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par Gérald TENENBAUM

Résumé. Désignons par $\{p_j(n)\}_{j=1}^{\omega(n)}$ la suite croissante des facteurs premiers distincts d'un entier n. Nous explicitions les détails de la preuve d'un énoncé d'Erdős impliquant que, pour toute fonction $\xi(n)$ tendant vers l'infini avec n, nous avons

$$f(n) := \max_{1\leqslant j<\omega(n)} \log\left(\frac{\log p_{j+1}(n)}{\log p_j(n)}\right) = \log_3 n + O(\xi(n))$$

pour presque tout entier n.

Abstract. Let $\{p_j(n)\}_{j=1}^{\omega(n)}$ denote the increasing sequence of distinct prime factors of an integer n. We provide details for the proof of a statement of Erdős implying that, for any function $\xi(n)$ tending to infinity with n, we have

$$f(n) := \max_{1\leqslant j<\omega(n)} \log\left(\frac{\log p_{j+1}(n)}{\log p_j(n)}\right) = \log_3 n + O(\xi(n))$$

for almost all integers n.

In private correspondence [4], E. Sofos asked for normal upper and lower bounds for the arithmetic function

$$f(n) := \max_{1\leqslant j<\omega(n)} \log\left(\frac{\log p_{j+1}(n)}{\log p_j(n)}\right),$$

where $\{p_j(n)\}_{j=1}^{\omega(n)}$ denotes the increasing sequence of distinct prime factors of an integer n. The answer to this question is actually provided by a statement of Erdős in [1], where it is asserted without proof that, for any $c > 0$, the density of the set of those integers n such that $\mathrm{e}^{f(n)} > c\log_2 n$ is $1 - \mathrm{e}^{-1/c}$. (Here and in the sequel, $\log_k$ denotes the k-fold iterated logarithm.) Indeed, given any function $\xi(n)$ tending to infinity and letting c tend sufficiently slowly to zero, we infer that, for almost all n, we have $f(n) > \log_3 n - \xi(n)$, while by letting c tend slowly to infinity we obtain that $f(n) \leqslant \log_3 n + \xi(n)$ also holds for a set of asymptotic density 1. Interesting related results appear in Erdős' articles [2] and [3].

In this short note, we provide the not so obvious details of the proof, in the spirit of the paper [1].

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Theorem 1 (Erdős [1]). *Let $c > 0$. Then the inequality*

$$f(n) > \log_3 n - \log(1/c)$$

holds on a set of integers n of asymptotic density $1 - \mathrm{e}^{-1/c}$. In particular, given any function $\xi(n) \to \infty$, we have

$$\log_3 n - \xi(n) \leqslant f(n) \leqslant \log_3 n + \xi(n)$$

for almost all integers n.

Proof. Let x be a large parameter, write $Z := c \log_2 x$, and let $\chi_p(n)$ denote the indicator function of the set of those integers $n \leqslant x$ that are divisible by the prime p but by no prime q such that $p < q \leqslant p^Z$. For squarefree m, put $\chi_m(n) := \prod_{p|m} \chi_p(n)$. Taking into account the slow growth of $\log_2 x$, it is sufficient to show that, as x tends to infinity,

$$N(x) := |\{n \leqslant x : p \mid n,\, p \leqslant x^{1/Z} \Rightarrow \chi_p(n) = 0\}| \sim \mathrm{e}^{-1/c} x. \tag{1}$$

Indeed, $n \leqslant x$ is counted by $N(x)$ if, and only if, $p_{j+1}(n) \leqslant p_j(n)^Z$ whenever $1 \leqslant j < \omega(n)$. Let $P^+(n)$ (resp. $P^-(n)$) denote the largest (resp. the smallest) prime factor of an integer n, with the convention that $P^+(1) = 1$ (resp. $P^-(1) = \infty$), and let μ designate the Möbius function. By the inclusion–exclusion principle (see, e.g., [6, p. 39]) or the Möbius inversion formula, we thus have, for $x \geqslant 1$,

$$N(x) = \sum_{\substack{m \leqslant x \\ m \in \mathcal{M}}} \mu(m) \sum_{n \leqslant x} \chi_m(n), \tag{2}$$

where $\mathcal{M}$ denotes the set of all squarefree integers $m \leqslant x$ such that

$$P^+(m) \leqslant x^{1/Z}, \text{ and } p_{j+1}(m) > p_j(m)^Z \quad (1 \leqslant j < \omega(m)).$$

Moreover, as is familiar in Brun's sieve method, we obtain a lower or an upper bound for the left-hand side by restricting the outer sum to integers m whose number of prime factors is at most equal to an odd or an even bound.

Let $k \geqslant 1$ be fixed. For each $m \in \mathcal{M}$, define $P_m := \prod_{p|m} \prod_{p<q\leqslant p^Z} q$. Then

$$N_k(x) := \sum_{\substack{m \in \mathcal{M} \\ \omega(m)=k}} \sum_{n \leqslant x} \chi_m(n) = \sum_{\substack{m \in \mathcal{M} \\ \omega(m)=k}} \sum_{\substack{\nu \leqslant x/m \\ (\nu, P_m)=1}} 1. \tag{3}$$

From a standard sieve result (see, e.g., the lemma in [5]), we infer that the inner sum is

$$\sim \frac{x\varphi(P_m)}{mP_m} = \frac{x}{mZ^k}\Big\{1 + O\Big(\frac{1}{\log P^-(m)}\Big) + o(1)\Big\} \tag{4}$$

provided $P^+(m) \leqslant x^{o(1/Z)}$. The contribution of the remaining integers m will be treated as an error term. Indeed, in this latter case we may plainly assume that $x^{\varepsilon_x} < P^+(m)^Z \leqslant x$, with ε_x tending to 0 arbitrarily slowly. Then the inner sum is classically (see, e.g., [6, Exercise 85])

$$\ll \frac{x\varphi(P_m)}{mP_m} \ll \frac{x}{mZ^k}$$

and the corresponding contribution to $N_k(x)$ is

$$\frac{x}{Z^k} \sum_{x^{\varepsilon_x/Z} < p \leqslant x^{1/Z}} \frac{1}{p} \sum_{\substack{P^+(h) \leqslant x \\ \omega(h) = k-1}} \frac{1}{h} \ll \frac{x \log(1/\varepsilon_x)}{\log_2 x} = o(x).$$

Carrying (4) back into (3) and summing over m, we obtain, for $k \geqslant 0$,

$$N_k(x) = \frac{x}{Z^k} \sum_{\substack{p_k^Z \leqslant x \\ p_j^Z \leqslant p_{j+1}\, (1 \leqslant j < k)}} \frac{1 + O(1/\log p_1) + o(1)}{p_1 \cdots p_k} \sim \frac{x (\log_2 x)^k}{k! Z^k} \sim \frac{x}{c^k k!}.$$

Thus, for arbirary $\ell \geqslant 1$, we have

$$x \sum_{0 \leqslant k \leqslant 2\ell+1} \frac{(-1)^k}{c^k k!} + o(x) \leqslant N(x) \leqslant x \sum_{0 \leqslant k \leqslant 2\ell} \frac{(-1)^k}{c^k k!} + o(x),$$

and the required result follows by selecting ℓ arbitrarily large. □

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