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On the Harborth constant of $C_3 \oplus C_{3p}$

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Résumé. Soit $(G, +, 0)$ un groupe abélien fini. La constante de Harborth de G, notée $\mathsf{g}(G)$, est le plus petit entier k tel que toute suite d'éléments deux à deux distincts de G de longueur k, de manière équivalente tout sous-ensemble de G de cardinal au moins k, admet une sous-suite de longueur $\exp(G)$ dont la somme soit 0. Dans cet article, il est démontré que $\mathsf{g}(C_3 \oplus C_{3p}) = 3p + 3$ pour tout nombre premier $p \neq 3$ et que $\mathsf{g}(C_3 \oplus C_9) = 13$.

Abstract. For a finite abelian group $(G, +, 0)$ the Harborth constant $\mathsf{g}(G)$ is the smallest integer k such that each squarefree sequence over G of length k, equivalently each subset of G of cardinality at least k, has a subsequence of length $\exp(G)$ whose sum is 0. In this paper, it is established that $\mathsf{g}(C_3 \oplus C_{3p}) = 3p + 3$ for prime $p \neq 3$ and $\mathsf{g}(C_3 \oplus C_9) = 13$.

1. Introduction

For $(G, +, 0)$ a finite abelian group, a zero-sum constant of G is often defined as the smallest integer k such that each set (or sequence, resp.) of elements of G of cardinality (or length, resp.) at least k has a subset (or subsequence, resp.) whose elements sum to 0, the neutral element of the group, and that possibly fulfills some additional condition (typically on its size). We refer to the survey article [6] for an overview of zero-sum constants of this and related forms. It is technically advantageous to work with squarefree sequences, that is, sequences where all terms are distinct, instead of sets.

Harborth [11] considered the constants that arise, for sequences and for squarefree sequences, when the additional condition on the subsequence is that its length is equal to the exponent of the group. His original motivation was a problem on lattice points. Considering these constants can be seen as an extension of the problem settled in the Theorem of Erdős–Ginzburg–Ziv [5] from cyclic groups to general finite abelian groups.

The constant nowadays called the Harborth constant of G, denoted $\mathsf{g}(G)$, is the constant that arises when considering squarefree sequences in the above mentioned problem. That is, $\mathsf{g}(G)$ is the smallest integer k such that

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every squarefree sequences over G of length at least k has a subsequence of length $\exp(G)$ that sums to 0. The exact value of $\mathsf{g}(G)$ is only known for a few types of groups. We refer to the monograph by Bajnok [1], in particular Chapter F.3, for a detailed exposition. We recall some known results that are relevant for our current investigations.

1.1. Some known results. For G an elementary 2-group, that is, the exponent of the group is 2, the problem admits a direct solution: there are no squarefree sequences of length 2 that sum to 0, and it follows that $\mathsf{g}(G) = |G|+1$ as there are no squarefree sequences of length strictly greater than the cardinality of G and therefore the condition is vacuously true for these sequences. For elementary 3-groups the problem of determining $\mathsf{g}(G)$ is particularly popular as it is equivalent to several other well-investigated problems such as cap-sets and sets without 3-term arithmetic progressions. Nevertheless, the exact value for elementary 3-groups is only known up to rank 6 (see [3] for a detailed overview and [15] for the result for rank 6); recently Ellenberg and Gijswijt [4], building on the work of Croot, Lev, and Pach [2], obtained a major improvement on asymptotic upper bounds for elementary 3-groups.

If G is a cyclic group, then the problem again admits a direct solution: the only squarefree sequence of length exponent is the one containing each element of the group G and it suffices to check whether the sum of all elements of G is 0 or not. More concretely, for n a strictly positive integer and C_n a cyclic group of order n, one has:

$$\mathsf{g}(C_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$

For groups of rank two the problem of determining $\mathsf{g}(G)$ is wide open. It is known that $\mathsf{g}(C_p \oplus C_p) = 2p-1$ for prime $p \geq 47$ and for $p \in \{3,5,7\}$. The latter is due to Kemnitz [12], the former due to Gao and Thangadurai [8], with an additional minor improvement from the original $p \geq 67$ to $p \geq 47$ in [7]. Furthermore, Gao and Thangadurai [8] determined $\mathsf{g}(C_4 \oplus C_4) = 9$ and then made the following conjecture:

$$\mathsf{g}(C_n \oplus C_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd} \\ 2n+1 & \text{if } n \text{ is even.} \end{cases}$$

Moreover, Ramos and some of the present authors [14] determined the value for groups of the form $C_2 \oplus C_{2n}$:

$$\mathsf{g}(C_2 \oplus C_{2n}) = \begin{cases} 2n+3 & \text{if } n \text{ is odd} \\ 2n+2 & \text{if } n \text{ is even.} \end{cases}$$

Finally, Kiefer [13] (also see [1, Proposition F.104]) showed that $\mathsf{g}(C_3 \oplus C_{3n}) \geq 3n+3$ for $n \geq 2$, which for n odd, is larger by one then what might be expected (we refer to Section 3.1 for further details).

1.2. Main result. In the current paper, we determine $\mathsf{g}(C_3 \oplus C_{3p})$ when p is a prime number. It turns out that the bound by Kiefer is usually sharp, yet there is one exception, namely $p = 3$. Specifically, we will show:

$$\mathsf{g}(C_3 \oplus C_{3p}) = \begin{cases} 3p+3 & \text{if } p \neq 3 \text{ is prime} \\ 3p+4 & \text{if } p = 3. \end{cases}$$

The proof makes use of various addition theorems, namely the Theorems of Cauchy–Davenport, Dias da Silva–Hamidoune, and Vosper. These are applied to 'projections' of the set to the subgroup C_p of $C_3 \oplus C_{3p}$. This is a reason why our investigations are limited to groups where p is prime. We also obtain some results by computational means. In particular, we confirm the conjecture by Gao and Thangadurai that we mentioned above for $C_6 \oplus C_6$.

2. Preliminaries

The notation used in this paper follows [9]. We recall some key notions and results. For $a, b \in \mathbb{R}$ the interval of *integers* is denoted by $[a, b]$, that is, $[a, b] = \{z \in \mathbb{Z} \colon a \leq z \leq b\}$. A cyclic group of order n is denoted by C_n.

Let G be a finite abelian group; we use additive notation. There are uniquely determined non-negative integers r and $1 < n_1 \mid \cdots \mid n_r$ such that $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$. The integer r is called the rank of G. Moreover, if $|G| > 1$, then n_r is the exponent of G, denoted $\exp(G)$; for the sake of completeness, we recall that for a group of cardinality 1 the exponent is 1.

A sequence over G is an element of the free abelian monoid over G. Multiplicative notation is used for this monoid and its neutral element, the empty sequence, is denoted by 1. That is, for S a sequence over G, for each $g \in G$ there exists a unique non-negative integer v_g such that $S = \prod_{g \in G} g^{v_g}$; we call v_g the multiplicity of g in S. For each sequence S over G there exist not necessarily distinct $g_1, \dots, g_\ell$ in G such that $S = g_1 \dots g_\ell$; these elements are determined uniquely up to ordering. The sequence S is called squarefree if $v_g \leq 1$ for each $g \in G$, equivalently, all the g_i are distinct.

The length of S is $\ell = \sum_{g \in G} v_g$; it is denoted by $|S|$. The sum of S is $\sum_{i=1}^{\ell} g_i = \sum_{g \in G} v_g g$; it is denoted by $\sigma(S)$. The support of the sequence S, denoted $\operatorname{supp}(S)$, is the *set* of elements appearing in S, that is, $\operatorname{supp}(S) = \{g \in G \colon v_g > 0\}$. A subsequence of S is a sequence T that divides S in the monoid of sequences, that is $T = \prod_{i \in I} g_i$ for some $I \subset [1, \ell]$. Moreover, $T^{-1}S$ denotes the sequence fulfilling $(T^{-1}S)T = S$.

Let G and G' be two groups and let f be a map from G to G'. We denote by f also the homomorphic extension of f to the monoid of sequences, that is, if $S = g_1 \dots g_\ell$ is a sequence over G, then $f(S) = f(g_1) \dots f(g_\ell)$ is a sequence over G'. Note that $|S| = |f(S)|$ always holds, even if the map f is not injective. This highlights a difference between working with sequences and working with sets. The image of a squarefree sequence might not be squarefree anymore, but it always has the same length as the original sequence. By contrast, for $A \subset G$ a subset $f(A) = \{f(a) \colon a \in A\}$ can have a cardinality strictly smaller than A.

If f is a group homomorphism, then $\sigma(f(S)) = f(\sigma(S))$. In particular, if f is an isomorphism, then S has a zero-sum subsequence of length k if and only if $f(S)$ has a zero-sum subsequence of length k. Moreover, for $g \in G$ and $S = g_1 \dots g_l$, the sequence $(g+g_1) \dots (g+g_\ell)$ is denoted by $g+S$. Note that S has a zero-sum subsequence of length $\exp(G)$ if and only if $g+S$ has a zero-sum subsequence of length $\exp(G)$.

The set $\Sigma(S) = \{\sigma(T) \colon 1 \neq T \mid S\}$ is the set of (nonempty) subsums of S. A sequence is called zero-sum free if $0 \notin \Sigma(S)$. Moreover, for a non-negative integer h, let $\Sigma_h(S) = \{\sigma(T) \colon T \mid S \text{ with } |T| = h\}$ denote the set of h-term subsums. These notations are also used for sets with the analogous meaning.

Using this notation the definition of the Harborth constant can be stated as follows: $\mathsf{g}(G)$ is the smallest integer k such that for each squarefree sequence S over G with length $|S| \geq k$ one has $0 \in \Sigma_{\exp(G)}(S)$. We also need the Davenport constant $\mathsf{D}(G)$, which is defined as the smallest integer k such that for each sequence S over G with length $|S| \geq k$ one has $0 \in \Sigma(S)$.

Let A and B be subsets of G. Then $A+B$ denotes the set $\{a+b \colon a \in A, b \in B\}$, called the sumset of A and B.

We recall some well-known results on set-addition in cyclic groups of prime order. We start with the classical Theorem of Cauchy–Davenport (see for example [10, Theorem 6.2]).

Theorem 2.1 (Cauchy–Davenport)**.** *Let p be a prime number and let $A, B \subset C_p$ be non-empty sets, then:*

$$|A+B| \geq \min\{p, |A|+|B|-1\}.$$

This yields immediately that for non-empty sets $A_1, \dots, A_h \subset C_p$ one has:

$$|A_1 + \dots + A_h| \geq \min\left\{p, \sum_{i=1}^{h} |A_i| - (h-1)\right\}.$$

The associated inverse problem, that is, the characterization of sets where the bound is sharp, is solved by the Theorem of Vosper (see for example [10, Theorem 8.1]).

Theorem 2.2 (Vosper). *Let p be a prime number and let $A, B \subset C_p$. Suppose that $|A|, |B| \geq 2$ and $|A+B| = |A| + |B| - 1$.*

- *If $|A+B| \leq p-2$, then A and B are arithmetic progressions with common difference, that is there is some $d \in C_p$ and there are $a, b \in C_p$ such that $A = \{a+id \colon i \in [0, |A|-1]\}$ and $B = \{b+id \colon i \in [0, |B|-1]\}$.*
- *If $|A+B| = p-1$, then $A = \{c - a \colon a \in C_p \setminus B\}$ for some $c \in C_p$.*

We also need the analogue of the Theorem of Cauchy–Davenport for restricted set addition. It is called the Theorem of Dias da Silva–Hamidoune (see for example [10, Theorem 22.5]).

Theorem 2.3 (Dias da Silva–Hamidoune). *Let p be a prime number. Let $A \subset C_p$ be a non-empty subset and let $h \in [1, |A|]$. Then:*

$$|\Sigma_h(A)| \geq \min\{p, h(|A| - h) + 1\}.$$

We end this section with two technical lemmas. The first asserts that, except for some corner-cases, the difference of an arithmetic progression in a cyclic group of prime order is, up to sign, uniquely determined. We include a proof as we could not find a suitable reference.

Lemma 2.4. *Let $p \geq 5$ be a prime number and let $A \subset C_p$ be a set such that $|A| = k$ with $2 \leq k \leq p-2$. Assume that A is an arithmetic progression, that is, there are some $r, a \in C_p$, such that $A = \{a + ir \colon i \in [0, k-1]\}$. The difference r is determined uniquely up to sign, that is, if there are some $s, b \in C_p$ such that $A = \{b + is \colon i \in [0, k-1]\}$, then $s \in \{r, -r\}$.*

Proof. Since A is an arithmetic progression with difference r if and only if the complement of A in C_p is an arithmetic progression with difference r, we can assume that $|A| \leq \frac{p-1}{2}$. Let e be some non-zero element of C_p.

As the problem is invariant under affine transformations, we can assume without loss of generality that $A = \{0, e, 2e, \ldots, (k-1)e\}$. Suppose for a contradiction that $A = \{a+ir \colon i \in [0, k-1]\}$ with $a, r \in C_p$ and $r \notin \{e, -e\}$. Without loss of generality we can assume that $r = r'e$ with $r' \in [2, \frac{p-1}{2}]$.

As

$$k - 1 < k - 1 + r' \leq \frac{p-1}{2} + \frac{p-1}{2} - 1 = p - 2 < p,$$

it follows that $(k-1)e + r \notin A$. It follows that $(k-1)e$ is also the last element of the arithmetic progression A when represented with respect to the difference r. That is, $(k-1)e = a + (k-1)r$.

The same reasoning shows that, when removing the element $(k-1)e$ from A then $(k-2)e$ is the last element of arithmetic progression $A \setminus \{(k-1)e\}$ both with respect to the difference r and e. Consequently, $(k-2)e + r = (k-1)e$. Thus, $r = e$. □

When trying to establish the existence of zero-sum subsequences whose length is close to that of the full sequence, it can be advantageous to work instead with the few elements of the sequence not contained in the putative subsequence. We formulate the exact link in the lemma below.

Lemma 2.5. *Let G be a finite abelian group. Let $0 \leq r \leq k$. The following statements are equivalent.*

- *Every squarefree sequence S over G of length k has a subsequence R of length r with $\sigma(S) = \sigma(R)$.*
- *Every squarefree sequence S over G of length k has a zero-sum subsequence T of length $k - r$.*

Proof. Let S be a squarefree sequence of length k. Now, let R be a subsequence of length r with $\sigma(R) = \sigma(S)$. Then the sequence $T = R^{-1}S$ is a sequence of length $k - r$ with sum $\sigma(S) - \sigma(R) = 0$. Conversely, let T be a zero-sum subsequence of length $k - r$. Then the sequence $R = T^{-1}S$ is a sequence of length $k-(k-r) = r$ with sum $\sigma(R) = \sigma(S)-\sigma(T) = \sigma(S)$. □

3. Main result

As mentioned in the introduction our main result is the exact value of the Harborth constant for groups of the form $C_3 \oplus C_{3p}$ where p is prime.

Theorem 3.1. *Let p be a prime number. Then*

$$\mathsf{g}(C_3 \oplus C_{3p}) = \begin{cases} 3p+3 & \text{for } p \neq 3 \\ 3p+4 & \text{for } p = 3. \end{cases}$$

We start by establishing that those values are lower bounds for the Harborth constant. Then, we establish the existence of the zero-sum subsequences that we need under several additional assumptions on the sequences. Finally, we combine all these results.

3.1. Lower bounds. In this section we establish lower bounds for the Harborth constant. We start with a general lemma. An interesting aspect of this lemma is that it mixes constants for squarefree sequences and sequences; it improves the result [14, Lemma 3.2], where instead of the Davenport constant the Olson constant was used.

Lemma 3.2. *Let G_1, G_2 be finite abelian groups with* $\exp(G_2) \mid \exp(G_1)$*. Then*

$$\mathsf{g}(G_1 \oplus G_2) \geq \mathsf{g}(G_1) + \mathsf{D}(G_2) - 1.$$

Proof. Let S_1 be a squarefree sequence over G_1 of length $\mathsf{g}(G_1) - 1$ that has no zero-sum subsequence of length $\exp(G_1)$. Let S_2' be a sequence over G_2 of length $\mathsf{D}(G_2) - 1$ that has no zero-sum subsequence. Suppose $S_2' = \prod_{g \in G_2} g^{v_g}$. Since S_2' is zero-sumfree $v_g < \exp(G_2) \le \exp(G_1)$ for each $g \in G_2$. Let $\{h_1, \dots, h_{\exp(G_1)-1}\}$ be distinct elements in G_1, and let

$$S_2 = \prod_{g \in G_2} \left(\prod_{i=1}^{v_g} (g + h_i) \right).$$

Then, S_2 is a squarefree zero-sum free sequence over $G_1 \oplus G_2$. Note that $S_1 S_2$ also is a squarefree sequence over $G_1 \oplus G_2$. To show our claim, it suffices to show that $S_1 S_2$ has no zero-sum subsequence of length $\exp(G_1 \oplus G_2)$. Assume to the contrary that $T \mid S_1 S_2$ is a zero-sum subsequence of length $\exp(G_1 \oplus G_2)$. Let $T = T_1 T_2$ with $T_i \mid S_i$. Since $\exp(G_1 \oplus G_2) = \exp(G_1)$, it follows that T is not a subsequence of S_1, that is, T_2 is not the empty sequence. Let $\pi : G \to G_2$ be the projection map of G on G_2 with respect to the decomposition $G = G_1 \oplus G_2$. Since $\sigma(\pi(T_1)) = 0$, it follows that $\sigma(\pi(T_2)) = 0$. Yet this is a contradiction, as $\pi(T_2)$ is a non-empty zero-sum subsequence of S_2', while by assumption S_2' has no non-empty zero-sum subsequence. □

Using this lemma in combination with the result for cyclic groups, yields the following bound, which is given in [1, Proposition F.102].

Lemma 3.3. *Let n_1, n_2 be strictly positive integers with $n_1 \mid n_2$. Then*

$$\mathsf{g}(C_{n_1} \oplus C_{n_2}) \ge \begin{cases} n_1 + n_2 - 1 & \text{if } n_2 \text{ is odd} \\ n_1 + n_2 & \text{if } n_2 \text{ is even.} \end{cases}$$

In particular,

$$\mathsf{g}(C_3 \oplus C_{3n}) \ge \begin{cases} 3n + 2 & \text{if } n \text{ is odd} \\ 3n + 3 & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Lemma 3.2 we have $\mathsf{g}(C_{n_1} \oplus C_{n_2}) \ge \mathsf{g}(C_{n_2}) + \mathsf{D}(C_{n_1}) - 1$. The claim follows using that

$$\mathsf{g}(C_{n_2}) = \begin{cases} n_2 & \text{if } n_2 \text{ is odd} \\ n_2 + 1 & \text{if } n_2 \text{ is even} \end{cases}$$

and $\mathsf{D}(C_{n_1}) = n_1$ (see, e.g., [10, Theorem 10.2]). The claim for $C_3 \oplus C_{3n}$ is a direct consequence. □

The bound for $\mathsf{g}(C_3 \oplus C_{3n})$ can be improved for odd n. This was initially done by Kiefer [13] (also see [1, Proposition F.104]). We include the argument, as our construction is slightly different.

Lemma 3.4. *Let $G = C_3 \oplus C_{3n}$ with an integer $n \geq 2$. Then $\mathsf{g}(G) \geq 3n+3$.*

Proof. For even n the claim is known by Lemma 3.3. Thus, we assume that n is odd. To prove this lemma, it suffices to give an example of a squarefree sequence of length $3n + 2$ that does not admit a zero-sum subsequence of length $\exp(G) = 3n$. Let $G = \langle e_1 \rangle \oplus \langle e_2 \rangle$ with $\operatorname{ord}(e_1) = 3$ and $\operatorname{ord}(e_2) = 3n$. Let π_1 and π_2 denote the projection maps $\pi_1 : G = \langle e_1 \rangle \oplus \langle e_2 \rangle \to \langle e_1 \rangle$ and $\pi_2 : G = \langle e_1 \rangle \oplus \langle e_2 \rangle \to \langle e_2 \rangle$.

Further, let

$$T_1 = \prod_{g \in \langle e_2 \rangle \setminus \{0, -e_2, e_2\}} (e_1 + g)$$

and $T_2 = 0(e_2)(2e_2)(3e_2)(-6e_2)$. Then $T = T_1 T_2$ is a squarefree sequence and $|T| = 3n - 3 + 5 = 3n + 2$.

To obtain the claimed bound, it suffices to assert that T does not have a zero-sum subsequence of length $3n$. Assume for a contradiction that T has a zero-sum subsequence R of length $3n$. Clearly, one has $\sigma(\pi_1(R)) = \sigma(\pi_2(R)) = 0$. Let $R = R_1 R_2$ with $R_1 | T_1$ and $R_2 | T_2$. Note that $\sigma(\pi_1(R)) = \sigma(\pi_1(R_1)) = |R_1| e_1$. Consequently, as $\sigma(\pi_1(R)) = 0$ it is necessary that 3 divides $|R_1|$. Moreover to obtain $|R| = 3n$ it is necessary that $3n - 5 \leq |R_1| \leq 3n - 3$. It follows that $|R_1| = 3n - 3$, that is, $R_1 = T_1$. Consequently $|R_2| = 3$.

Now, $\sigma(\pi_1(R_1)) = |R_1| e_1 = 0$. Furthermore

$$\sigma(\pi_2(R_1)) = \sum_{h \in \langle e_2 \rangle \setminus \{-e_2, e_2, 0\}} h = \left(\sum_{h \in \langle e_2 \rangle} h \right) - (-e_2 + e_2 + 0),$$

which is also equal to 0, since the sum of all elements of the cyclic group $\langle e_2 \rangle$ is 0 (here it is used that $3n$ is odd). Thus, $\sigma(R_1) = 0$, and it follows that: $\sigma(R) = 0$ if and only if $\sigma(R_2) = 0$. However, T_2 has no subsequence of length 3 with sum 0. Thus T has no zero-sum subsequence of length $3n$. □

It turns out that for $p = 3$, there is a better construction.

Lemma 3.5. *One has $\mathsf{g}(C_3 \oplus C_9) \geq 13$.*

Proof. To prove this lemma, as $\exp(G) = 9$, it suffices to give an example of a squarefree sequence T of length 12 over G that does not admit any zero-sum subsequence T_1 of length 9. Let $G = \langle e_1 \rangle \oplus \langle e_2 \rangle$ with $\operatorname{ord}(e_1) = 3$, and $\operatorname{ord}(e_2) = 9$.

Let us consider the following sequence:

$$T = R(e_1 + R)(e_2 + R)(e_1 + e_2 + R), \text{ with } R = 0(3e_2)(6e_2).$$

This is a squarefree sequence of length 12 that satisfies $\sigma(T) = 0 + 0 + 3e_2 + 3e_2 = 6e_2$. By Lemma 2.5 with $k = 12$ and $r = 9$, the sequence T has a zero-sum subsequence of length 9 if and only if T has a subsequence

T_2 with $|T_2| = 3 = 12 - 9$ and $\sigma(T) = \sigma(T_2) = 6e_2$. For a contradiction let us assume such a subsequence T_2 exists. Let $H = \{0, 3e_2, 6e_2\}$ and let $\pi : G \to G/H$ be the standard epimorphism. One has $G/H \cong C_3 \oplus C_3$, and this group is generated by $f_1 = \pi(e_1)$ and $f_2 = \pi(e_2)$.

Since $\sigma(T_2) = 6e_2$, one has that $\pi(T_2)$ is a zero-sum subsequence of $\pi(T)$ and $\pi(\sigma(T)) = \pi(6e_2) = 0$.

But, note that the only subsequences of $\pi(T) = 0^3 {f_1}^3 {f_2}^3 (f_1 + f_2)^3$ of length 3 which have sum zero are 0^3, ${f_1}^3$, ${f_2}^3$ and $(f_1 + f_2)^3$. It remains to check if any of the corresponding subsequences of T has sum $6e_2$. This is not the case. Concretely, we have $\sigma(R) = 0$, $\sigma(e_1 + R) = 0$, $\sigma(e_2 + R) = 3e_2$, and $\sigma((e_1 + e_2) + R) = 3e_2$. Thus, the sequence T does not have any subsequence of length 3 with sum $6e_2$. This establishes the claimed bound. □

3.2. Establishing the existence of zero-sum subsequence of length $\exp(G)$ under various assumptions. Let us fix some notation that will be used throughout the subsection. Let $G = C_3 \oplus C_{3p}$ with $p \neq 3$ a prime number. We note that $G = H_1 \oplus H_2$ where $H_1 \cong C_3^2$ is the subgroup of elements of order dividing 3 and $H_2 \cong C_p$ is the subgroup of elements of order dividing p. For $i \in \{1, 2\}$, let $\pi_i : G \to H_i$ the projection from G to H_i with respect to the decomposition $G = H_1 \oplus H_2$.

For a sequence S over G there exists a unique decomposition $S = \prod_{h \in H_1} S_h$ where S_h is the subsequence of elements of S with $\pi_1(g) = h$. If S is squarefree then for each $h \in H_1$ the sequence $\pi_2(S_h)$ is a squarefree sequence over H_2.

To establish the bound $\mathsf{g}(G) \leq 3p + 3$ we need to show that every squarefree sequence S of length $3p+3$ over G has a zero-sum subsequence of length $3p$. By Lemma 2.5 this is equivalent to establishing that every squarefree sequence S of length $3p + 3$ over G has a subsequence R of length 3 with the same sum as S.

To obtain such a sequence of length 3 we typically first restrict our considerations to finding a subsequence for which $\pi_1(\sigma(S)) = \pi_1(\sigma(R))$; this condition can be established via explicit arguments, as the group H_1 is fixed and small. Then, using tools from additive combinatorics recalled in Section 2, we show that among the sequences with $\pi_1(\sigma(S)) = \pi_1(\sigma(R))$ there is one for which we also have $\pi_2(\sigma(S)) = \pi_2(\sigma(R))$ and thus satisfy $\sigma(S) = \sigma(R)$ as needed.

We formulate a technical lemma that is a key tool in our argument. Note that for the proof of this lemma it is crucial that p is prime. Later on, for example in Proposition 3.8, the case of small primes creates extra difficulty in the proof. Since we can deal with the case $p = 2$ by computational means, we exclude this one right away to avoid considering corner cases.

Lemma 3.6. *Let $p \geq 5$ be a prime and let S be a squarefree sequence of length $3p+3$ over $G = C_3 \oplus C_{3p}$. Let $S = \prod_{h \in H_1} S_h$ where S_h is the subsequence of elements of S with $\pi_1(g) = h$.*

1. *If there exist distinct $x, y, z \in H_1$ with $x + y + z = \pi_1(\sigma(S))$ such that S_x, S_y, S_z are all non-empty and $|S_x| + |S_y| + |S_z| - 2 \geq p$, then S has a zero-sum subsequence of length $3p$.*
2. *If there exist distinct $x, y \in H_1$ with $2x + y = \pi_1(\sigma(S))$ such that $|S_x| \geq 2$ and $|S_y| \geq 1$ and $2|S_x| + |S_y| - 4 \geq p$, then S has a zero-sum subsequence of length $3p$.*
3. *If there exist $x \in H_1$ with $3x = \pi_1(\sigma(S))$ such that $|S_x| \geq 3$ and $3|S_x| - 8 \geq p$, then S has a zero-sum subsequence of length $3p$.*

We use and combine the theorems of Cauchy–Davenport and Dias da Silva–Hamidoune.

Proof. In each case we show that under the assumptions of the lemma, S has a subsequence R of length 3 with the same sum. By Lemma 2.5 with $k = 3p + 3$ and $r = 3$ this establishes our claim.

(1). Let $x, y, z \in H_1$ be distinct with $x + y + z = \pi_1(\sigma(S))$. If $g_x, g_y, g_z \in G$ are such that g_x divides S_x, g_y divides S_y and g_z divides S_z, then $g_x g_y g_z$ is a subsequence of S and $\pi_1(\sigma(g_x g_y g_z)) = x + y + z = \pi_1(\sigma(S))$.

Thus, to show that S has a subsequence R of length 3 it suffices to show that there exist elements g_x, g_y and g_z in G such that g_x divides S_x, g_y divides S_y and g_z divides S_z with $\pi_2(\sigma(g_x g_y g_z)) = \pi_2(\sigma(S))$.

Let Ω denote the set of all sequence $g_x g_y g_z$ of length 3 with $g_x \mid S_x$, $g_y \mid S_y$, $g_z \mid S_z$. We note that

$$\{\pi_2(\sigma(R)) \colon R \in \Omega\} = \operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z)).$$

From Theorem 2.1, the Theorem of Cauchy–Davenport,

$$|\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z))| \\ \geq \min\{p, |\operatorname{supp}(\pi_2(S_x))| + |\operatorname{supp}(\pi_2(S_y))| + |\operatorname{supp}(\pi_2(S_z))| - 2\}$$

As the sequence S is squarefree, for each $h \in H_1$, the sequence $\pi_2(S_h)$ is squarefree as well. Consequently, $|\operatorname{supp}(\pi_2(S_h))| = |S_h|$. Thus, if $|S_x| + |S_y| + |S_z| - 2 \geq p$, then $\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z))$ must be equal to the full group H_2. In particular, there exits a sequence $R \in \Omega$ with $\pi_2(\sigma(R)) = \pi_2(\sigma(S))$, and the proof is complete.

(2). The argument is similar to the one in the first part, yet we need to use the Theorem of Dias da Silva–Hamidoune in addition. Let $x, y \in H_1$ be distinct with $2x + y = \pi_1(\sigma(S))$. If $g_x g'_x \mid S_x$ and $g_y \mid S_y$, then $g_x g'_x g_y$ is a subsequence of S and $\pi_1(\sigma(g_x g'_x g_y)) = 2x + y = \pi_1(\sigma(S))$.

Let Ω denote the set of all sequence $g_x g'_x g_y$ with $g_x g'_x$ divides S_x and g_y divides S_y. We note that $\{\pi_2(\sigma(R)) \colon R \in \Omega\} = \Sigma_2(\mathrm{supp}(\pi_2(S_x))) + \mathrm{supp}(\pi_2(S_y))$.

By the Theorems of Dias da Silva–Hamidoune and Cauchy–Davenport (see Theorems 2.1 and 2.3) we get that, as p is assumed to be prime,

$$|\Sigma_2(\mathrm{supp}(\pi_2(S_x))) + \mathrm{supp}(\pi_2(S_y))| \geq \min\{p, 2|\mathrm{supp}(\pi_2(S_x))| + |\mathrm{supp}(\pi_2(S_y))| - 4\}.$$

As in (1), if $2|S_x| + |S_y| - 4 \geq p$, then there exists some $R \in \Omega$ with $\pi_2(\sigma(R)) = \pi_2(\sigma(S))$, and the proof is complete.

(3). In this part we consider $x \in H_1$ with $3x = \pi_1(\sigma(S))$. The remainder of the argument is similar to the preceding parts, and we skip the details. □

In the present context there exists essentially two types of sequences S over G of length $3p+3$: those for which $\pi_1(\sigma(S))$ equals zero and those for which it is non-zero. It is clear that this property is preserved under automorphisms of the group, and when the length of the sequence is a multiple of 3 it is also preserved under translations. We treat these two types of sequences separately. In either case, it will be relevant to understand subsequences of lengths three of $\pi_1(S)$ that have sum $\pi_1(\sigma(S))$. In the former case, the sequence is formed by the three elements from a coset of H_1 or it contains only a unique element with multiplicity three.

The latter type is treated in Proposition 3.10. For the former type, two cases are distinguished: the case where the support of $\pi_1(S)$ is the full group H_1 (see Proposition 3.7) and the cases where it is not (see Proposition 3.8).

Proposition 3.7. *Let $p \geq 5$ be a prime and let S be a squarefree sequence of length $3p+3$ over $G = C_3 \oplus C_{3p}$. If $\sigma(\pi_1(S)) = 0$ and $\mathrm{supp}(\pi_1(S)) = H_1$, then S has a zero-sum subsequence of length $3p$.*

Proof. To simplify the subsequent considerations, we note that we can assume that $\sigma(\pi_2(S)) = 0$ (and thus $\sigma(S) = 0$). Indeed, it suffices to note that if, for any $h \in G$, the shifted sequence $h + S$ contains a zero-sum subsequence of length $3p$, then the sequence S contains a zero-sum subsequence of length $3p$. There is some $h' \in H_2$ such that $(3p+3)h' = -\sigma(\pi_2(S))$; note that as p and $3p+3$ are co-prime, the multiplication $h \mapsto (3p+3)h$ is an isomorphism on H_2. Now, one can consider $h' + S$ instead of S provided the additional condition $\sigma(\pi_1(S)) = 0$ is not altered. Since $\sigma(\pi_1(h'+S)) = |S|h' + \sigma(\pi_1(S))$ and since $|S|h' = (3p+3)h' = 0$, this is indeed true and $\mathrm{supp}(\pi_1(h'+S)) = H_1$ still holds.

Let H'_1 be a nontrivial cyclic subgroup of H_1 and let $g \in H_1$, and $\{x, y, z\} = g + H'_1$ a coset. Since $x + y + z = 0 = \sigma(\pi_1(S))$, it follows that if $|S_x S_y S_z| - 2 \geq p$, then from Lemma 3.6 (1), the result holds.

It remains to consider the case where for each coset of H_1 of cardinality three, denoted $\{x, y, z\}$, one has $|S_x S_y S_z| \le p+1$. We note that this is only possible if for every coset $\{x, y, z\}$, one has $|S_x S_y S_z| = p+1$. Indeed, H_1 can be partitioned as the disjoint union of three such cosets, say, $H_1 = \{x_1, y_1, z_1\} \cup \{x_2, y_2, z_2\} \cup \{x_3, y_3, z_3\}$. Then, on the one hand

$$|S_{x_i} S_{y_i} S_{z_i}| \le p+1, \text{ for each } i \in [1,3]$$

yet on the other hand $|S_{x_1} S_{y_1} S_{z_1}| + |S_{x_2} S_{y_2} S_{z_2}| + |S_{x_3} S_{y_3} S_{z_3}| = |S| = 3p+3$. Therefore it is necessary that for each $i \in [1,3]$ one has$|S_{x_i} S_{y_i} S_{z_i}| = p+1$.

Next we assert that this is only possible if each of the 9 sequences has the same length. Let $H_1 = \{q_1, q_2, \dots, q_9\}$ such that $|S_{q_1}| \ge |S_{q_2}| \ge \dots \ge |S_{q_9}|$. Let $v_i = |S_{q_i}|$. There is a some $j \in [3,9]$ such that $\{q_1, q_2, q_j\}$ is a coset and there is some $i \in [1,7]$ such that $\{q_8, q_9, q_i\}$ is a coset. Then one has $v_1 + v_2 + v_j = p+1$ and $v_8 + v_9 + v_i = p+1$.

It follows that $(v_1 - v_9) + (v_2 - v_8) + (v_j - v_i) = 0$, and hence $(v_1 - v_9) + (v_2 - v_8) = v_i - v_j$. Yet, as $(v_1 - v_9) \ge (v_i - v_j)$ we get that $v_2 - v_8 = 0$. Consequently, one has $v_2 = v_8$ and further $v_2 = v_3 = \dots = v_8 = v$. This common value needs to be $\frac{p+1}{3}$. It remains to show that $v_1 = v$ and $v_9 = v$. There exists a coset of H_1 of cardinality 3 that contains q_9 and that does not contain q_1, so $v_9 + 2v = p+1$ and thus $v_9 = \frac{p+1}{3}$. In the same way we get that $v_1 = v$.

We now reconsider, for a coset $\{x, y, z\}$, the cardinality of the set

$$\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z)).$$

By Theorem 2.1, the Theorem of Cauchy–Davenport, one has

$$\begin{aligned}
&|\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z))| \\
&\ge \min\{p, |\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y))| + |\operatorname{supp}(\pi_2(S_z))| - 1\} \\
&\ge \min\{p, \min\{p, |\operatorname{supp}(\pi_2(S_x))| + |\operatorname{supp}(\pi_2(S_y))| - 1\} + |\operatorname{supp}(\pi_2(S_z))| - 1\}.
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\min\{p, |\operatorname{supp}(\pi_2(S_x))| + |\operatorname{supp}(\pi_2(S_y))| + |\operatorname{supp}(\pi_2(S_z))| - 2\} \\
= \min\{p, |S_x| + |S_y| + |S_z| - 2\} = p - 1.
\end{aligned}$$

If one has $|\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z))| \ge p$, then

$$\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z)) = H_2,$$

and we can conclude as in Lemma 3.6.

Thus, it remains to consider the case that $|\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y)) + \operatorname{supp}(\pi_2(S_z))| = p - 1$. Now, this is only possible when

$$|\operatorname{supp}(\pi_2(S_x)) + \operatorname{supp}(\pi_2(S_y))| = |\operatorname{supp}(\pi_2(S_x))| + |\operatorname{supp}(\pi_2(S_y))| - 1,$$

as otherwise the second inequality in the displayed equation above would be strict. In the same way, we get that $|\text{supp}(\pi_2(S_x)) + \text{supp}(\pi_2(S_z))| = |\text{supp}(\pi_2(S_x))| + |\text{supp}(\pi_2(S_z))| - 1$.

The Theorem of Vosper, see Theorem 2.2, yields that $\text{supp}(\pi_2(S_x))$ and $\text{supp}(\pi_2(S_y))$ are arithmetic progressions with a common difference, and the sets $\text{supp}(\pi_2(S_x))$ and $\text{supp}(\pi_2(S_z))$ are arithmetic progressions with a common difference. From Lemma 2.4, it follows that there is a common difference for all three progressions $\text{supp}(\pi_2(S_x))$, $\text{supp}(\pi_2(S_y))$, $\text{supp}(\pi_2(S_z))$. Indeed, one has this for any pair of the 9 sets, as the argument can be applied for any coset. Thus, all 9 sets are arithmetic progressions with a common difference. Let us denote this difference by e; of course, this is a generating element of H_2.

If there is some $h \in H_1$ such that $\pi_2(S_h)$ has a zero-sum subsequence of length 3, then in fact S_h has a zero-sum sequence of length 3. Since we assumed at the start that $\sigma(S) = 0$, invoking Lemma 2.5 our claim is complete.

Thus, we assume that for no $h \in H_1$ the sequence $\pi_2(S_h)$ has a zero-sum subsequence of length 3. In particular $\pi_2(S_h)$ does not have $(-e)e0$, as a subsequence. Thus, for each $h \in H_1$ one has $\pi_2(S_h) = \prod_{j=s_h}^{s_h+\frac{p-2}{3}} (je)$ for some integer s_h with $0 \le s_h \le s_h + \frac{p-2}{3} < p-1$. It is easy to see that:

$$\Sigma_3(\pi_2(S_h)) = \{je \colon j \in [3s_h + 3, 3s_h - 3 + (p-2)]\}.$$

For this set not to contain 0, we need $3s_h - 3 + (p-2) < p$. So $3s_h < 5$, that is $s_h \in \{0, 1\}$.

If there is a coset $\{x, y, z\}$ such that $s_x = s_y = s_z = 0$, then clearly $\pi_2(S_x)+\pi_2(S_y)+\pi_2(S_z)$ contains 0. Yet if there is no coset $\{x, y, z\}$ such that $s_x = s_y = s_z = 0$, then there is a coset $\{x', y', z'\}$ such that $s_{x'}+s_{y'}+s_{z'} \ge 2$; indeed, it suffices to note that by the former condition there must be at least two elements $h, h' \in H_1$ with $s_h \ge 1$ and $s_{h'} \ge 1$. However, this gives that $\pi_2(S_{x'})+\pi_2(S_{y'})+\pi_2(S_{z'})$ will contain $(2\cdot\frac{p+1}{3} + \frac{p-2}{3})e = pe = 0$. Thus, the argument is complete. □

For the next result, we keep the condition that $\sigma(\pi_1(S)) = 0$, yet consider the case $\text{supp}(\pi_1(S)) \ne H_1$ instead.

Proposition 3.8. *Let $p \ge 5$ be a prime and let S be a squarefree sequence of length $3p+3$ over $G = C_3 \oplus C_{3p}$. If $\sigma(\pi_1(S)) = 0$ and $\text{supp}(\pi_1(S)) \ne H_1$, then S has a zero-sum subsequence of length $3p$.*

Proof. Let $h \in H_1$ such that $|S_h| = 0$; such an element exists by assumption. Now, as recalled in Section 2, the sequence S contains a zero-sum subsequence T of length $3p$ then the sequence $S - h$ contains $T - h$ as a zero-sum subsequence of length $3p$. Since $\text{supp}(\pi_1(-h + S)) = -h + \text{supp}(\pi_1(S))$ it follows from $h \notin \text{supp}(\pi_1(S))$ that $0 \notin \text{supp}(\pi_1(-h + S))$.

Since $\sigma(\pi_1(-h+S)) = |S|(-h)+\sigma(\pi_1(S))$ and since $|S|h = (3p+3)h = 0$, we can consider $-h+S$ instead of S, as the additional condition $\sigma(\pi_1(S)) = 0$ is not altered. Thus, by translation, it can be assumed without loss of generality that $|S_0| = 0$.

Now, we distinguish cases according to the cardinality of $\operatorname{supp}(\pi_1(S))$. By assumption it is strictly less than $|H_1| = 9$.

Suppose that $|\operatorname{supp}(\pi_1(S))| = 8$. We note that H_1 has exactly 8 cosets of cardinality 3 that do not contain 0. Each non-zero element is contained in exactly 3 of them. Thus there exists as coset $\{x, y, z\}$ such that $|S_xS_yS_z| \geq \frac{3}{8}|S| = \frac{9p+9}{8}$. The existence of the required subsequence now follows from Lemma 3.6 (1) as $(9p+9)/8 > p+1$.

Suppose that $|\operatorname{supp}(\pi_1(S))| = 7$. Let $-x \in H_1$ be the non-zero element such that $|S_{-x}| = 0$. We note that there are 4 cosets of cardinality 3 that contain x, and 3 of those contain neither $-x$ nor 0. It thus follows that there exists a coset $\{x, y, z\}$ such that such that $|S_xS_yS_z| \geq |S_x| + \frac{1}{3}(|S_x^{-1}S|) = (p+1) + \frac{2|S_x|}{3} > p+1$. Using Lemma 3.6 (1) again, the existence of the required subsequence follows.

Suppose that $|\operatorname{supp}(\pi_1(S))| = 6$. Let $g, h \in H_1$ be the two non-zero elements such that $|S_g| = |S_h| = 0$. If $g = -h$, then there is a coset $\{x, y, z\}$ with respect to the subgroup $\{0, g, -g\}$ such that $|S_xS_yS_z| \geq \frac{1}{2}|S| = \frac{3p+3}{2} > p+1$ (note that the two cosets other than $\{0, g, -g\}$ itself cover the 6 remaining elements of H_1). Again, from Lemma 3.6 (1) the argument is complete.

If $g \neq -h$, then for $\{x, y, z\} = \{-g, -g+h, -g-h\}$ or for $\{x, y, z\} = \{-h, -h+g, -h-g\}$, we have $|S_xS_yS_z| \geq |S_{-g-h}| + \frac{1}{2}|(S_{g+h}S_{-g-h})^{-1}S| = (3p+3-|S_{g+h}|+|S_{-g-h}|)/2$; note that both sets are cosets that contain $-g-h$ and the union of the two cosets contains all element of $H_1 \setminus \{0, g, h\}$ except for $g+h$. Since $|S_{g+h}| \leq p$, it follows that $|S_xS_yS_z| > p+1$ and again from Lemma 3.6 (1) the argument is complete.

Suppose that $|\operatorname{supp}(\pi_1(S))| \leq 5$. In this case there exists some $h \in H_1$ such that $|S_h| \geq \frac{1}{5}|S| = \frac{3p+3}{5}$. If $p > 5$, by applying Lemma 3.6 (3) to S_h, then we can complete the proof; for $p \geq 11$ this is direct and for $p = 7$ we observe that one has $|S_h| \geq 5$. It remains to consider the special case $p = 5$. Lemma 3.6 (3) can be applied if there exists some $h \in H_1$ with $|S_h| = 5$. Thus assume that $|S_h| \leq 4$ for all $h \in H_1$. This implies $|\operatorname{supp}(\pi_1(S))| = 5$ since otherwise there would exist some $h \in H_1$ with $|S_h| \geq 18/4 > 4$.

Let $\{h_1, \dots, h_5\} \subset H_1$ such that $|S_{h_i}| \neq 0$ for each $i \in [1, 5]$. Since $\mathsf{g}(C_3^2) = 5$, as recalled in the Introduction, there exist distinct $i, j, k \in [1, 5]$ such that $h_i + h_j + h_k = 0$. Now, $|S_{h_i}S_{h_j}S_{h_k}| = |S| - 2\max\{|S_h| \colon h \in H_1\} \geq 18 - 2 \cdot 4 = 10$. Again we can apply Lemma 3.6 (1) to complete the argument. □

Remark 3.9. For p sufficiently large a shorter argument is available. There exists some $h \in H_1$ such that $|S_h| \geq \frac{1}{8}|S| = \frac{3p+3}{8}$. We can apply Lemma 3.6 (3) if $3|S_h| - 8 \geq p$. This is true provided that $3 \cdot \frac{3p+3}{8} - 8 \geq p$, which is equivalent to $\frac{p}{8} - \frac{55}{8} \geq 0$. Hence for $p \geq 55$ we can complete the argument in this way.

We now turn to the case $\sigma(\pi_1(S)) \neq 0$.

Proposition 3.10. *Let $p \geq 5$ be a prime and let S be a squarefree sequence of length $3p+3$ over $G = C_3 \oplus C_{3p}$. If $\sigma(\pi_1(S)) \neq 0$, then S has a zero-sum subsequence of length $3p$.*

Proof. Let $c = \sigma(\pi_1(S))$. Without loss of generality, one can assume that $|S_c|$ is maximal among all $|S_h|$ for $h \in H_1$. The argument is the same as in the proof of Proposition 3.8.

Notice that $\frac{|S|}{9} = \frac{3p+3}{9} = \frac{p+1}{3}$ and thus $|S_c| \geq \frac{p+1}{3}$. Let us show that $|S_c| > \frac{p+1}{3}$. If for each $h \in H_1$ one has $|S_h| = \frac{p+1}{3}$, then $\sigma(\pi_1(S)) = \frac{p+1}{3} \sum_{h \in H_1} h$. Yet $\sum_{h \in H_1} h = 0$, which contradicts $\sigma(\pi_1(S)) = c \neq 0$. Thus $|S_h| \neq \frac{p+1}{3}$ for some $h \in H_1$, and thus $|S_c| > \frac{p+1}{3}$.

The strategy of the proof is again to apply Lemma 3.6. To this end we need to find a subsequence of $\pi_1(S)$ of length 3 that has sum c.

One possibility is to consider such a subsequence formed by elements from the cyclic subgroup of $C = \{-c, 0, c\}$ only. Thus the subsequences of this subgroup of length 3 which have sum c are: 0^2c and $(-c)^2 0$ and $c^2(-c)$. This approach works if sufficiently many elements from the sequence S are contained in this subgroup. This is detailed in case 1 below.

Another possibility is to consider subsequences of the form $ch(-h)$ with $h \notin C$. While not phrased explicitly in this form, the distinction of subcases in Case 2 corresponds to the number (counted without multiplicity) of distinct subsequences of this form in $\pi_1(S)$.

Let $v_C = |S_0 S_c S_{-c}|$.

Case 1: $v_C \geq p+3$. If $|S_{-c}| = 0$, then $|S_0| + |S_c| = v_C \geq p+3$. Thus, as $|S_c| \leq p$, we have $|S_0| \geq 3$. Thus $|S_c| + 2|S_0| = |S_c| + |S_0| + |S_0| \geq p+3+2 = p+5$. Since $|S_c| > \frac{p+1}{3} \geq 2$, from Lemma 3.6 (2) applied with $x = 0$ and $y = c$, the claim follows.

If $|S_0| \leq 1$, then $|S_{-c}| \geq 2$ and thus $2 \cdot |S_c| + |S_{-c}| \geq v_C - 1 + |S_c| \geq p+4$. The claim follows from Lemma 3.6 (2), applied with $x = c$ and $y = -c$.

If $|S_{-c}| \geq 1$ and $|S_0| \geq 2$, and one of $|S_c| + 2|S_0| \geq p+4$ or $2|S_c| + |S_{-c}| \geq p+4$ are true, then the claim follows from Lemma 3.6 (2); notice that $c + 2 \cdot 0 = c$ and $2c + (-c) = c$. Thus, assume $|S_c| + 2|S_0| \leq p+3$ and $2|S_c| + |S_{-c}| \leq p+3$.

Summing the two inequalities, it follows that $3|S_c| + 2|S_0| + |S_{-c}| \leq 2p+6$. Since $v_C = |S_c| + |S_0| + |S_{-c}| \geq p+3$, it follows that $v_C = p+3$ and

$|S_c| = |S_{-c}|$. Since $|S_c| \geq \frac{v_C}{3} = \frac{p+3}{3}$, which is not an integer, it follows that in fact $|S_c| \geq \frac{p+4}{3}$. Yet, then $2|S_c| + |S_{-c}| = 3|S_c| \geq p+4$, and the claim follows again.

Case 2: $v_C \leq p+2$. The set $H_1 \setminus C$, can be partitioned into three subsets of size two, each containing an element and its opposite, say $H_1 \setminus C = \{g_1, -g_1, g_2, -g_2, g_3, -g_3\}$. Possibly exchanging the role of g_i and $-g_i$, one can assume that for each $i \in [1,3]$ one has $|S_{g_i}| \geq |S_{-g_i}|$. In addition, by renumbering if necessary, one can assume that $|S_{-g_1}| \geq |S_{-g_2}| \geq |S_{-g_3}|$. Adopting this convention we get that $|S_{-g_3}| > 0$ implies that in fact all six sequence S_h for $h \in H_1 \setminus C$ are non-empty. However, note that we do not know if, say, $|S_{g_1}| \geq |S_{g_2}|$; we only know $|S_{g_1}| \geq |S_{-g_1}| \geq |S_{-g_2}|$ and $|S_{g_2}| \geq |S_{-g_2}|$.

Case 2.1: $|S_{-g_3}| > 0$. Let $i \in [1,3]$ such that $|S_{g_i}S_{-g_i}|$ is maximal among $|S_{g_1}S_{-g_1}|$, $|S_{g_2}S_{-g_2}|$, and $|S_{g_3}S_{-g_3}|$. Thus $|S_{g_i}S_{-g_i}| \geq \frac{3p+3-v_C}{3}$.

Hence

$$|S_c| + |S_{g_i}| + |S_{-g_i}| \geq \frac{3p+3-v_C}{3} + |S_c| = (p+1) + \left(|S_c| - \frac{v_C}{3}\right).$$

Thus, $|S_c| + |S_{g_i}| + |S_{-g_i}| \geq p+1$ with equality if and only if $|S_c| = \frac{v_C}{3}$ and $|S_{g_i}| + |S_{-g_i}| = \frac{3p+3-v_C}{3}$. If equality does not hold, then the claim follows from Lemma 3.6 (1) as one has $|S_c| + |S_{g_i}| + |S_{-g_i}| > p+1$.

Thus assume that one has equality, that is, assume $|S_c| = \frac{v_C}{3}$ and $|S_{g_i}| + |S_{-g_i}| = \frac{3p+3-v_C}{3}$. The latter implies that in fact $|S_{g_j}| + |S_{-g_j}| = \frac{3p+3-v_C}{3}$ for each $j \in [1,3]$, while the former implies that $|S_c| = |S_{-c}| = |S_0|$.

Since $|S_c| \geq \frac{p+2}{3}$ (recall the argument at the very beginning of the proof) while $v_C \leq p+2$ (this is the assumption of Case 2) we get that in fact $v_C = p+2$, and thus $|S_c| = \frac{p+2}{3}$. Furthermore, we can now infer that $|S_{g_j}| + |S_{-g_j}| = \frac{2p+1}{3}$ for each $j \in [1,3]$. Yet since $|S_{g_j}| \leq |S_c|$, this is only possible if $|S_{g_j}| = \frac{p+2}{3}$ and $|S_{-g_j}| = \frac{p-1}{3}$. Therefore, one has

$$\begin{aligned} c &= \sigma(\pi_1(S)) \\ &= \frac{p+2}{3}(c + (-c) + 0 + g_1 + g_2 + g_3) + \frac{p-1}{3}(-g_1 - g_2 - g_3) \\ &= g_1 + g_2 + g_3. \end{aligned}$$

Now, we can apply Lemma 3.6 (1) with g_1, g_2, g_3; note that $|S_{g_1}| + |S_{g_2}| + |S_{g_3}| = 3 \cdot \frac{p+2}{3} = p+2$. (In fact it can be seen that $g_1 + g_2 + g_3 = c$ is impossible. To assert this would be another way to conclude the argument.)

Case 2.2: $|S_{-g_2}| > 0$ *and* $|S_{-g_3}| = 0$. One has $|S_{g_1} S_{-g_1} S_{g_2} S_{-g_2}| = 3p + 3 - v_C - |S_{g_3}| \geq 3p + 3 - v_C - |S_c|$. Let $i \in \{1, 2\}$ such that $|S_{g_i} S_{-g_i}|$ is maximal among $|S_{g_1} S_{-g_1}|$ and $|S_{g_2} S_{-g_2}|$. Then,

$$|S_c| + |S_{g_i}| + |S_{-g_i}| \geq \frac{3p + 3 - v_C - |S_c|}{2} + |S_c| = p + 1 + \frac{p + 1 - v_C + |S_c|}{2}.$$

Now, since $v_C \leq p + 2$ and $|S_c| \geq \frac{p+2}{3} \geq 2$, it follows that $|S_c| + |S_{g_i}| + |S_{-g_i}| \geq p + 2$ and one can apply Lemma 3.6 (1) with $c, g_i, -g_i$.

Case 2.3: $|S_{-g_1}| > 0$ *and* $|S_{-g_2}| = |S_{-g_3}| = 0$. One has $|S_{g_1} S_{-g_1}| = 3p+3-v_C - |S_{g_2} S_{g_3}| \geq 3p+3-v_C-2 \cdot |S_c|$. Thus, $|S_c| + |S_{g_1}| + |S_{-g_1}| \geq 3p+3-v_C-|S_c|$. Since $v_C \leq p + 2$ and $|S_c| \leq p$, it follow that $|S_c| + |S_{g_1}| + |S_{-g_1}| \geq p$ with equality if and only if $v_C = p + 2$ and $|S_c| = p$. If equality does not hold, the claim follows from Lemma 3.6 (1) with $c, g_1, -g_1$. Thus, we assume $v_C = p + 2$ and $|S_c| = p$. As $2|S_c| + |S_{-c}| \geq 2p \geq p + 4$, we have that if $|S_{-c}| \neq 0$, then the claim follows from Lemma 3.6 (2) with $x = c$ and $y = -c$. If $|S_{-c}| = 0$, then $|S_0| = v_C - |S_c| = 2$ and the claim follows from Lemma 3.6 (2) with $x = 0$ and $y = c$ as $2|S_0| + |S_c| = p + 4$ and we are done again.

Case 2.4: $|S_{-g_1}| = |S_{-g_2}| = |S_{-g_3}| = 0$. One has $|S_{g_1} S_{g_2} S_{g_3}| = 3p+3-v_C \geq 2p + 1$ If $|S_c| = p$, then we can assume $v_C \leq p + 1$ (see the argument at the end of the preceding case). Thus, in this case $|S_{g_1} S_{g_2} S_{g_3}| \geq 2p + 2$. It follows that for each $i \in [1, 3]$, one has $|S_{g_i}| \geq 2$, and thus $2|S_{g_i}| + |S_{g_j}| \geq 2 + (2p + 2 - p) = p + 4$, for each choice of distinct $i, j \in [1, 3]$.

If $|S_c| \leq p - 1$, then it follows that for each $i \in [1, 3]$, one has$|S_{g_i}| \geq 2p+1-2(p-1) = 3$, and thus $2|S_{g_i}| + |S_{g_j}| \geq 3 + (2p+1-(p-1)) = p+5$, for each choice of distinct $i, j \in [1, 3]$.

Thus, if there is a choice of i, j such that $2g_i + g_j = c$, applying Lemma 3.6 (2), yields the claimed result as $2|S_{g_i}| + |S_{g_j}| \geq p + 4$.

By inspection we can see that indeed there always is such a choice. To wit, for d an element in H_1 such that $H_1 = \langle c \rangle \oplus \langle d \rangle$ we note that

$$\{\{g_1, -g_1\}, \{g_2, -g_2\}, \{g_3, -g_3\}\} = \{\{d, -d\}, \{c+d, -c-d\}, \{c-d, -c+d\}\}.$$

There are eight possibilities for the set $\{g_1, g_2, g_3\}$ (note that the order of the elements is not relevant), and for each of these eight choices we find a relation of the form $2 \cdot g_i + 1 \cdot g_j + 0 \cdot g_k = c$ with $\{i, j, k\} = \{1, 2, 3\}$. Specifically:

- $2 \cdot d + 1 \cdot (c + d) + 0 \cdot (c - d) = c$
- $1 \cdot d + 0 \cdot (c + d) + 2 \cdot (-c + d) = c$
- $0 \cdot d + 1 \cdot (-c - d) + 2 \cdot (c - d) = c$
- $1 \cdot d + 0 \cdot (-c - d) + 2 \cdot (-c + d) = c$
- $2 \cdot (-d) + 0 \cdot (c + d) + 1 \cdot (c - d) = c$
- $0 \cdot (-d) + 2 \cdot (c + d) + 1 \cdot (-c + d) = c$

- $1 \cdot (-d) + 2 \cdot (-c - d) + 0 \cdot (c - d) = c$
- $1 \cdot (-d) + 2 \cdot (-c - d) + 0 \cdot (-c + d) = c$

The claim is established. □

3.3. Proof of Theorem 3.1. To establish our main result we combine the partial results obtained thus far.

By Lemmas 3.3 and 3.4 we know that $\mathsf{g}(C_3 \oplus C_{3n}) \geq 3n + 3$ for each $n \geq 2$.

Now, assume that $p \geq 5$ is prime. We want to show that $\mathsf{g}(C_3 \oplus C_{3p}) \leq 3p + 3$. Let S be a squarefree sequence of length $3p + 3$ over $C_3 \oplus C_{3p}$. We need to show that S has a zero-sum subsequence of length $3p$. We continue to use the maps π_1 and π_2 introduce in the preceding subsection.

If $\sigma(\pi_1(S)) \neq 0$, then S has a zero-sum subsequence by Proposition 3.10. If $\sigma(\pi_1(S)) = 0$, then S has a zero-sum subsequence either by Proposition 3.7 or by Proposition 3.8.

Thus, in any case, S has a zero-sum subsequence of length $3p$ and therefore $\mathsf{g}(C_3 \oplus C_{3p}) \leq 3p+3$. In combination with the lower bound this implies that indeed $\mathsf{g}(C_3 \oplus C_{3p}) = 3p + 3$ for each prime $p \geq 5$.

It remains to determine the value of $\mathsf{g}(C_3 \oplus C_6)$ and of $\mathsf{g}(C_3 \oplus C_9)$. We know by Lemmas 3.3 and 3.5 that the respective values are lower bounds. To show that these values are the exact values of the Harborth constant we used an algorithm for determining the Harborth constant that we discuss in the last section.

4. An algorithm for determining the Harborth constant and some computational results

For the description of the algorithm we use the language of sets rather than that of sequences, as the description feels slightly more natural. To determine the Harborth constant of G means to find the smallest k such that each subset of G of cardinality k has a subset of cardinality $\exp(G)$ with sum 0. We outline the algorithm we used below.

In the first step, all subsets of G of cardinality $\exp(G)$ with sum 0 are constructed (see the discussion at the end of the section for some details on this). If the subsets of cardinality $\exp(G)$ with sum 0 happen to be all the subsets of G of cardinality $\exp(G)$, then this means that the Harborth constant is $\exp(G)$. If not, then we consider all subsets of G that are direct successors of a set of cardinality equal to $\exp(G)$ with sum 0; in other words, we extend each subset of cardinality $\exp(G)$ with sum 0 in all possible ways to a subset of cardinality $\exp(G) + 1$. Thus, we obtain all subsets of G of cardinality $\exp(G)+1$ that contain a subset of cardinality $\exp(G)$ with sum 0. If the subsets of G obtained in this way are all subsets of G of cardinality $\exp(G) + 1$, then we have established that the Harborth constant of G is $\exp(G) + 1$. If not, then we continue as above until for some k the set of subsets of cardinality k obtained in this way coincides with the set of all subsets of cardinality k of G.

Below we detail the steps of the algorithm a bit more. However, a more complete investigation of the algorithmic problem will be presented elsewhere, and we gloss over more technical aspects here. The source code is available at `https://github.com/Zerdoum/Harborth_constant`.

4.1. The steps of the algorithm.

Input: A finite abelian group G of order n and exponent e.

`[Initialization]`: Let $\mathcal{Z}(e)$ denote the collection of all subsets of G of cardinality e that have sum 0. Set $k = e$.

`[Check]`: If $|\mathcal{Z}(k)| = \binom{n}{k}$, then return $\mathsf{g}(G) = k$ and end. Else, increment k to $k + 1$.

`[Extend]`: Let $\mathcal{Z}(k)$ denote the collection of all subsets of cardinality k of G that have subset that is in $\mathcal{Z}(k-1)$. Go to `[Check]`.

Output: $\mathsf{g}(G)$, the Harborth constant of the group G.

We add some further explanations and remarks.

- The group intervenes only in the step `[Initialization]`. (The rest of the algorithm operates merely with subsets of a given ambient set.) To find all subsets of cardinality e with sum 0, we browse all subsets of cardinality $e - 1$. For each of these sets, we check if the inverse of the $e - 1$ elements belongs to the set; if it does not, we add it to the set to obtain a set of cardinality e and sum 0. For this step the subsets of G are represented by a bitmap.
- For the latter parts of the algorithm in addition to the representations as bitmaps a judiciously chosen numbering of the subsets of G is used. In particular, the numbering is chosen in such a way that for every subset its cardinal is at least as large as the cardinal of all its predecessors. This is useful as in this way at each step, our search can be efficiently limited to the $\binom{n}{k}$ subsets of a given cardinal k instead of having to consider all 2^n subsets at each step.
- The subsets of cardinality k that are not in $\mathcal{Z}(k)$ are all the subsets of G of cardinality k that have no zero-sum subset of e elements. Thus, in the final step before the algorithm terminates we effectively have all the subsets of G of cardinality $\mathsf{g}(G) - 1$ that have no zero-sum subset of e elements. That is, the algorithm can be immediately modified to solve the inverse problem associated to $\mathsf{g}(G)$ as well.
- The algorithm is valid for any finite abelian group. With the hardware at our disposal it is possible to compute the Harborth constant for finite abelian groups of order up to about 45. The main limiting factor is memory. In order to increase the size of accessible groups,

we are currently working on a more efficient subset-representation based on data compression.

- The fact that e is equal to the exponent of the group is not relevant for the algorithm. It can be directly modified to compute related constants.

We end by mentioning two further computational results.

Proposition 4.1.

(1) $\mathsf{g}(C_6 \oplus C_6) = 13$.
(2) $\mathsf{g}(C_3 \oplus C_{12}) = 15$.

The former confirms the conjecture by Gao and Thangadurai, mentioned in the introduction, $\mathsf{g}(C_n \oplus C_n) = 2n + 1$ for even n in case $n = 6$. The latter shows that $\mathsf{g}(C_3 \oplus C_{3n}) = 3n+3$ also holds for $n = 4$, which supports the idea that $\mathsf{g}(C_3 \oplus C_{3n}) = 3n + 3$ might hold for n that are not prime as well.

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