$ of k and denote by $G_k := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k.

For any number field k and places v of $\mathbb{Q}$, we denote by $S_v(k)$ the set of places of k above v. For a finite set S of places of k, we denote by k_S the maximal extension of k contained in $\overline{k}$ which is unramified outside S. Set

$$G_{k,S} := \mathrm{Gal}(k_S/k).$$

For a prime $\mathfrak{q}$ of k, we denote by $k_{\mathfrak{q}}$ the completion of k at $\mathfrak{q}$. We fix a place $\widetilde{\mathfrak{q}}$ of $\overline{k}$ lying above $\mathfrak{q}$ and identify the decomposition group of $\widetilde{\mathfrak{q}}$ in G_k with $G_{k_{\mathfrak{q}}}$.

For a pro-finite group G and a topological G-module M, let $C^\bullet(G, M)$ denote the complex of inhomogeneous continuous cochains of G with values in M. For each integer $i \geq 0$, we write $H^i(G, M)$ for its i-th cohomology group.

For an R-module M and an ideal I of R, we set

$$M[I] := \{m \in M \mid Im = 0\}.$$

2. Selmer structure

Throughout this paper, we use the following notations. Let R be a complete noetherian local ring with finite residue field of characteristic $p > 2$ and $\mathfrak{m}_R$ denotes the maximal ideal of R. Let k be a number field. Fix a finite set of places S of k satisfying

$$S_p(k) \cup S_\infty(k) \subseteq S.$$

Let T be a free R-module of finite rank with an R-linear continuous action of $G_{k,S}$.

2.1. Definition of Selmer structure. The contents of this subsection are based on [11, Chapters 1 and 2].

Definition 2.1 ([11, Definition 2.1.1])**.** A Selmer structure $\mathcal{F}$ on T is a collection of the following data:

- a finite set $S(\mathcal{F})$ of places of k containing S,
- a choice of R-submodule $H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T)$ of $H^1(G_{k_{\mathfrak{q}}}, T)$ for each prime $\mathfrak{q} \in S(\mathcal{F})$.

For each prime $\mathfrak{q} \notin S(\mathcal{F})$, we set

$$H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T) := H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) := \mathrm{im}\Big(H^1(\mathrm{Gal}(k_{\mathfrak{q}}^{\mathrm{ur}}/k_{\mathfrak{q}}), T) \hookrightarrow H^1(G_{k_{\mathfrak{q}}}, T)\Big),$$

where $k_{\mathfrak{q}}^{\mathrm{ur}}$ denotes the maximal unramified extension of $k_{\mathfrak{q}}$. We call $H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T)$ the local condition of $\mathcal{F}$ at a prime $\mathfrak{q}$ of k.

Remark 2.2. Since $p > 2$, the module $H^1(G_{k_v}, T)$ vanishes for any $v \in S_\infty(k)$. Therefore, in this paper, we ignore the local condition at any infinite place of k.

Remark 2.3. Let $R \to R'$ be a surjective ring homomorphism. A Selmer structure $\mathcal{F}$ on T induces a Selmer structure on $T \otimes_R R'$, that we will denote by $\mathcal{F}_{R'}$. If there is no risk of confusion, then we also write $\mathcal{F}$ instead of $\mathcal{F}_{R'}$.

For a Selmer structure $\mathcal{F}$ on T, we define the Selmer module $H^1_{\mathcal{F}}(k, T) \subseteq H^1(G_{k,S(\mathcal{F})}, T)$ associated with $\mathcal{F}$ by

$$H^1_{\mathcal{F}}(k, T) := \ker \left(H^1(G_{k,S(\mathcal{F})}, T) \longrightarrow \bigoplus_{\mathfrak{q} \in S(\mathcal{F})} H^1_{/\mathcal{F}}(G_{k_{\mathfrak{q}}}, T) \right).$$

Here we set

$$H^1_{/\mathcal{F}}(G_{k_{\mathfrak{q}}}, T) := H^1(G_{k_{\mathfrak{q}}}, T)/H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T)$$

for any prime $\mathfrak{q}$ of k.

Definition 2.4.

(i) For a topological $\mathbb{Z}_p$-module M, we define the Pontryagin dual $M^\vee$ of M by
$$M^\vee := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p).$$
(ii) For any integer $n > 0$, let μ_n denote the group of all n-th roots of unity.

(iii) The Cartier dual of T is defined by
$$T^\vee(1) := T^\vee \otimes_{\mathbb{Z}_p} \varprojlim_{n>0} \mu_{p^n} = \mathrm{Hom}_{\mathrm{cont}}(T, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \varprojlim_{n>0} \mu_{p^n}.$$

Let $\mathcal{F}$ be a Selmer structure on T. For a prime $\mathfrak{q}$ of k, we have the local Tate paring

$$\langle \cdot, \cdot \rangle_{\mathfrak{q}} \colon H^1(G_{k_{\mathfrak{q}}}, T) \times H^1(G_{k_{\mathfrak{q}}}, T^\vee(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

We define an R-submodule $H^1_{\mathcal{F}^*}(G_{k_{\mathfrak{q}}}, T^\vee(1))$ of $H^1(G_{k_{\mathfrak{q}}}, T^\vee(1))$ to be the orthogonal complement of $H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T)$ under the local Tate paring $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$. Hence we obtain the dual Selmer module $H^1_{\mathcal{F}^*}(k, T^\vee(1))$ associated with $\mathcal{F}$:

$$H^1_{\mathcal{F}^*}(k, T^\vee(1)) := \ker \left(H^1(G_{k,S(\mathcal{F})}, T^\vee(1)) \longrightarrow \bigoplus_{\mathfrak{q} \in S(\mathcal{F})} H^1_{/\mathcal{F}^*}(G_{k_{\mathfrak{q}}}, T^\vee(1)) \right).$$

Theorem 2.5 ([11, Theorem 2.3.4])**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Selmer structures on T satisfying*

$$H^1_{\mathcal{F}_1}(G_{k_{\mathfrak{q}}}, T) \subseteq H^1_{\mathcal{F}_2}(G_{k_{\mathfrak{q}}}, T)$$

for any prime $\mathfrak{q}$ of k. Then there is an exact sequence of R-modules

$$\begin{aligned} 0 \longrightarrow H^1_{\mathcal{F}_1}(k, T) \longrightarrow H^1_{\mathcal{F}_2}(k, T) \longrightarrow \bigoplus_{\mathfrak{q}} H^1_{\mathcal{F}_2}(G_{k_{\mathfrak{q}}}, T)/H^1_{\mathcal{F}_1}(G_{k_{\mathfrak{q}}}, T) \\ \longrightarrow H^1_{\mathcal{F}_1^*}(k, T^\vee(1))^\vee \longrightarrow H^1_{\mathcal{F}_2^*}(k, T^\vee(1))^\vee \longrightarrow 0, \end{aligned}$$

where $\mathfrak{q}$ runs over all the primes of k satisfying $H^1_{\mathcal{F}_1}(G_{k_{\mathfrak{q}}}, T) \neq H^1_{\mathcal{F}_2}(G_{k_{\mathfrak{q}}}, T)$.

2.2. Hypotheses.

Definition 2.6. For a G_k-module M, we define the field $k(T)$ to be the minimal Galois extension of k such that $G_{k(T)}$ acts trivially on T.

We denote by $\mathcal{H}$ the Hilbert class field of k. We set

$$k_{p^n} := \mathcal{H}k((\mathcal{O}_k^\times)^{1/p^n}) \text{ and } k(T)_{p^n} := k_{p^n}k(T).$$

Here $(\mathcal{O}_k^\times)^{1/p^n}$ denotes the set $\{u \in \bar{k} \mid u^{p^n} \in \mathcal{O}_k^\times\}$. We also put

$$k_{p^\infty} := \bigcup_{n>0} k_{p^n} \text{ and } k(T)_{p^\infty} := k_{p^\infty}k(T).$$

To simplify the notation, we set

$$\overline{T} = T/\mathfrak{m}_R T.$$

In this paper, we assume the following hypotheses for the R-module T (introduced by Mazur and Rubin in [11, Section 3.5]).

Hypothesis 2.7.

(i) The $R/\mathfrak{m}_R[G_k]$-module $\overline{T}$ is irreducible.

(ii) There exists an element $\tau \in G_{k_{p^\infty}}$ such that the R-module $T/(1-\tau)T$ is free of rank 1.

(iii) The modules $H^1(\mathrm{Gal}(k(T)_{p^\infty}/k), \overline{T})$ and $H^1(\mathrm{Gal}(k(T)_{p^\infty}/k), \overline{T}^\vee(1))$ vanish.

(iv) If $p = 3$, then $\overline{T}$ and $\overline{T}^\vee(1)$ have no nonzero isomorphic $\mathbb{Z}_p[G_k]$-subquotients.

Remark 2.8. If T satisfies hypothesis 2.7, then so does T/IT for any ideal I of R.

Definition 2.9. Let $n > 0$ be an integer.

(i) We define $\mathcal{P}_n(T)$ to be the set of primes $\mathfrak{q} \notin S$ of k such that the Frobenius $\mathrm{Fr}_\mathfrak{q}$ at $\mathfrak{q}$ is conjugate to τ in $\mathrm{Gal}(k(T)_{p^n}/k)$.

(ii) For a Selmer structure $\mathcal{F}$ on T, we put $\mathcal{P}_n(T, \mathcal{F}) := \mathcal{P}_n(T) \setminus S(\mathcal{F})$.

(iii) For a set $\mathcal{Q}$ of primes of k, we denote by $\mathcal{N}(\mathcal{Q})$ the set of square-free products of primes in $\mathcal{Q}$.

Definition 2.10. For a prime $\mathfrak{q}$ of k, we denote by $k(\mathfrak{q})$ the maximal p-extension of k inside the ray class field modulo $\mathfrak{q}$. We also denote by $k(\mathfrak{q})_\mathfrak{q}$ the completion of $k(\mathfrak{q})$ at the (fixed) prime lying above $\mathfrak{q}$.

Let us suppose that R is artinian. Fix an integer $n > 0$ with $p^n R = 0$. A prime $\mathfrak{q} \in \mathcal{P}_n(T)$ splits completely in k_{p^n}, and hence $k(\mathfrak{q})_\mathfrak{q}/k_\mathfrak{q}$ is a totally tamely ramified cyclic extension such that $[k(\mathfrak{q})_\mathfrak{q} \colon k_\mathfrak{q}]$ annihilates R. Thus we can define the subgroup $H^1_{\mathrm{tr}}(G_{k_\mathfrak{q}}, T)$ by

$$H^1_{\mathrm{tr}}(G_{k_\mathfrak{q}}, T) := \ker\Big(H^1(G_{k_\mathfrak{q}}, T) \longrightarrow H^1(G_{k(\mathfrak{q})_\mathfrak{q}}, T)\Big)$$

(see Definition A.1). Furthermore, by Hypothesis 2.7 (ii), the R-module $T/(1-\mathrm{Fr}_{\mathfrak{q}})T$ is free of rank 1, which implies that

$$H^1(G_{k_{\mathfrak{q}}}, T) = H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \oplus H^1_{\mathrm{tr}}(G_{k_{\mathfrak{q}}}, T)$$

and $H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T)$, $H^1_{\mathrm{tr}}(G_{k_{\mathfrak{q}}}, T)$, $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T)$, and $H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}}, T)$ are free R-modules of rank 1 (see Lemmas A.2 and A.4).

Let $\mathcal{F}$ be a Selmer structure on T, and let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be pairwise relatively prime integral ideals in $\mathcal{N}(\mathcal{P}_n(T, \mathcal{F}))$. Define a new Selmer structure $\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})$ on T by the following data:

- $S(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})) := S(\mathcal{F}) \cup \{\mathfrak{q} \mid \mathfrak{abc}\}$,
- define $H^1_{\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(G_{k_{\mathfrak{q}}}, T) := \begin{cases} H^1(G_{k_{\mathfrak{q}}}, T) & \text{if } \mathfrak{q} \mid \mathfrak{a}, \\ 0 & \text{if } \mathfrak{q} \mid \mathfrak{b}, \\ H^1_{\mathrm{tr}}(G_{k_{\mathfrak{q}}}, T) & \text{if } \mathfrak{q} \mid \mathfrak{c}, \\ H^1_{\mathcal{F}}(G_{k_{\mathfrak{q}}}, T) & \text{otherwise.} \end{cases}$

We note that $(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c}))^* = (\mathcal{F}^*)^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{c})$ by [11, Proposition 1.3.2].

2.3. Core rank and cartesian condition. In this subsection, suppose that

- R is a zero dimensional Gorenstein ring, and
- T satisfies Hypothesis 2.7.

We fix an R-isomorphism $R/\mathfrak{m}_R \xrightarrow{\sim} R[\mathfrak{m}_R]$, and it induces an injection

$$\overline{T} \hookrightarrow T.$$

Let $\mathcal{F}$ be a Selmer structure on T.

Definition 2.11 ([13, Definition 3.8])**.** We say that the Selmer structure $\mathcal{F}$ on T is cartesian if the map

$$H^1_{/\mathcal{F}}(G_{k_{\mathfrak{q}}}, \overline{T}) \longrightarrow H^1_{/\mathcal{F}}(G_{k_{\mathfrak{q}}}, T)$$

induced by the injection $\overline{T} \hookrightarrow T$ is injective for any prime $\mathfrak{q} \in S(\mathcal{F})$.

Remark 2.12.

(i) When R is a principal artinian local ring, the cartesian condition in the sense of this paper is equivalent to that of [11, Definition 1.1.4]. Moreover, under the setting of [6], the conditions (C2) and (C3) in [6, Definition 2.5] are equivalent to the cartesian condition in the sense of this paper.

(ii) The definition of cartesian condition is independent of the choice of the isomorphism $R/\mathfrak{m}_R \xrightarrow{\sim} R[\mathfrak{m}_R]$.

(iii) Let R' be a zero dimensional Gorenstein local ring and $\pi\colon R \to R'$ a surjective ring homomorphism. If $\mathcal{F}$ is cartesian, then so is $\mathcal{F}_{R'}$ ([13, Lemma 3.11]).

Fix an integer $n > 0$ with $p^n R = 0$.

Lemma 2.13 ([13, Corollary 3.18]). *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{N}(\mathcal{P}_n(T, \mathcal{F}))$ be pairwise relatively prime integral ideals. If $\mathcal{F}$ is cartesian, then so is $\mathcal{F}_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{c})$.*

Lemma 2.14 ([1, §3.2], [11, Lemma 3.5.3], [13, Lemmas 3.13 and 3.14]).

(i) *For an ideal I of R, the inclusion map $(T/IT)^\vee(1) = T^\vee(1)[I] \hookrightarrow T^\vee(1)$ induces an isomorphism*
$$H^1_{\mathcal{F}^*}(k, (T/IT)^\vee(1)) \xrightarrow{\sim} H^1_{\mathcal{F}^*}(k, T^\vee(1))[I].$$

(ii) *If $\mathcal{F}$ is cartesian, then the fixed injection $\overline{T} \hookrightarrow T$ induces an isomorphism*
$$H^1_{\mathcal{F}}(k, \overline{T}) \xrightarrow{\sim} H^1_{\mathcal{F}}(k, T)[\mathfrak{m}_R].$$

Definition 2.15. We define the core rank $\chi(\mathcal{F})$ of the Selmer structure $\mathcal{F}$ on T by
$$\chi(\mathcal{F}) := \dim_{R/\mathfrak{m}_R} H^1_{\mathcal{F}}(k, \overline{T}) - \dim_{R/\mathfrak{m}_R} H^1_{\mathcal{F}^*}(k, \overline{T}^\vee(1)).$$

Lemma 2.16 ([13, Corollary 3.21]). *For any pairwise relatively prime integral ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{N}(\mathcal{P}_n(T, \mathcal{F}))$, we have*
$$\chi(\mathcal{F}_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{c})) = \chi(\mathcal{F}) + \nu(\mathfrak{a}) - \nu(\mathfrak{b}),$$
where $\nu(\mathfrak{n})$ denotes the number of prime divisors of $\mathfrak{n}$.

Lemma 2.17 ([11, Corollary 4.1.9]). *Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T, \mathcal{F})$ containing $\mathcal{P}_m(T, \mathcal{F})$ for some integer $m \geq n$. Suppose that $\chi(\mathcal{F}) \geq 0$. Then, for any ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$, there is an ideal $\mathfrak{m} \in \mathcal{N}(\mathcal{Q})$ with $\mathfrak{n} \mid \mathfrak{m}$ such that $H^1_{\mathcal{F}^*(\mathfrak{m})}(k, \overline{T}^\vee(1))$ vanishes.*

Lemma 2.18 ([13, Lemma 4.6]). *Let $\mathcal{F}$ be a cartesian Selmer structure on T and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{N}(\mathcal{P}_n(T, \mathcal{F}))$ pairwise relatively prime integral ideals. If the module $H^1_{(\mathcal{F}_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{c}))^*}(k, \overline{T}^\vee(1))$ vanishes and $\chi(\mathcal{F}) + \nu(\mathfrak{a}) - \nu(\mathfrak{b}) \geq 0$, then the R-module $H^1_{\mathcal{F}_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{c})}(k, T)$ is free of rank $\chi(\mathcal{F}) + \nu(\mathfrak{a}) - \nu(\mathfrak{b})$.*

Lemma 2.19 ([13, Lemma 3.25]). *Let M be a free R-submodule of $H^1(G_k, T)$ of rank $s \geq 0$. For a ring homomorphism $R \to R'$, the canonical map*
$$M \otimes_R R' \longrightarrow H^1(G_k, T \otimes_R R')$$
is a split injection.

2.4. Examples. In this subsection, we give two important examples of Selmer strucures.

Example 2.20 ([12, Section 5.2]). Let $\chi\colon G_k \to \overline{\mathbb{Q}}^\times$ be a prime-to-p finite order character. We set $R := \mathbb{Z}_p[\mathrm{im}(\chi)]$ and $T := R(1) \otimes \chi^{-1}$. In this setting, we define a canonical unramified Selmer structure $\mathcal{F}_{\mathrm{ur}}$ on T by the following data.

- We set $S(\mathcal{F}_{\mathrm{ur}}) := S_p(k) \cup S_\infty(k) \cup \{\mathfrak{q} \mid T \text{ is ramified at } \mathfrak{q}\}$,
- For a prime $\mathfrak{q} \in S(\mathcal{F}_{\mathrm{ur}})$ with $\mathfrak{q} \nmid p$, we set
$$H^1_{\mathcal{F}_{\mathrm{ur}}}(G_{k_\mathfrak{q}}, T) := \ker\Big(H^1(G_{k_\mathfrak{q}}, T) \longrightarrow H^1(G_{k_\mathfrak{q}^{\mathrm{ur}}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\Big).$$
- For a prime $\mathfrak{p} \in S_p(k)$, we set
$$H^1(G_{k_\mathfrak{p}}, T)^{\mathrm{u}} := \bigcap_{k_\mathfrak{p} \subseteq L \subseteq k_\mathfrak{p}^{\mathrm{ur}}} \mathrm{im}\Big(H^1(L, T) \longrightarrow H^1(G_{k_\mathfrak{p}}, T)\Big),$$
where L runs over all finite unramified extension of $k_\mathfrak{p}$. Put
$$H^1_{\mathcal{F}_{\mathrm{ur}}}(G_{k_\mathfrak{p}}, T) := \ker\Big(H^1(G_{k_\mathfrak{p}}, T) \longrightarrow (H^1(G_{k_\mathfrak{p}}, T)/H^1(G_{k_\mathfrak{p}}, T)^{\mathrm{u}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\Big).$$

We put $k_\chi := \overline{k}^{\ker(\chi)}$ and
$$e_\chi := \frac{1}{[k_\chi : k]} \sum_{g \in \mathrm{Gal}(k_\chi/k)} \chi(g) g^{-1}.$$

By [12, Proposition 5.5], we have the canonical isomorphisms
$$H^1_{\mathcal{F}_{\mathrm{ur}}}(k, T) \cong e_\chi(\mathcal{O}_{k_\chi}^\times \otimes_\mathbb{Z} R) \quad \text{and} \quad H^1_{\mathcal{F}_{\mathrm{ur}}^*}(k, T^\vee(1))^\vee \cong e_\chi(\mathrm{Cl}(k_\chi) \otimes_\mathbb{Z} R),$$
where $\mathrm{Cl}(k_\chi)$ denotes the ideal class group of k_χ. Furthermore, [12, Corollary 5.6] shows that
$$\chi(\mathcal{F}_{\mathrm{ur}}) = \mathrm{rank}_R\left(e_\chi(\mathcal{O}_{k_\chi}^\times \otimes_\mathbb{Z} R)\right) = \#\{v \in S_\infty(k) \mid \chi(G_{k_v}) = 1\}.$$
In particular, if k is a totally real field and χ is a totally odd character, then we have $\chi(\mathcal{F}_{\mathrm{ur}}) = 0$.

Example 2.21. Let E be an elliptic curve over k, and $T := T_p(E)$ denotes the p-adic Tate module of the elliptic curve E. Let $S_{\mathrm{bad}}(E)$ denote the set of the primes of k where E has bad reduction. We define a Selmer structure $\mathcal{F}_{\mathrm{cl}}$ on T by the following data:

- We set $S(\mathcal{F}_{\mathrm{cl}}) := S_p(k) \cup S_\infty(k) \cup S_{\mathrm{bad}}(E)$, where $S_{\mathrm{bad}}(E)$ denotes the set of primes of k where E has bad reduction.
- For a prime $\mathfrak{q} \in S(\mathcal{F}_{\mathrm{cl}})$, we define $H^1_{\mathcal{F}_{\mathrm{cl}}}(G_{k_\mathfrak{q}}, T)$ to be the image of the p-adic completion of $E(k_\mathfrak{q})$ under the Kummer map.

Then the module $H^1_{\mathcal{F}_{\mathrm{cl}}^*}(k, T^\vee(1))$ coincides with the classical Selmer group $\mathrm{Sel}(k, E[p^\infty])$. For any integer $n > 0$, we identify $E[p^n] = E[p^n]^\vee(1)$ by using the Weil pairing. We then have $(\mathcal{F}_{\mathrm{cl}})_{\mathbb{Z}_p/(p^n)} = (\mathcal{F}_{\mathrm{cl}}^*)_{\mathbb{Z}_p/(p^n)}$, and $\chi(\mathcal{F}_{\mathrm{cl}}) = 0$.

3. Review of the definition of Stark systems

In this section, we quickly review the definition of Stark system over a zero dimensional Gorenstein local ring developed in [2, 12, 13].

We assume that R is a zero dimensional Gorenstrein local ring and let $\mathcal{F}$ be a Selmer structure on T. Suppose that T satisfies Hypothesis 2.7.

Fix an integer $n > 0$ with $p^n R = 0$. To simplify the notation, we put

$$\mathcal{P}_m := \mathcal{P}_m(T, \mathcal{F}) \text{ and } \mathcal{N}_m := \mathcal{N}(\mathcal{P}_m(T, \mathcal{F})).$$

for an integer $m \geq n$. Let $\nu(\mathfrak{n})$ denote the number of prime divisors of $\mathfrak{n}$. For an R-module M, we put

$$M^* := \mathrm{Hom}_R(M, R).$$

For an integer $r \geq 0$, we define the r-th exteiror bi-dual $\bigcap_R^r M$ by

$$\bigcap\nolimits_R^r M := \Big(\bigwedge\nolimits_R^r M^*\Big)^*.$$

Definition 3.1. For an ideal $\mathfrak{n} \in \mathcal{N}_n$ and an integer $r \geq 0$, define

$$W_\mathfrak{n} := \bigoplus_{\mathfrak{q} | \mathfrak{n}} H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T)^*,$$

$$X^r_\mathfrak{n}(T, \mathcal{F}) := \bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^\mathfrak{n}}(k, T) \otimes_R \det(W_\mathfrak{n}).$$

Let $\mathfrak{n} \in \mathcal{N}_n$ be an ideal. For an ideal $\mathfrak{m} \mid \mathfrak{n}$, we have the following cartesian diagram:

$$\begin{tikzcd}
H^1_{\mathcal{F}^\mathfrak{m}}(k, T) \arrow[r, hook] \arrow[d] & H^1_{\mathcal{F}^\mathfrak{n}}(k, T) \arrow[d] \\
\bigoplus_{\mathfrak{q}|\mathfrak{m}} H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T) \arrow[r, hook] & \bigoplus_{\mathfrak{q}|\mathfrak{n}} H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T).
\end{tikzcd}$$

Hence by Definition B.2, we obtain an R-homomorphism

$$\Phi_{\mathfrak{n},\mathfrak{m}} \colon X^r_\mathfrak{n}(T, \mathcal{F}) \longrightarrow X^r_\mathfrak{m}(T, \mathcal{F}).$$

Furthermore, Proposition B.3 implies the following

Lemma 3.2. *We have* $\Phi_{\mathfrak{n}_1,\mathfrak{n}_3} = \Phi_{\mathfrak{n}_2,\mathfrak{n}_3} \circ \Phi_{\mathfrak{n}_1,\mathfrak{n}_2}$ *for any* $\mathfrak{n}_1 \in \mathcal{N}$, $\mathfrak{n}_2 \mid \mathfrak{n}_1$, *and* $\mathfrak{n}_3 \mid \mathfrak{n}_2$.

Definition 3.3. Let $r \geq 0$ be an integer. For a subset $\mathcal{Q}$ of $\mathcal{P}_n$, we define the module $\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q})$ of Stark systems of rank r for $(T, \mathcal{F}, \mathcal{Q})$ by

$$\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q}) := \varprojlim_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} X^r_\mathfrak{n}(T, \mathcal{F}),$$

where the inverse limit is taken with respect to the maps $\Phi_{\mathfrak{n},\mathfrak{m}}$. We call an element of $\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q})$ a Stark system of rank r associated with the tuple $(T, \mathcal{F}, \mathcal{Q})$.

Theorem 3.4 ([2, Theorem 3.19], [12, Theorem 6.7], [13, Theorem 4.7]). *Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n$ containing $\mathcal{P}_m$ for some integer $m \geq n$. Suppose that $\mathcal{F}$ is cartesian and $r = \chi(\mathcal{F}) \geq 0$.*

(i) *For any ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ with $H^1_{(\mathcal{F}^{\mathfrak{n}})^*}(k, \overline{T}^\vee(1)) = 0$, the projection map*

$$\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q}) \longrightarrow X^r_{\mathfrak{n}}(T, \mathcal{F})$$

is an isomorphism.

(ii) *The R-module $\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q})$ is free of rank 1.*

4. Higher rank Kolyvagin systems

In this section, we recall the definition of higher rank Kolyvagin systems developed in [2, 11, 12] and explain why it is not easy to define a Kolyvagin system of rank 0.

As in Section 3, R is a zero dimensional Gorenstrein local ring, $\mathcal{F}$ denotes a Selmer structure on T, and suppose that T satisfies Hypothesis 2.7. Fix an integer $n > 0$ with $p^n R = 0$. To simplify the notation, we put

$$\mathcal{P}_m := \mathcal{P}_m(T, \mathcal{F}) \text{ and } \mathcal{N}_m := \mathcal{N}(\mathcal{P}_m(T, \mathcal{F})).$$

for an integer $m \geq n$.

4.1. Definition of higher rank Kolyvagin systems.

For each prime $\mathfrak{q} \in \mathcal{P}_n$, we set

$$G_{\mathfrak{q}} := \mathrm{Gal}(k(\mathfrak{q})/k(1)) = \mathrm{Gal}(k(\mathfrak{q})_{\mathfrak{q}}/k_{\mathfrak{q}}).$$

For an ideal $\mathfrak{n} \in \mathcal{N}_n$, we also put

$$G_{\mathfrak{n}} := \bigotimes_{\mathfrak{q} \mid \mathfrak{n}} G_{\mathfrak{q}}.$$

Definition 4.1. Let $\mathfrak{q} \in \mathcal{P}_n$ be a prime.

(i) We define the map $v_{\mathfrak{q}}$ by the composite map

$$v_{\mathfrak{q}} \colon H^1(G_k, T) \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} H^1(G_{k_{\mathfrak{q}}}, T) \longrightarrow H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T).$$

For an ideal $\mathfrak{n} \in \mathcal{N}_n$ with $\mathfrak{q} \mid \mathfrak{n}$, by Definition B.2, this map induces an R-homomorphism

$$v_{\mathfrak{q}} \colon {\bigcap}^r_R H^1_{\mathcal{F}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}} \longrightarrow H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \otimes_R {\bigcap}^{r-1}_R H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}.$$

(ii) We define the finite-singular comparison map $\varphi^{\mathrm{fs}}_{\mathfrak{q}}$ by

$$\varphi^{\mathrm{fs}}_{\mathfrak{q}} \colon H^1(G_k, T) \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}}, T) \xleftarrow{\sim} H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \xrightarrow{\varphi^{\mathrm{fs}}} H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \otimes G_{\mathfrak{q}}.$$

For an ideal $\mathfrak{n} \in \mathcal{N}_n$ with $\mathfrak{q} \mid \mathfrak{n}$, by Definition B.2, this map induces an R-homomorphism

$$\varphi_{\mathfrak{q}}^{\mathrm{fs}} \colon \bigcap\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{n}/\mathfrak{q})}(k,T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}/\mathfrak{q}} \longrightarrow H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T) \otimes_R \bigcap\nolimits_R^{r-1} H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(k,T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}.$$

Definition 4.2. Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n$ and $r > 0$ an integer. We define the module $\mathrm{KS}_r(T, \mathcal{F}, \mathcal{Q})$ of rank r to be the set of elements

$$\{\kappa_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} \in \prod_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} \bigcap\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{n})}(k,T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$$

satisfying the finite-singular relation

$$v_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = \varphi_{\mathfrak{q}}^{\mathrm{fs}}(\kappa_{\mathfrak{n}/\mathfrak{q}})$$

for any ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ and prime $\mathfrak{q} \in \mathcal{Q}$ with $\mathfrak{q} \mid \mathfrak{n}$.

Remark 4.3. When $r = 0$, we can not consider the homomorphisms $v_{\mathfrak{q}}$ and $\varphi_{\mathfrak{q}}^{\mathrm{fs}}$. Hence we can not directly define a Kolyvagin system of rank 0.

4.2. Regulator map. Let us define the regulator map

$$\mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q}) \longrightarrow \mathrm{KS}_r(T, \mathcal{F}, \mathcal{Q})$$

introduced in [2, 12]. Take an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. There is an exact sequence of R-modules

$$0 \longrightarrow H^1_{\mathcal{F}(\mathfrak{n})}(k,T) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{n}}}(k,T) \longrightarrow \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T).$$

Applying Definition B.2 to this exact sequence, we obtain a natural homomorphism

$$\bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(k,T) \otimes_R \det\left(\bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T)^*\right) \longrightarrow \bigcap\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{n})}(k,T).$$

By Lemmas A.2 and A.4, we have the canonical isomorphism

$$H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T) \xleftarrow{\sim} H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}},T) \xrightarrow{\sim} H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T) \otimes_{\mathbb{Z}} G_{\mathfrak{q}}. \tag{4.1}$$

Since $G_{\mathfrak{q}}$ is a cyclic group with $(\# G_{\mathfrak{q}}) \cdot R = 0$, this isomorphism induces an isomorphism

$$H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T)^* \otimes_{\mathbb{Z}} G_{\mathfrak{q}} \cong H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T)^*,$$

and we obtain an isomorphism

$$\det\left(\bigoplus_{\mathfrak{q} \mid \mathfrak{n}} H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T)^*\right) \otimes_{\mathbb{Z}} G_{\mathfrak{n}} \cong W_{\mathfrak{n}}.$$

Hence we get a natural homomorphism

$$\Pi_{\mathfrak{n}} \colon X^r_{\mathfrak{n}}(T, \mathcal{F}) \longrightarrow \bigcap\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{n})}(k,T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}.$$

Lemma 4.4 ([2, Proposition 4.3], [12, Proposition 12.3])**.** *For a Stark system $\epsilon = \{\epsilon_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} \in \mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q})$, we have*

$$\{(-1)^{\nu(\mathfrak{n})}\Pi_{\mathfrak{n}}(\epsilon_{\mathfrak{n}})\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})} \in \mathrm{KS}_r(T, \mathcal{F}, \mathcal{Q}).$$

Definition 4.5. We define the regulator map

$$\mathrm{Reg}_r \colon \mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q}) \longrightarrow \mathrm{KS}_r(T, \mathcal{F}, \mathcal{Q})$$

by $\mathrm{Reg}_r(\{\epsilon_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})}) = \{(-1)^{\nu(\mathfrak{n})}\Pi_{\mathfrak{n}}(\epsilon_{\mathfrak{n}})\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q})}$.

When $r = 1$, the following theorem is proved by Mazur and Rubin in [12]. In the higher rank case, it is proved by Burns, Sakamoto and Sano in [1]. (In this paper, we only use the rank 1 case of the following theorem).

Theorem 4.6 ([1, Theorem 5.2(i)], [12, Theorem 12.4], [15, Theorem 3.17])**.** *Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n$ containing $\mathcal{P}_m$ for some integer $m \geq n$. Suppose that $\mathcal{F}$ is cartesian and $r := \chi(\mathcal{F}) > 0$. Then the regulator map*

$$\mathrm{Reg}_r \colon \mathrm{SS}_r(T, \mathcal{F}, \mathcal{Q}) \longrightarrow \mathrm{KS}_r(T, \mathcal{F}, \mathcal{Q})$$

is an isomorphism.

5. Kolyvagin systems of rank 0 over zero dimensional Gorenstein local rings

Throughout this section, suppose that R is a zero dimensional Gorenstein local ring and that T satisfies Hypothesis 2.7. Let $n > 0$ be an integer with $p^n R = 0$. We fix an isomorphism $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \cong R$ for each prime $\mathfrak{q} \in \mathcal{P}_n(T)$. Hence we have an isomorphisms

$$v_{\mathfrak{q}} \colon H^1(G_k, T) \longrightarrow H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \cong R,$$
$$\varphi^{\mathrm{fs}}_{\mathfrak{q}} \colon H^1(G_k, T) \longrightarrow H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T) \otimes_{\mathbb{Z}} G_{\mathfrak{q}} \cong R \otimes_{\mathbb{Z}} G_{\mathfrak{q}}.$$

For a subset $\mathcal{Q}$ of $\mathcal{P}_n(T)$, we put

$$\mathcal{M}(\mathcal{Q}) := \{(\mathfrak{n}, \mathfrak{q}) \in \mathcal{N}(\mathcal{Q}) \times \mathcal{Q} \mid \mathfrak{q} \text{ is coprime to } \mathfrak{n}\}.$$

Let $\mathcal{F}$ be a Selmer structure on T.

5.1. Definition of Kolyvagin systems of rank 0.

Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T, \mathcal{F})$.

Definition 5.1. A Kolyvagin system of rank 0 is an element

$$\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q}) \in \mathcal{M}(\mathcal{Q})} \in \prod_{(\mathfrak{n},\mathfrak{q}) \in \mathcal{M}(\mathcal{Q})} H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$$

which satisfies the following relations for any elements $(\mathfrak{n},\mathfrak{q}),(\mathfrak{n},\mathfrak{r}),(\mathfrak{n}\mathfrak{q},\mathfrak{r}) \in \mathcal{M}(\mathcal{Q})$:

$$v_{\mathfrak{q}}(\kappa_{\mathfrak{n}\mathfrak{q},\mathfrak{r}}) = \varphi_{\mathfrak{q}}^{\mathrm{fs}}(\kappa_{\mathfrak{n},\mathfrak{r}}), \tag{5.1}$$

$$v_{\mathfrak{r}}(\kappa_{\mathfrak{n}\mathfrak{q},\mathfrak{r}}) = -\varphi_{\mathfrak{q}}^{\mathrm{fs}}(\kappa_{\mathfrak{n},\mathfrak{q}}), \tag{5.2}$$

$$v_{\mathfrak{q}}(\kappa_{1,\mathfrak{q}}) = v_{\mathfrak{r}}(\kappa_{1,\mathfrak{r}}). \tag{5.3}$$

We denote by $\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$ the module of all Kolyvagin systems of rank 0.

Remark 5.2.

(i) The idea of the definition of Kolyvagin systems of rank 0 comes from the definition of Stark systems of rank 0.

(ii) A family of elements which satisfies the relations (5.1), (5.2) and (5.3) has already been considered by Kurihara in [8, 9, 10]. For an ideal $\mathfrak{m}$ satisfying a certain condition, Kurihara constructed a finite family $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{\mathfrak{n}\mathfrak{q}\mid\mathfrak{m}}$ satisfying the relations (5.1), (5.2) and (5.3) from an Euler system of Gauss sum type. However, in the definition of Kolyvagin system of rank 0, we consider $\kappa_{\mathfrak{n},\mathfrak{q}}$ for general $(\mathfrak{n},\mathfrak{q})$. It is worth noting that Theorem 5.5 shows that a non-trivial Kolyvagin system of rank 0 exists.

Remark 5.3. The relation (5.2) depends on the choice of the fixed isomorphism $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T) \cong R$ for each prime $\mathfrak{q} \in \mathcal{P}_n(T)$. Hence the module $\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$ also depends on these fixed isomorphisms.

By the relation (5.1), for a system $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q})\in\mathcal{M}(\mathcal{Q})} \in \mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$, we have $\{\kappa_{\mathfrak{n},\mathfrak{q}'}\}_{\mathfrak{n}\in\mathcal{N}(\mathcal{Q}\setminus\{\mathfrak{q}'\})} \in \mathrm{KS}_1(T,\mathcal{F}^{\mathfrak{q}'},\mathcal{Q}\setminus\{\mathfrak{q}'\})$ for each prime $\mathfrak{q}'$. Hence we have a natural homomorphism

$$\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q}) \longrightarrow \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{KS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}).$$

5.2. Regulator map. Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T,\mathcal{F})$. In this subsection, let us define the regulator map

$$\mathrm{Reg}_0\colon \mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \longrightarrow \mathrm{KS}_0(T,\mathcal{F},\mathcal{Q}).$$

Recall that

$$W_{\mathfrak{n}} = \det\left(\bigoplus_{\mathfrak{q}\mid\mathfrak{n}} H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T)^*\right)$$

for an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{P}_n(T))$. For each prime $\mathfrak{q} \in \mathcal{P}_n(T)$, the fixed isomorphism $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}},T) \cong R$ induces an isomorphism $W_{\mathfrak{q}} \cong R$, and we also obtain an isomorphism

$$W_{\mathfrak{n}} \cong W_{\mathfrak{n}} \otimes_R W_{\mathfrak{q}} \xrightarrow{x_{\mathfrak{n}}\otimes x_{\mathfrak{q}} \mapsto x_{\mathfrak{n}}\wedge x_{\mathfrak{q}}} W_{\mathfrak{n}\mathfrak{q}}$$

for any element $(\mathfrak{n},\mathfrak{q}) \in \mathcal{M}$. Hence we get an isomorphism

$$\begin{aligned} X^0_{\mathfrak{n}\mathfrak{q}}(T,\mathcal{F}) &= \bigcap\nolimits_R^{\nu(\mathfrak{n}\mathfrak{q})} H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(k,T) \otimes_R W_{\mathfrak{n}\mathfrak{q}} \\ &\cong \bigcap\nolimits_R^{1+\nu(\mathfrak{n})} H^1_{(\mathcal{F}^{\mathfrak{q}})^{\mathfrak{n}}}(k,T) \otimes_R W_{\mathfrak{n}} \\ &= X^1_{\mathfrak{n}}(T,\mathcal{F}^{\mathfrak{q}}). \end{aligned}$$

By the construction of the transition map $\Phi_{\mathfrak{n},\mathfrak{m}}$, this isomorphism is compatible with $\Phi_{\mathfrak{n},\mathfrak{m}}$, and so we obtain an injective homomorphism

$$\mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \hookrightarrow \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{SS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}).$$

Lemma 5.4. *We have the canonical homomorphism*

$$\mathrm{Reg}_0\colon\ \mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \longrightarrow \mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \arrow[r, hook] \arrow[d] & \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{SS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}) \arrow[d, "\prod \mathrm{Reg}_1"] \\ \mathrm{KS}_0(T,\mathcal{F},\mathcal{Q}) \arrow[r, hook] & \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{KS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}) \end{tikzcd} \tag{5.4}$$

Proof. Let $\epsilon = \{\epsilon_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathcal{N}(\mathcal{Q})} \in \mathrm{SS}_0(T,\mathcal{F},\mathcal{Q})$ be a Stark system. Take an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ and a prime $\mathfrak{q} \in \mathcal{Q}$ with $\mathfrak{q} \nmid \mathfrak{n}$. For each prime $\mathfrak{r} \in \mathcal{Q}$, fix a basis $e_{\mathfrak{r}}$ of $H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{r}}},T)$. Then by using the isomorphism (4.1), we obtain bases $f_{\mathfrak{r}}$ and $\sigma_{\mathfrak{r}}$ of $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{r}}},T)$ and $G_{\mathfrak{r}}$, respectively. We may assume that $f_{\mathfrak{r}}$ corresponds to 1 under the fixed isomorphism $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{r}}},T) \cong R$. Let $f^*_{\mathfrak{r}}$ denote the dual basis of $H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{r}}},T)^*$ with respect to $f_{\mathfrak{r}}$. For each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$, we fix an order on the set of prime divisors of $\mathfrak{n}$. We then have the basis $f^*_{\mathfrak{n}} = \wedge_{\mathfrak{q}\mid\mathfrak{n}} f^*_{\mathfrak{q}}$ of $W_{\mathfrak{n}}$. Put

$$\epsilon_{\mathfrak{n}\mathfrak{q}} = \bar{\epsilon}_{\mathfrak{n},\mathfrak{q}} \otimes (f^*_{\mathfrak{n}} \wedge f^*_{\mathfrak{q}}) \ \text{ in } \bigcap\nolimits_R^{\nu(\mathfrak{n}\mathfrak{q})} H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(k,T) \otimes_R W_{\mathfrak{n}\mathfrak{q}}.$$

Denote $e^*_{\mathfrak{q}}$ by the composite map

$$H^1(G_k,T) \longrightarrow H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{q}}},T) \xrightarrow{e_{\mathfrak{q}}\mapsto 1} R.$$

We set

$$\kappa_{\mathfrak{n},\mathfrak{q}} := (-1)^{\nu(\mathfrak{n})} \bar{\epsilon}_{\mathfrak{n},\mathfrak{q}}(\wedge_{\mathfrak{s}\mid\mathfrak{n}} e^*_{\mathfrak{s}}) \otimes \bigotimes_{\mathfrak{s}\mid\mathfrak{n}} \sigma_{\mathfrak{s}}.$$

Then $\{\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{\mathfrak{n}\in\mathcal{N}(\mathcal{Q}\setminus\{\mathfrak{q}\})}\}_{\mathfrak{q}\in\mathcal{Q}}$ is the image of ϵ under the composite map

$$\mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \longrightarrow \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{SS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}) \longrightarrow \prod_{\mathfrak{q}\in\mathcal{Q}} \mathrm{KS}_1(T,\mathcal{F}^{\mathfrak{q}},\mathcal{Q}\setminus\{\mathfrak{q}\}).$$

Hence $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q})\in\mathcal{M}(\mathcal{Q})}$ satisfies the relation (5.1). Furthermore, since $v_{\mathfrak{q}}(\kappa_{1,\mathfrak{q}}) = v_{\mathfrak{q}}(\bar{\epsilon}_{1,\mathfrak{q}}) = \epsilon_1$, $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q})\in\mathcal{M}(\mathcal{Q})}$ also satisfies the relation (5.3).

Let us show that $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q})\in\mathcal{M}(\mathcal{Q})}$ satisfies the relation (5.2), that is,

$$v_{\mathfrak{r}}(\kappa_{\mathfrak{nq},\mathfrak{r}}) = -\varphi_{\mathfrak{q}}^{\mathrm{fs}}(\kappa_{\mathfrak{n},\mathfrak{q}})$$

for any elements $(\mathfrak{n},\mathfrak{q}),(\mathfrak{nq},\mathfrak{r}) \in \mathcal{M}(\mathcal{Q})$. We may assume that $f_{\mathfrak{nq}}^* = f_{\mathfrak{n}}^* \wedge f_{\mathfrak{q}}^*$. Then, by the definition of Stark system, we have

$$v_{\mathfrak{r}}(\bar{\epsilon}_{\mathfrak{nq},\mathfrak{r}}) = (-1)^{\nu(\mathfrak{nq})}\bar{\epsilon}_{\mathfrak{n},\mathfrak{q}}.$$

Hence we conclude that

$$\begin{aligned} v_{\mathfrak{r}}(\kappa_{\mathfrak{nq},\mathfrak{r}}) &= (-1)^{\nu(\mathfrak{nq})}\bar{\epsilon}_{\mathfrak{nq},\mathfrak{r}}(\wedge_{\mathfrak{s}\mid\mathfrak{n}} e_{\mathfrak{s}}^* \wedge e_{\mathfrak{q}}^* \wedge v_{\mathfrak{r}}) \otimes \bigotimes_{\mathfrak{s}\mid\mathfrak{nq}} \sigma_{\mathfrak{s}} \\ &= (-1)^{\nu(\mathfrak{nq})}\bar{\epsilon}_{\mathfrak{n},\mathfrak{q}}(\wedge_{\mathfrak{s}\mid\mathfrak{n}} e_{\mathfrak{s}}^* \wedge e_{\mathfrak{q}}^*) \otimes \bigotimes_{\mathfrak{s}\mid\mathfrak{nq}} \sigma_{\mathfrak{s}} \\ &= -\bar{\epsilon}_{\mathfrak{n},\mathfrak{q}}(e_{\mathfrak{q}}^* \wedge \wedge_{\mathfrak{s}\mid\mathfrak{n}} e_{\mathfrak{s}}^*) \otimes \bigotimes_{\mathfrak{s}\mid\mathfrak{nq}} \sigma_{\mathfrak{s}} \\ &= -\varphi_{\mathfrak{q}}^{\mathrm{fs}}(\kappa_{\mathfrak{n},\mathfrak{q}}). \end{aligned}$$

□

5.3. Rigidity of Kolyvagin systems. Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T,\mathcal{F})$ containing $\mathcal{P}_m(T,\mathcal{F})$ for some integer $m \geq n$.

Theorem 5.5. *Suppose that*

- *the Selmer structure $\mathcal{F}$ is cartesian, and*
- *$\chi(\mathcal{F}) = 0$.*

Then the regulator map

$$\mathrm{Reg}_0\colon\ \mathrm{SS}_0(T,\mathcal{F},\mathcal{Q}) \longrightarrow \mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$$

is an isomorphism. In particular, the R-module $\mathrm{KS}_0(T,\mathcal{F},\mathcal{P})$ is free of rank 1.

Proof. Take an element $(\mathfrak{n},\mathfrak{q}) \in \mathcal{M}(\mathcal{Q})$ with $H^1_{(\mathcal{F}^*)_{\mathfrak{q}}(\mathfrak{n})}(k,\overline{T}^\vee(1)) = 0$. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{SS}_0(T,\mathcal{F},\mathcal{P}) \arrow[rr, "\mathrm{Reg}_0"] \arrow[d, "\cong"] & & \mathrm{KS}_0(T,\mathcal{F},\mathcal{P}) \arrow[d] \\ X^0_{\mathfrak{nq}}(T,\mathcal{F}) \arrow[r, "\cong"] & X^1_{\mathfrak{n}}(T,\mathcal{F}^{\mathfrak{q}}) \arrow[r, "(-1)^{\nu(\mathfrak{n})}\Pi_{\mathfrak{n}}"] & H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k,T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}. \end{tikzcd}$$

The left vertical arrow is an isomorphism by Theorem 3.4 (i). Since $\mathcal{F}$ is cartesian and the module $H^1_{(\mathcal{F}^*)_{\mathfrak{q}}(\mathfrak{n})}(k,\overline{T}^\vee(1))$ vanishes, Lemma 2.18 implies that the R-module $H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k,T)$ is free of rank $\chi(\mathcal{F}) + \nu(\mathfrak{q}) = 1$. Hence applying Theorem 2.5 with $\mathcal{F}_1 = \mathcal{F}^{\mathfrak{q}}(\mathfrak{n})$ and $\mathcal{F}_2 = \mathcal{F}^{\mathfrak{n}\mathfrak{q}}$, we get a split exact sequence of free R-modules

$$0 \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{q}}}(k, T) \longrightarrow \bigoplus_{\mathfrak{r} \mid \mathfrak{n}} H^1_{/\mathrm{tr}}(G_{k_{\mathfrak{r}}}, T) \longrightarrow 0.$$

Hence we see that the composite

$$\mathrm{SS}_0(T, \mathcal{F}, \mathcal{P}) \longrightarrow \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q}) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$$

is an isomorphism, and this theorem follows from Proposition 5.6 below. $\square$

Proposition 5.6. *Suppose that*

- *the Selmer structure $\mathcal{F}$ is cartesian, and*
- $\chi(\mathcal{F}) = 0$.

For any element $(\mathfrak{n}, \mathfrak{q}) \in \mathcal{M}$ with $H^1_{(\mathcal{F}^*)_{\mathfrak{q}}(\mathfrak{n})}(k, \overline{T}^\vee(1)) = 0$, the projection map

$$\mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q}) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$$

is an isomorphism.

Proof. Take an element $(\mathfrak{n}, \mathfrak{q}) \in \mathcal{M}(\mathcal{Q})$ with $H^1_{(\mathcal{F}^*)_{\mathfrak{q}}(\mathfrak{n})}(k, \overline{T}^\vee(1)) = 0$. In the proof of Theorem 5.5, we proved that the composite

$$\mathrm{SS}_0(T, \mathcal{F}, \mathcal{P}) \longrightarrow \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q}) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$$

is an isomorphism. Hence the projection $\mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q}) \to H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$ is surjective, and we only need to show that this homomorphism is injective. To see this, take a Kolyvagin system $\kappa = \{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{(\mathfrak{n},\mathfrak{q}) \in \mathcal{M}(\mathcal{Q})} \in \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q})$ with $\kappa_{\mathfrak{n},\mathfrak{q}} = 0$. Let $(\mathfrak{m}, \mathfrak{r}) \in \mathcal{M}(\mathcal{Q})$ be an element. It suffices to prove that $\kappa_{\mathfrak{m},\mathfrak{r}} = 0$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{SS}_1(T, \mathcal{F}^{\mathfrak{q}}, \mathcal{Q} \setminus \{\mathfrak{q}\}) \arrow[r, "\cong", "\mathrm{Reg}_1"'] \arrow[d, "\cong"] & \mathrm{KS}_1(T, \mathcal{F}^{\mathfrak{q}}, \mathcal{Q} \setminus \{\mathfrak{q}\}) \arrow[d] \\
X^1_{\mathfrak{n}}(T, \mathcal{F}^{\mathfrak{q}}) \arrow[r, "(-1)^{\nu(\mathfrak{n})}\Pi_{\mathfrak{n}}"] & H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}.
\end{tikzcd}$$

Since $\mathcal{F}^{\mathfrak{q}}$ is cartesian and $\chi(\mathcal{F}^{\mathfrak{q}}) = 1$ by Lemmas 2.13 and 2.16, Theorem 3.4 (i) shows that the left vertical arrow is an isomorphism. Moreover, Reg_1 is also an isomorphism by Theorem 4.6. In the proof of Theorem 5.5, we show that $\Pi_{\mathfrak{n}}$ is an isomorphism. Hence the projection map

$$\mathrm{KS}_1(T, \mathcal{F}^{\mathfrak{q}}, \mathcal{Q} \setminus \{\mathfrak{q}\}) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{n}}. \tag{5.5}$$

is also an isomorphism. Since $\{\kappa_{\mathfrak{n},\mathfrak{q}}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q} \setminus \{\mathfrak{q}\})} \in \mathrm{KS}_1(T, \mathcal{F}^{\mathfrak{q}}, \mathcal{Q} \setminus \{\mathfrak{q}\})$ and $\kappa_{\mathfrak{n},\mathfrak{q}} = 0$, we see that $\kappa_{\mathfrak{d},\mathfrak{q}} = 0$ for any ideal $\mathfrak{d} \in \mathcal{N}(\mathcal{Q} \setminus \{\mathfrak{q}\})$.

By Lemma 2.17, one can take an ideal $\mathfrak{d} \in \mathcal{N}(\mathcal{Q} \setminus \{\mathfrak{q}\})$ with $\mathfrak{r} \mid \mathfrak{d}$ such that $H^1_{\mathcal{F}^*(\mathfrak{d})}(k, \overline{T}^\vee(1))$ vanishes. Since $\chi(\mathcal{F}(\mathfrak{d})) = \chi(\mathcal{F}) = 0$ by Lemma 2.16, the module $H^1_{\mathcal{F}(\mathfrak{d})}(k, \overline{T})$ also vanishes. Hence, by Lemma 2.14, we conclude that

$$H^1_{\mathcal{F}(\mathfrak{d})}(k, T) = H^1_{\mathcal{F}^*(\mathfrak{d})}(k, T) = 0$$

Applying Theorem 2.5 with $\mathcal{F}_1 = \mathcal{F}(\mathfrak{d})$ and $\mathcal{F}_2 = \mathcal{F}^{\mathfrak{r}}(\mathfrak{d}/\mathfrak{r})$, we have an isomorphism

$$\varphi^{\mathrm{fs}}_{\mathfrak{r}} \colon H^1_{\mathcal{F}^{\mathfrak{r}}(\mathfrak{d}/\mathfrak{r})}(k, T) \xrightarrow{\sim} H^1_{/\mathrm{tr}}(k_{\mathfrak{r}}, T) \cong R \otimes_{\mathbb{Z}} G_{\mathfrak{r}}.$$

Since $\kappa_{\mathfrak{d},\mathfrak{q}} = 0$, the relation (5.2) shows that

$$\varphi^{\mathrm{fs}}_{\mathfrak{r}}(\kappa_{\mathfrak{d}/\mathfrak{r},\mathfrak{r}}) = -v_{\mathfrak{q}}(\kappa_{\mathfrak{d},\mathfrak{q}}) = 0,$$

which implies $\kappa_{\mathfrak{d}/\mathfrak{r},\mathfrak{r}} = 0$. The same argument as the proof of the fact that the projection (5.5) is an isomorphism shows that the projection map

$$\mathrm{KS}_1(T, \mathcal{F}^{\mathfrak{r}}, \mathcal{Q} \setminus \{\mathfrak{r}\}) \longrightarrow H^1_{\mathcal{F}^{\mathfrak{r}}(\mathfrak{d}/\mathfrak{r})}(k, T) \otimes_{\mathbb{Z}} G_{\mathfrak{d}/\mathfrak{r}}.$$

is an isomorphism. Since $\{\kappa_{\mathfrak{n},\mathfrak{r}}\}_{\mathfrak{n} \in \mathcal{N}(\mathcal{Q}\setminus\{\mathfrak{r}\})} \in \mathrm{KS}_1(T, \mathcal{F}^{\mathfrak{r}}, \mathcal{Q} \setminus \{\mathfrak{r}\})$ and $\kappa_{\mathfrak{d}/\mathfrak{r},\mathfrak{r}} = 0$, we conclude that $\kappa_{\mathfrak{m},\mathfrak{r}} = 0$. □

5.4. Kolyvagin systems and Selmer modules.

Definition 5.7. Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T, \mathcal{F})$. Take an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ and $\mathfrak{q} \in \mathcal{Q}$ a prime with $\mathfrak{q} \nmid \mathfrak{n}$. For a Kolyvagin system $\kappa \in \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q})$, we put

$$\delta_{\mathfrak{n}}(\kappa) := v_{\mathfrak{q}}(\kappa_{\mathfrak{n},\mathfrak{q}}) \in R \otimes_{\mathbb{Z}} G_{\mathfrak{n}}.$$

By the relations (5.2) and (5.3), the element $\delta_{\mathfrak{n}}(\kappa)$ does not depend on the choice of the prime $\mathfrak{q} \in \mathcal{Q}$.

We fix a generator of $G_{\mathfrak{q}}$ for each prime $\mathfrak{q} \in \mathcal{Q}$ and identify $R \otimes_{\mathbb{Z}} G_{\mathfrak{n}}$ with R for each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$. For an integer $i \geq 0$ and a Kolyvagin system $\kappa \in \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q})$, define an ideal $I_i(\kappa)$ of R by

$$I_i(\kappa) := \langle\, \delta_{\mathfrak{n}}(\kappa) \mid \mathfrak{n} \in \mathcal{N}(\mathcal{Q}),\, \nu(\mathfrak{n}) \leq i \,\rangle.$$

The ideal $I_i(\kappa)$ does not depend on the choice of the fixed generator of $G_{\mathfrak{q}}$.

Let $\mathcal{Q}$ be a subset of $\mathcal{P}_n(T, \mathcal{F})$ containing $\mathcal{P}_m(T, \mathcal{F})$ for some integer $m \geq n$.

Theorem 5.8. *Suppose that*

- *the Selmer structure $\mathcal{F}$ is cartesian, and*
- $\chi(\mathcal{F}) = 0$.

Take a Kolyvagin system $\kappa \in \mathrm{KS}_0(T, \mathcal{F}, \mathcal{Q})$.

(i) *For each ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$, we have*

$$R \cdot \delta_{\mathfrak{n}} \subseteq \mathrm{Fitt}^0_R(H^1_{\mathcal{F}(\mathfrak{n})^*}(k, T^\vee(1))^\vee),$$

with equality if κ is a basis of $\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$*. In particular, if κ is a basis, then we have*

$$R\cdot\delta_1=\mathrm{Fitt}^0_R(H^1_{\mathcal{F}^*}(k,T^\vee(1))^\vee).$$

(ii) *For each integer $i\geq 0$, we have*

$$I_i(\kappa)\subseteq\mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(k,T^\vee(1))^\vee).$$

Remark 5.9. Suppose that R is a principal ideal ring. Mazur and Rubin proved in [11, Theorem 4.5.9] that Kolyvagin systems of rank 1 controlled all the higher Fitting ideals of the dual Selmer module. In the higher rank case, Burns, Sakamoto and Sano proved in [1, Theorem 5.2 (iii)] a similar result (see also [1, Remark 5.3]). However, a similar argument of the proof of [11, Theorem 4.5.9] does not work in the rank 0 case. It is an interesting problem to understand conditions under which the inclusion in Theorem 5.8 (ii) becomes an equality.

Proof. Since the R-module $\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$ is free of rank 1 by Theorem 5.5, we may assume that κ is a basis of $\mathrm{KS}_0(T,\mathcal{F},\mathcal{Q})$. Then one can take a basis ϵ of $\mathrm{SS}_0(T,\mathcal{F},\mathcal{P})$ such that $\mathrm{Reg}_0(\epsilon)=\kappa$ by Theorem 5.5.

Let us prove claim (i). To do this, we take an ideal $\mathfrak{n}\in\mathcal{N}(\mathcal{Q})$ and fix a generator of $G_\mathfrak{q}$ for each $\mathfrak{q}\in\mathcal{Q}$. Hence we can regard $\delta_\mathfrak{n}$ as an element of R for each ideal $\mathfrak{n}\in\mathcal{N}(\mathcal{Q})$. By using the fixed isomorphisms $\{H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}},T)\cong R\}_{\mathfrak{q}\in\mathcal{Q}}$, we also regard $\epsilon_\mathfrak{n}$ as an element of $\bigcap_R^{\nu(\mathfrak{n})}H^1_{\mathcal{F}^\mathfrak{n}}(k,T)$ for each ideal $\mathfrak{n}\in\mathcal{N}(\mathcal{Q})$.

Let $\mathfrak{n}\in\mathcal{N}(\mathcal{Q})$ be an ideal. By Lemmas 2.14 and 2.17, one can take an ideal $\mathfrak{m}\in\mathcal{N}(\mathcal{Q})$ with $\mathfrak{n}\mid\mathfrak{m}$ and $H^1_{(\mathcal{F}^\mathfrak{m})^*}(k,T^\vee(1))=0$. Applying Theorem 2.5 with $\mathcal{F}_1=\mathcal{F}(\mathfrak{n})$ and $\mathcal{F}_2=\mathcal{F}^\mathfrak{m}$, we get an exact sequence of R-modules

$$0\longrightarrow H^1_{\mathcal{F}(\mathfrak{n})}(k,T)\longrightarrow H^1_{\mathcal{F}^\mathfrak{m}}(k,T)\longrightarrow X\longrightarrow H^1_{\mathcal{F}^*(\mathfrak{n})}(k,T^\vee(1))^\vee\longrightarrow 0,$$

where

$$X=\bigoplus_{\mathfrak{q}\mid\mathfrak{n}}H^1_{/\mathrm{tr}}(G_{k_\mathfrak{q}},T)\oplus\bigoplus_{\mathfrak{q}\mid\frac{\mathfrak{m}}{\mathfrak{n}}}H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}},T).$$

The fixed isomorphisms induce an isomorphism $X\cong R^{\nu(\mathfrak{m})}$. Applying Definition B.2 to the exact sequence $0\to H^1_{\mathcal{F}(\mathfrak{n})}(k,T)\to H^1_{\mathcal{F}^\mathfrak{m}}(k,T)\to X\cong R^{\nu(\mathfrak{m})}$, we obtain an R-homomorphism

$$\bigcap\nolimits_R^{\nu(\mathfrak{m})}H^1_{\mathcal{F}^\mathfrak{m}}(k,T)\longrightarrow R.$$

Furthermore, we see that the image of $\epsilon_\mathfrak{m}$ under this homomorphism is $\delta_\mathfrak{n}$. We note that the R-module $H^1_{\mathcal{F}^\mathfrak{m}}(k,T)$ is free of rank $\nu(\mathfrak{m})$ by Lemma 2.18 and that $\epsilon_\mathfrak{m}$ is a basis of $\bigcap_R^{\nu(\mathfrak{m})}H^1_{\mathcal{F}^\mathfrak{m}}(k,T)$ by Theorem 3.4. Hence claim (i) follows from Lemma B.4.

Let us show claim (ii). Take an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{Q})$ and an integer $i \geq \nu(\mathfrak{n})$. By Lemmas 2.14 and 2.17, one can take an ideal $\mathfrak{m} \in \mathcal{N}(\mathcal{Q})$ with $i \leq \nu(\mathfrak{m})$ and $\mathfrak{n} \mid \mathfrak{m}$ such that the module $H^1_{(\mathcal{F}^{\mathfrak{m}})^*}(k, T^\vee(1))$ vanishes. In the proof of [13, Theorem 4.10], we show that

$$\mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee) = \sum_{\mathfrak{d} \mid \mathfrak{m},\, \nu(\mathfrak{d}) = i} \mathrm{Fitt}^0_R(H^1_{(\mathcal{F}^{\mathfrak{d}})^*}(k, T^\vee(1))^\vee).$$

Let $\mathfrak{d} \mid \mathfrak{m}$ be an ideal with $\mathfrak{n} \mid \mathfrak{d}$ and $\nu(\mathfrak{d}) = i$. By the definition of Selmer structures $\mathcal{F}^{\mathfrak{d}}$ and $\mathcal{F}(\mathfrak{n})$, we have the canonical surjection

$$H^1_{(\mathcal{F}^{\mathfrak{d}})^*}(k, T^\vee(1))^\vee \longrightarrow H^1_{\mathcal{F}(\mathfrak{n})^*}(k, T^\vee(1))^\vee.$$

Claim (i) implies that

$$\begin{aligned} \delta_{\mathfrak{n}} &\in \mathrm{Fitt}^0_R(H^1_{\mathcal{F}(\mathfrak{n})^*}(k, T^\vee(1))^\vee) \\ &\subseteq \mathrm{Fitt}^0_R(H^1_{(\mathcal{F}^{\mathfrak{d}})^*}(k, T^\vee(1))^\vee) \\ &\subseteq \mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee). \end{aligned}$$

In particular, we have $I_i(\kappa) \subseteq \mathrm{Fitt}^i_R(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee)$. □

6. Kolyvagin systems of rank 0 over Gorenstein local rings

In this section, suppose that R is a complete Gorenstein local ring and that T satisfies Hypothesis 2.7. Let $\mathcal{I}$ denote the set of ideals I of R such that R/I is a zero dimensional Gorenstein ring.

The R-module $T/(1-\tau)T$ is free of rank 1 by Hypothesis 2.7 (ii). In this section, we fix an isomorphism

$$T/(1-\tau)T \cong R. \tag{6.1}$$

For each prime $\mathfrak{q} \notin S$ of k, we also fix an isomorphism

$$G_{\mathfrak{q}} \cong \mathbb{Z}/(\# G_{\mathfrak{q}}).$$

Let $I \in \mathcal{I}$ be an ideal. Since R/I is artinian, one can take an integer $n > 0$ with $p^n \in I$. For a prime $\mathfrak{q} \in \mathcal{P}_n(T/IT)$, we have an isomorphism

$$\begin{aligned} H^1_{\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T/IT) &\cong T/(1-\mathrm{Fr}_{\mathfrak{q}})T \otimes_R R/I \\ &= T/(1-\tau)T \otimes_R R/I \\ &\cong R/I, \end{aligned}$$

where the last isomorphism is induced by the isomorphism (6.1). Since we fix an isomorphism $G_{\mathfrak{q}} \cong \mathbb{Z}/(\# G_{\mathfrak{q}})$, we also have an isomorphism

$$H^1_{/\mathrm{ur}}(G_{k_{\mathfrak{q}}}, T/IT) \cong R/I$$

by Lemma A.4. Hence we get the R/I-module

$$\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})).$$

We set

$$\mathrm{KS}_0(T/IT, \mathcal{F}) := \varinjlim_{n>0,\, p^n \in I} \mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})).$$

For a prime $\mathfrak{q} \in \mathcal{P}_n(T/IT)$, the fixed isomorphism $H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T/IT) \cong R/I$ is compatible with base change, namely, for any ideal $J \in \mathcal{I}$ with $I \subseteq J$, the following diagram commutes:

$$\begin{tikzcd}
H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T/IT) \arrow[r, "\cong"] \arrow[d] & R/I \arrow[d] \\
H^1_{/\mathrm{ur}}(G_{k_\mathfrak{q}}, T/JT) \arrow[r, "\cong"] & R/J.
\end{tikzcd}$$

Hence we have the canonical homomorphism

$$\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})) \longrightarrow \mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F}))$$

for any integer $n > 0$ with $p^n \in I$.

Definition 6.1. We say that a Selmer structure $\mathcal{G}$ on T is cartesian if, for any ideal $I \in \mathcal{I}$, the induced Selmer structure $\mathcal{G}_{R/I}$ on T/IT is cartesian.

Lemma 6.2. *Let $n > 0$ be an integer with $p^n \in I$. If $\mathcal{F}$ is cartesian, then we have the canonical isomorphism*

$$\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})) \otimes_R R/J \xrightarrow{\sim} \mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_n(T/JT, \mathcal{F}))$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})) \otimes_R R/J \arrow[d, "\cong"] \arrow[dr] & \\
\mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_n(T/JT, \mathcal{F})) \arrow[r] & \mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F})).
\end{tikzcd}$$

(Note that $\mathcal{P}_n(T/IT, \mathcal{F}) \subseteq \mathcal{P}_n(T/JT, \mathcal{F})$).

Proof. To simplify the notation, we set $\mathcal{P}_{n,I} := \mathcal{P}_n(T/IT, \mathcal{F})$. By Lemma 2.17, there is an element $(\mathfrak{n}, \mathfrak{q}) \in \mathcal{M}(\mathcal{P}_n(T/IT, \mathcal{F}))$ such that the R-module $H^1_{(\mathcal{F}^\mathfrak{q}(\mathfrak{n}))^*}(k, \overline{T}^\vee(1))$ vanishes. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_{n,I}) \otimes_R R/J \arrow[r] \arrow[d, "\cong"] & \mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_{n,I}) \arrow[d] & \mathrm{KS}_0(T/JT, \mathcal{F}, \mathcal{P}_{n,J}) \arrow[l] \arrow[d, "\cong"] \\
H^1_{\mathcal{F}^\mathfrak{q}(\mathfrak{n})}(k, T/IT) \otimes_R R/J \arrow[r] & H^1_{\mathcal{F}^\mathfrak{q}(\mathfrak{n})}(k, T/JT) & H^1_{\mathcal{F}^\mathfrak{q}(\mathfrak{n})}(k, T/JT). \arrow[l, "="]
\end{tikzcd}$$

Note that the left and right vertical arrows are isomorphism by Proposition 5.6. Lemma 2.18 shows that the R/I-module $H^1_{\mathcal{F}^\mathfrak{q}(\mathfrak{n})}(k, T/IT)$ and the

R/J-module $H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T/JT)$ are free of rank 1. Hence Lemma 2.19 implies that the canonical map

$$H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T/IT) \otimes_R R/J \longrightarrow H^1_{\mathcal{F}^{\mathfrak{q}}(\mathfrak{n})}(k, T/JT)$$

is an isomorphism. Therefore, we obtain the desired isomorphism. $\square$

For a cartesian Selmer structure $\mathcal{F}$ on T, we define the module $\mathrm{KS}_0(T, \mathcal{F})$ of Kolyvagin systems of rank 0 for $(T, \mathcal{F})$ by

$$\mathrm{KS}_0(T, \mathcal{F}) := \varprojlim_{I \in \mathcal{I}} \mathrm{KS}_0(T/IT, \mathcal{F}),$$

where the transition maps are defined by using Lemma 6.2.

Theorem 6.3. *Suppose that*

- *the Selmer structure* $\mathcal{F}$ *is cartesian, and*
- $\chi(\mathcal{F}) = 0$.

Then the following claims are valid.

(i) *The R-module* $\mathrm{KS}_0(T, \mathcal{F})$ *is free of rank* 1.
(ii) *For an ideal $I \in \mathcal{I}$, the canonical homomorphism*

$$\mathrm{KS}_0(T, \mathcal{F}) \otimes_R R/I \xrightarrow{\sim} \mathrm{KS}_0(T/IT, \mathcal{F})$$

is an isomorphism.

Proof. Let $I, J \in \mathcal{I}$ be ideals of R with $I \subseteq J$. By Proposition 5.6, we see that, for any integer $n > 0$ with $p^n \in I$, the canonical homomorphism

$$\mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT)) \longrightarrow \mathrm{KS}_0(T/IT, \mathcal{F})$$

is an isomorphism. Hence the R/J-module $\mathrm{KS}_0(T/JT, \mathcal{F})$ is free of rank 1 by Theorem 5.5. Furthermore, Lemma 6.2 shows that the canonical homomorphism

$$\mathrm{KS}_0(T/IT, \mathcal{F}) \otimes_R R/J \xrightarrow{\sim} \mathrm{KS}_0(T/JT, \mathcal{F}).$$

is an isomorphism. This theorem follows from these two facts. $\square$

Take an ideal $I \in \mathcal{I}$ of R and an integer $n > 0$ with $p^n \in I$. Let $\kappa \in \mathrm{KS}_0(T/IT, \mathcal{F})$ be a Kolyvagin system. Take a representative $\widetilde{\kappa} \in \mathrm{KS}_0(T/IT, \mathcal{F}, \mathcal{P}_n(T/IT, \mathcal{F}))$ of κ. For an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{P}_n(T/IT))$, we have an element

$$\delta_{\mathfrak{n}}(\kappa) := \delta_{\mathfrak{n}}(\widetilde{\kappa}) \in R/I \otimes_{\mathbb{Z}} G_{\mathfrak{n}} \cong R/I.$$

For each integer $i \geq 0$, we define an ideal $I_i(\kappa) \subseteq R/I$ by

$$I_i(\kappa) := \bigcap_{n>0,\, p^n \in I} \langle\, \delta_{\mathfrak{n}}(\kappa) \mid \mathfrak{n} \in \mathcal{N}(\mathcal{P}_n(T/IT, \mathcal{F})),\, \nu(\mathfrak{n}) \leq i \,\rangle.$$

Let $\kappa = \{\kappa_I\}_{I \in \mathcal{I}} \in \mathrm{KS}_0(T, \mathcal{F})$ be a Kolyvagin system. For an ideal $\mathfrak{n} \in \mathcal{N}(\mathcal{P}_n(T/IT, \mathcal{F}))$, we see that

$$\delta_{\mathfrak{n}}(\kappa_I) \bmod J = \delta_{\mathfrak{n}}(\kappa_J)$$

for any ideal $J \in \mathcal{I}$ with $I \subseteq J$. In particular, for each integer $i \geq 0$, we have $I_i(\kappa_I) \bmod J \subseteq I_i(\kappa_J)$. Hence one can define an ideal $I_i(\kappa)$ of R by

$$I_i(\kappa) := \varprojlim_{I \in \mathcal{I}} I_i(\kappa_I).$$

Theorem 6.4. *Suppose that*

- *the Selmer structure $\mathcal{F}$ is cartesian, and*
- $\chi(\mathcal{F}) = 0$.

Let $\kappa \in \mathrm{KS}_0(T, \mathcal{F})$ be a Kolyvagin system.

(i) *For each integer $i \geq 0$, we have*

$$I_i(\kappa) \subseteq \mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee).$$

(ii) *If κ is a basis, then we have*

$$R \cdot \delta_1(\kappa) = \mathrm{Fitt}_R^0(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee).$$

Remark 6.5. Consider the situation in [14]:

- k is a totally real field,
- R is the cyclotomic Iwasawa algebra,
- T is the cyclotomic deformation of $\mathbb{Z}_p(1) \otimes \chi^{-1}$, where $\chi\colon G_k \to \mathbb{Z}_p^\times$ is a nontrivial even character of finite order,
- $\mathcal{F}$ is the strict Selmer structure (see [14, Example A.7]).

In this case, the argument in [14, Section 5] (that is based on Kurihara's works [8, 10]) shows that $I_i(\kappa) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee)$ for a basis $\kappa \in \mathrm{KS}_0(T, \mathcal{F})$ and an integer $i \geq 0$. The author believes that the argument in [14, Section 5] still works in more general situations, namely, the author expects that if $\dim R \geq 1$ and $H^1_{\mathcal{F}}(k, T) = 0$, then we have

$$I_i(\kappa) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee)$$

for a basis $\kappa \in \mathrm{KS}_0(T, \mathcal{F})$ and an integer $i \geq 0$.

Proof. Let $i \geq 0$ be an integer. For any ideal $I \in \mathcal{I}$ of R, we have

$$\mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee) R/I = \mathrm{Fitt}_{R/I}^i(H^1_{\mathcal{F}^*}(k, (T/IT)^\vee(1))^\vee)$$

by Lemma 2.14 (i). Since R is a complete noetherian local ring, any ideal of R is closed, and we obtain

$$\mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(k, T^\vee(1))^\vee) = \varprojlim_{I \in \mathcal{I}} \mathrm{Fitt}_{R/I}^i(H^1_{\mathcal{F}^*}(k, (T/IT)^\vee(1))^\vee).$$

Hence this theorem follows from Theorem 5.8. $\square$

Appendix A. Local Galois cohomology

In this appendix, we recall the results proved by Mazur and Rubin in [11, Chapter 1].

Let $\ell \neq p$ be a prime and K a finite extension of $\mathbb{Q}_\ell$. We denote by $\mathbb{F}$ the residue field of K. Let R be an artinian local ring with residue characteristic $p > 2$, and T denotes a free R-module of finite rank with an R-linear continuous G_K-action. Suppose that

- $(\#\mathbb{F}^\times) \cdot R = 0$, and
- the G_K-action on T is unramified.

Let $\mathcal{I}_K$ be the inertia subgroup of G_K and Fr the Frobenius element of $G_K/\mathcal{I}_K$.

Definition A.1 ([11, Definition 1.1.6 (iv)])**.** We fix a totally tamely ramified cyclic extension L of K such that $[L\colon K]$ annihilates R. Define

$$\begin{aligned} H^1_{\mathrm{ur}}(G_K, T) &:= \ker\Big(H^1(G_K, T) \longrightarrow H^1(\mathcal{I}_K, T)\Big), \text{ and} \\ H^1_{\mathrm{tr}}(G_K, T) &:= \ker\Big(H^1(G_K, T) \longrightarrow H^1(G_L, T)\Big). \end{aligned}$$

We also set $H^1_{/*}(G_K, T) := H^1(G_K, T)/H^1_*(G_K, T)$ for $* \in \{\mathrm{ur}, \mathrm{tr}\}$.

Lemma A.2 ([11, Lemmas 1.2.1 and 1.2.4])**.**

(i) *We have the canonical isomorphisms*

$$\begin{aligned} &H^1_{\mathrm{ur}}(G_K, T) \xrightarrow{\sim} T/(1-\mathrm{Fr})T \\ \textit{and} \quad &H^1_{/\mathrm{ur}}(G_K, T) \otimes_{\mathbb{Z}} \mathrm{Gal}(L/K) \xrightarrow{\sim} T^{\mathrm{Fr}=1}. \end{aligned}$$

(ii) *The composite map $H^1_{\mathrm{tr}}(G_K, T) \to H^1(G_K, T) \to H^1_{/\mathrm{ur}}(G_K, T)$ is an isomorphism.*

Suppose that $\det(1 - \mathrm{Fr} \mid T) = 0$. Define the characteristic polynomial $P(x) \in R[x]$ of the Frobenius by

$$P(x) = \det(1 - \mathrm{Fr} \cdot x \mid T).$$

Since $P(1) = 0$, there is a unique polynomial $Q(x) \in R[x]$ satisfying $P(x) = (x-1)Q(x)$. Since $P(\mathrm{Fr}^{-1}) \cdot T = 0$ by the Cayley–Hamilton theorem, we have

$$Q(\mathrm{Fr}^{-1}) \cdot T \subseteq T^{\mathrm{Fr}=1}.$$

Definition A.3 ([11, Definition 1.2.2])**.** We define the finite-singular comparison map φ^{fs} on T to be the composition

$$H^1_{\mathrm{ur}}(G_K, T) \xrightarrow{\sim} T/(1-\mathrm{Fr})T \xrightarrow{Q(\mathrm{Fr}^{-1})} T^{\mathrm{Fr}=1} \xleftarrow{\sim} H^1_{/\mathrm{ur}}(G_K, T) \otimes_{\mathbb{Z}} \mathrm{Gal}(L/K).$$

Lemma A.4 ([11, Lemma 1.2.3]). *Suppose that the R-module $T/(1-\mathrm{Fr})T$ is free of rank 1 and that $\det(1-\mathrm{Fr} \mid T)=0$. Then the finite-singular comparison map*

$$\varphi^{\mathrm{fs}}\colon H^1_{\mathrm{ur}}(G_K,T) \longrightarrow H^1_{/\mathrm{ur}}(G_K,T) \otimes_{\mathbb{Z}} \mathrm{Gal}(L/K)$$

is an isomorphism. In particular, the R-modules $H^1_{\mathrm{ur}}(G_K,T)$, $H^1_{\mathrm{tr}}(G_K,T)$, $H^1_{/\mathrm{ur}}(G_K,T)$, and $H^1_{/\mathrm{tr}}(G_K,T)$ are free of rank 1.

Appendix B. Exterior bi-dual

In this appendix, we recall the basis properties of the exterior bi-dual proved in [13].

Let R be a zero dimensional Gorenstein local ring. Note that all free R-modules are injective. For each R-module M we set

$$M^* := \mathrm{Hom}_R(M,R).$$

The functor $(\,\cdot\,)^* = \mathrm{Hom}_R(\,\cdot\,,R)$ is exact since R is an injective R-module. Furthermore, Matlis duality tells us that the canonical map

$$M \longrightarrow M^{**}$$

is an isomorphism for any finitely generated R-module M. For an integer $r \geq 0$, we define the r-th exterior bi-dual $\bigcap^r_R M$ by

$$\bigcap\nolimits^r_R M := \Big(\bigwedge\nolimits^r_R M^*\Big)^*.$$

Lemma B.1 ([13, Lemma 2.1]). *Let $r \geq 0$ be an integer and*

$$F \xrightarrow{h} M \xrightarrow{g} N \longrightarrow 0$$

an exact sequence of R-modules. If the R-module F is free of rank $s \leq r$, then there is a unique R-homomorphism

$$\varphi\colon \bigwedge\nolimits^{r-s}_R N \otimes_R \det(F) \longrightarrow \bigwedge\nolimits^r_R M$$

such that $\varphi\left(\wedge^{r-s} g(b) \otimes a\right) = (\wedge^s h(a)) \wedge b$ for any $a \in \det(F)$ and $b \in \bigwedge^{r-s}_R M$.

For $i \in \{1,2\}$, let M_i be an R-module and F_i a free R-module of rank s_i. Suppose that we have the following cartesian diagram:

$$\begin{tikzcd} M_1 \arrow[r, hook] \arrow[d] & M_2 \arrow[d] \\ F_1 \arrow[r, hook] & F_2, \end{tikzcd}$$

where the horizontal arrows are injective. The R-module F_2/F_1 is free of rank $s_2 - s_1$ since F_1 is an injective R-module. Applying Lemma B.1

to the exact sequence $(F_2/F_1)^* \to M_2^* \to M_1^* \to 0$, we obtain an R-homomorphism

$$\widetilde{\Phi}\colon \bigcap\nolimits_R^r M_2 \otimes_R \det((F_2/F_1)^*) \longrightarrow \bigcap\nolimits_R^{r-s_2+s_1} M_1$$

for any integer $r \geq s_2 - s_1$.

Definition B.2 ([13, Definition 2.3])**.** For any cartesian diagram as above and an integer $r \geq s_2 - s_1$, we have an R-homomorphism

$$\begin{aligned}\Phi\colon \bigcap\nolimits_R^r M_2 \otimes_R \det(F_2^*) &\xrightarrow{\sim} \bigcap\nolimits_R^r M_2 \otimes_R \det((F_2/F_1)^*) \otimes_R \det(F_1^*) \\ &\longrightarrow \bigcap\nolimits_R^{r-s_2+s_1} M_1 \otimes_R \det(F_1^*),\end{aligned}$$

where the first map is induced by the isomorphism

$$\det((F_2/F_1)^*) \otimes_R \det(F_1^*) \xrightarrow{\sim} \det(F_2^*); a \otimes b \longrightarrow a \wedge \widetilde{b},$$

where $\widetilde{b}$ is a lift of b in $\bigwedge_R^{s_1} F_2^*$ and the second map is induced by $\widetilde{\Phi}$.

Proposition B.3 ([13, Proposition 2.4])**.** *Suppose that we have the following commutative diagram of R-modules*

$$\begin{array}{ccccc} M_1 & \hookrightarrow & M_2 & \hookrightarrow & M_3 \\ \downarrow & & \downarrow & & \downarrow \\ F_1 & \hookrightarrow & F_2 & \hookrightarrow & F_3 \end{array}$$

such that F_1, F_2, and F_3 are free of finite rank and the two squares are cartesian. Let $s_i = \mathrm{rank}_R(F_i)$ and $r \geq s_3 - s_1$. For $i, j \in \{1, 2, 3\}$ with $j < i$, we denote by

$$\Phi_{ij}\colon \bigcap\nolimits_R^{r-s_3+s_i} M_i \otimes_R \det(F_i^*) \longrightarrow \bigcap\nolimits_R^{r-s_3+s_j} M_j \otimes_R \det(F_j^*)$$

the map given by Definition B.2. Then we have

$$\Phi_{31} = \Phi_{21} \circ \Phi_{32}.$$

Lemma B.4 ([2, Proposition A.2 (ii)], [13, Lemma 4.8])**.** *For $i \in \{1, 2\}$, let F_i be a free R-module of rank s_i. Suppose that $s_2 \leq s_1$ and there is an exact sequence of R-modules*

$$0 \longrightarrow N \longrightarrow F_1 \longrightarrow F_2 \longrightarrow M \longrightarrow 0.$$

Let

$$\phi\colon \bigwedge\nolimits_R^{s_1-s_2} N^* \longrightarrow R$$

be the image of a basis of $\bigcap_R^{s_1} F_1 \otimes_R \det(F_2^*)$ under the map

$$\bigcap\nolimits_R^{s_1} F_1 \otimes_R \det(F_2^*) \longrightarrow \bigcap\nolimits_R^{s_1-s_2} N$$

defined in Definition B.2. Then we have $\mathrm{im}(\phi) = \mathrm{Fitt}_R^0(M)$.

References

[1] D. BURNS, R. SAKAMOTO & T. SANO, "On the theory of higher rank Euler, Kolyvagin and Stark systems, II", https://arxiv.org/abs/1805.08448, submitted, 2018.

[2] D. BURNS & T. SANO, "On the Theory of Higher Rank Euler, Kolyvagin and Stark Systems", *Int. Math. Res. Not.* **2021** (2021), no. 13, p. 10118-10206.

[3] K. BÜYÜKBODUK, "Kolyvagin systems of Stark units", *J. Reine Angew. Math.* **631** (2009), p. 85-107.

[4] ———, "Stark units and the main conjectures for totally real fields", *Compos. Math.* **145** (2009), no. 5, p. 1163-1195.

[5] ———, "On Euler systems of rank r and their Kolyvagin systems", *Indiana Univ. Math. J.* **59** (2010), no. 4, p. 1277-1332.

[6] ———, "Λ-adic Kolyvagin systems", *Int. Math. Res. Not.* **2011** (2011), no. 14, p. 3141-3206.

[7] ———, "Stickelberger elements and Kolyvagin systems", *Nagoya Math. J.* **203** (2011), p. 123-173.

[8] M. KURIHARA, "Refined Iwasawa theory and Kolyvagin systems of Gauss sum type", *Proc. Lond. Math. Soc.* **104** (2012), no. 4, p. 728-769.

[9] ———, "The structure of Selmer groups of elliptic curves and modular symbols", in *Iwasawa theory 2012*, Contributions in Mathematical and Computational Sciences, vol. 7, Springer, 2012, p. 317-356.

[10] ———, "Refined Iwasawa theory for p-adic representations and the structure of Selmer groups", *Münster J. Math.* **7** (2014), no. 1, p. 149-223.

[11] B. MAZUR & K. RUBIN, *Kolyvagin systems*, Memoirs of the American Mathematical Society, vol. 799, American Mathematical Society, 2004.

[12] ———, "Controlling Selmer groups in the higher core rank case", *J. Théor. Nombres Bordeaux* **28** (2016), no. 1, p. 145-183.

[13] R. SAKAMOTO, "Stark systems over Gorenstein local rings", *Algebra Number Theory* **12** (2018), no. 10, p. 2295-2326.

[14] ———, "A higher rank Euler system for $\mathbb{G}_m$ over a totally real field", https://arxiv.org/abs/2002.04871, to appear in *Am. J. Math.*, 2020.

[15] ———, "On the theory of higher rank Euler, Kolyvagin and Stark systems: a research announcement", *RIMS Kôkyûroku Bessatsu* **B83** (2020), p. 141-159, Algebraic Number Theory and Related Topics 2017.

Ryotaro SAKAMOTO
Department of Mathematics Faculty of Science and Technology
Keio University, 3-14-1 Hiyoshi
Kohoku-ku, Yokohama, 223-8522, Japan
E-mail: r-sakamoto@keio.jp