JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Dan ABRAMOVICH et Anthony VÁRILLY-ALVARADO

Campana points, Vojta's conjecture, and level structures on semistable abelian varieties

Tome 30, nº 2 (2018), p. 525-532.

<http://jtnb.cedram.org/item?id=JTNB_2018__30_2_525_0>

© Société Arithmétique de Bordeaux, 2018, tous droits réservés.

L'accès aux articles de la revue « Journal de Théorie des Nombres de Bordeaux » (http://jtnb.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://jtnb.cedram.org/legal/). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Journal de Théorie des Nombres
de Bordeaux **30** (2018), 525–532

Campana points, Vojta's conjecture, and level structures on semistable abelian varieties

par Dan ABRAMOVICH et Anthony VÁRILLY-ALVARADO

Résumé. Nous introduisons une conjecture qualitative, dans l'esprit de Campana, qui prédit que certains sous-ensembles de points rationnels sur une variété sur un corps de nombres, ou un champ de Deligne–Mumford sur un anneau de S-entiers, ne peut pas être dense pour la topologie de Zariski. La conjecture interpole, d'une manière que nous précisérons, entre la conjecture de Lang pour les points rationnels sur les variétés de type général sur un corps de nombres, et la conjecture de Lang et Vojta qui affirme que les points S-entiers sur une variété de type général logarithmique ne sont pas denses pour la topologie de Zariski. Nous montrons que notre conjecture découle de la conjecture de Vojta. En supposant la conjecture, nous prouvons le théorème suivant: étant donné un corps de nombres K, un ensemble fini S de places de K contenant les places infinies, et un nombre entier positif g, il existe un entier m_0 tel que, pour tout $m > m_0$, il n'y a pas de variétés abélienes A/K de dimension g avec réduction semistable en dehors de S avec une structure pleine de niveau m.

Abstract. We introduce a qualitative conjecture, in the spirit of Campana, to the effect that certain subsets of rational points on a variety over a number field, or a Deligne–Mumford stack over a ring of S-integers, cannot be Zariski-dense. The conjecture interpolates, in a way that we make precise, between Lang's conjecture for rational points on varieties of general type over number fields, and the conjecture of Lang and Vojta that asserts that S-integral points on a variety of logarithmic general type are not Zariski-dense. We show our conjecture follows from Vojta's conjecture. Assuming our conjecture, we prove the following theorem: Fix a number field K, a finite set S of places of K containing the infinite places, and a positive integer g. Then there is an integer m_0 such that, for any $m > m_0$, no principally polarized abelian variety A/K of dimension g with semistable reduction outside of S has full level-m structure.

Manuscrit reçu le 20 octobre 2016, révisé le 23 novembre 2016, accepté le 2 décembre 2016.
2010 *Mathematics Subject Classification.* 11J97, 14K10, 14K15, 11G18.
Mots-clefs. Campana points, Vojta's conjecture, abelian varieties, level structures.
Research by D. A. partially supported by NSF grant DMS-1500525. Research by A. V.-A. partially supported by NSF CAREER grant DMS-1352291. This continues work of the authors that began during the workshop "Explicit methods for modularity of K3 surfaces and other higher weight motives", held at ICERM in October, 2015. We thank the organizers of the workshop and the staff at ICERM for creating the conditions that sparked this project. We thank David Zureick-Brown for comments and Keerthi Padapusi Pera for the appendix of [2].

1. Introduction

Fix a number field K with ring of integers $\mathcal{O}_K$, as well as a finite set S of places of K that contains the archimedean places. Denote by $\mathcal{O}_{K,S}$ the corresponding ring of S-integers. The purpose of this short note is to introduce a qualitative conjecture, in the spirit of Campana, to the effect that certain subsets of rational points on a variety over K or a Deligne–Mumford stack over $\mathcal{O}_{K,S}$ cannot be Zariski-dense; see Conjecture 1.2. This conjecture interpolates, in a way that we make precise, between Lang's conjecture for rational points on varieties over K of general type, and the conjecture of Lang and Vojta that asserts that $\mathcal{O}_{K,S}$-points on a variety of logarithmic general type are not Zariski-dense. One might thus expect our conjecture to follow from Vojta's quantitative conjecture on integral points; we show this is the case. As an application we show, assuming Conjecture 1.2, that for a fixed positive integer g, there is an integer m_0 such that, for any $m > m_0$, no principally polarized abelian variety A/K of dimension g with semistable reduction outside of S has full level-m structure.

Let $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K,S}$ be a smooth proper morphism from a scheme or Deligne–Mumford stack, and let $\mathscr{D}$ be a fiber-wise normal crossings divisor on $\mathscr{X}$. We say that $(\mathscr{X}, \mathscr{D})$ is *normal crossings model* of its generic fiber (X, D). Write $\mathscr{D} = \sum_i \mathscr{D}_i$ and let D_i be the generic fiber of $\mathscr{D}_i$. For each D_i appearing in D, choose $0 \le \epsilon_i \le 1$; write $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$, $D_{\vec{\epsilon}} = \sum_i \epsilon_i D_i$, and $\mathscr{D}_{\vec{\epsilon}} = \sum_i \epsilon_i \mathscr{D}_i$. Given a maximal ideal q of $\mathcal{O}_{K,S}$ with localization $\mathcal{O}_{K,q}$, and a point $x \in \mathscr{X}(\mathcal{O}_{K,q})$, we denote by $n_q(\mathscr{D}_i, x)$ the intersection multiplicity of x and $\mathscr{D}_i$, and for real numbers a_i, we define $n_q(\sum_i a_i \mathscr{D}_i, x)$ as $\sum_i a_i n_q(\mathscr{D}_i, x)$; see §2.3 for details.

Definition 1.1. A point $x \in \mathscr{X}(\mathcal{O}_{K,S})$ is called an *$\vec{\epsilon}$-Campana point* of $(\mathscr{X}, \mathscr{D})$ if for every maximal ideal $q \subset \mathcal{O}_{K,S}$ such that $n_q(\mathscr{D}, x) > 0$ we have $n_q(\mathscr{D}_{\vec{\epsilon}}, x) \ge 1$. We write $\mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ for the set of $\vec{\epsilon}$-Campana points of $(\mathscr{X}, \mathscr{D})$.

A $\mathbb{Q}$-divisor B on X is said to be *big* if it is the pullback of a big $\mathbb{Q}$-divisor on the coarse moduli space $\underline{X}$. With this terminology we state:

Conjecture 1.2 ($\vec{\epsilon}$-Campana Conjecture)**.** *If $K_X + \sum(1 - \epsilon_i)D_i$ is big, then $\mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ is not Zariski-dense in X.*

Remark 1.3. Campana stated the case of Conjecture 1.2 for curves in [4, Conj. 4.5 and Rem. 4.6]. Abramovich gave a higher dimensional statement of it in [1, Conj. 2.4.19]. Conjecture 1.2 is a streamlined version of this generalization.

Remark 1.4. Setting $\epsilon_i = 1$ for all i in Conjecture 1.2, the condition $n_q(\mathscr{D}_{\vec{\epsilon}}, x) \ge 1$ is automatically satisfied for all rational points and all q. Also $K_X + \sum(1 - \epsilon_i)D_i = K_X$, hence we recover Lang's conjecture for rational points on varieties of general type. At the other end of the spectrum, setting all the $\epsilon_i = 0$, the condition $n_q(\mathscr{D}_{\vec{\epsilon}}, x) \geq 1$ can only be satisfied if $n_q(\mathscr{D}, x) = 0$ at all $q \subset \mathcal{O}_{K,S}$, so x is S-integral on $\mathcal{X} \setminus \mathcal{D}$. Hence we get the Lang–Vojta conjecture: S-integral points on a variety of logarithmic general type are not Zariski-dense. We show in §3 that Conjecture 1.2 follows from Vojta's conjecture.

1.1. Application. We give an application of Conjecture 1.2, in the spirit of our recent work [3, 2]. Recall that, for a positive integer m, a *full level-m* structure on an abelian variety A/K of dimension g is an isomorphism of group schemes on the m-torsion subgroup

$$\phi\colon A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g. \tag{1.1}$$

Theorem 1.5. *Let K be a number field, S a finite set of places, and let g be a positive integer. Assume Conjecture 1.2. Then there is an integer m_0 such that, for any $m > m_0$, no principally polarized abelian variety A/K of dimension g with semistable reduction outside S has full level-m structure.*

Remark 1.6. In [2], we prove a version of Theorem 1.5 without a semistability assumption on the abelian variety A, at the cost of assuming Vojta's conjecture.

The idea behind Theorem 1.5 is the following. Let $(\widetilde{\mathcal{A}}_g)_K$ denote the moduli stack parametrizing principally polarized abelian varieties of dimension g over K. We show in Proposition 2.2 that for $\epsilon > 0$ and $\vec{\epsilon} = (\epsilon, \epsilon, \dots)$, if $X \subseteq (\widetilde{\mathcal{A}}_g)_K$ is closed, then the set $X(K,S)_{\geq m_0}$ of K-rational points of X corresponding to abelian varieties A/K admitting full level-m structure for some $m \geq m_0$ and having semistable reduction outside of S lies inside an $\vec{\epsilon}$-Campana set, for $m_0 \gg 0$. We then use a result on logarithmic hyperbolicity [3, Thm. 1.6] to verify the hypothesis of Conjecture 1.2 in this case, and a Noetherian induction argument to conclude the proof of Theorem 1.5.

2. Proof of Theorem 1.5

2.1. Moduli spaces and toroidal compactifications. We follow the notation of [2, §4], working over $\operatorname{Spec} \mathbb{Z}$:

- $\widetilde{\mathcal{A}}_g \subset \overline{\widetilde{\mathcal{A}}}_g$**:** a toroidal compactification of the moduli *stack* of principally polarized abelian varieties of dimension g;
- $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$**:** the resulting compactification of the moduli *space* of principally polarized abelian varieties of dimension g;
- $\widetilde{\mathcal{A}}_g^{[m]} \subset \overline{\widetilde{\mathcal{A}}}_g^{[m]}$**:** a compatible toroidal compactification of the moduli *stack* of principally polarized abelian varieties of dimension g with full level-m structure;

$\mathcal{A}_g^{[m]} \subset \overline{\mathcal{A}}_g^{[m]}$**:** the resulting compactification of the moduli space of principally polarized abelian varieties of dimension g with full level-m structure.

As noted in [2, §4], we may use a construction by Faltings and Chai [5, p. 128, IV Rem. 6.12, IV.6.9] of the stack $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$, which is a priori smooth over $\operatorname{Spec}\mathbb{Z}[1/m, \zeta_m]$, where ζ_m is a primitive m-th root of unity, to obtain a Deligne–Mumford stack we denote by $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m]}$, smooth over $\mathbb{Z}[1/m]$. This is extended as a Deligne–Mumford stack over all of $\operatorname{Spec}\mathbb{Z}$ by defining $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ as the normalization of $\overline{\widetilde{\mathcal{A}}}_g$ in the finite $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m]} \to (\overline{\widetilde{\mathcal{A}}}_g)_{\mathbb{Z}[1/m]}$. Unfortunately, even the interior of this stack over primes dividing m does not have a modular interpretation. See, however, the results of Madapusi Pera in [2, Appendix A]. Note that in the reference [5], Specifically Definition I.4.6 (ii), all algebraic stacks are Deligne–Mumford.

2.2. Semistability and integrality. We require the following statement, essentially contained in [5].

Proposition 2.1. *Let K be a number field, S a finite set of places. Let A/K be a principally polarized abelian variety with full level-m structure, and with semistable reduction outside of S. Then the point $x_m\colon \operatorname{Spec} K \to \widetilde{\mathcal{A}}_g^{[m]}$ associated to A extends to an integral point $\xi_m\colon \operatorname{Spec}\mathcal{O}_{K,S} \to \overline{\widetilde{\mathcal{A}}}_g^{[m]}$.*

Proof. First consider the case $m = 1$. By [5, Thm. IV.5.7 (5)] the extension exists if and only if, for every prime $q \notin S$ and for any strictly henselization V of $\mathcal{O}_{K,q}$ with valuation v, the bimultiplicative form $v \circ b$ corresponds to a point of a cone only depending on q. (Here b is the symmetric bimultiplicative form associated to the degeneration of A at q by the theory of degenerations, as indicated in [5, Prop. IV.5.1].) This condition is automatic for a Dedekind domain such as $\mathcal{O}_{K,q}$, see [5, Rem. IV.5.3], hence our proposition holds in case $m = 1$.

To prove the statement in general, consider the point $x\colon \operatorname{Spec} K \to \widetilde{\mathcal{A}}_g$ obtained by composing x_m with $\widetilde{\mathcal{A}}_g^{[m]} \to \widetilde{\mathcal{A}}_g$. Since the proposition holds for $m = 1$, the point x extends to $\xi\colon \operatorname{Spec}\mathcal{O}_{K,S} \to \overline{\widetilde{\mathcal{A}}}_g$. Since $\overline{\widetilde{\mathcal{A}}}_g^{[m]} \to \overline{\widetilde{\mathcal{A}}}_g$ is representable and finite, the stack $Z := \operatorname{Spec}\mathcal{O}_{K,S} \times_{\overline{\widetilde{\mathcal{A}}}_g} \overline{\widetilde{\mathcal{A}}}_g^{[m]}$, where the projection on the left is ξ, is in fact a scheme finite over $\operatorname{Spec}\mathcal{O}_{K,S}$. The point x_m defines a point $\operatorname{Spec} K \to Z$, which extends to a point $\operatorname{Spec}\mathcal{O}_{K,S} \to Z$ by the valuative criterion for properness. Composing with the projection $Z \to \overline{\widetilde{\mathcal{A}}}_g^{[m]}$ gives the desired point ξ_m. □

2.3. Intersection multiplicities. Fix a normal crossings model $(\mathscr{X}, \mathscr{D})$, and let $I_{\mathscr{D}_i}$ denote the ideal of $\mathscr{D}_i$. Given a maximal ideal q of $\mathcal{O}_{K,S}$ with localization $\mathcal{O}_{K,q}$, and a point $x \in \mathscr{X}(\mathcal{O}_{K,q})$, define $n_q(\mathscr{D}_i, x)$ through the equality of ideals in $\mathcal{O}_{K,q}$

$$I_{\mathscr{D}_i}\big|_x = q^{n_q(\mathscr{D}_i,x)}.$$

We call $n_q(\mathscr{D}_i, x)$ the intersection multiplicity of x and $\mathscr{D}_i$.

2.4. Notation for substacks. Let $X \subseteq (\widetilde{\mathcal{A}}_g)_K$ be a closed substack, let $X' \to X$ be a resolution of singularities, $X' \subset \overline{X}'$ a smooth compactification with $D = \overline{X}' \setminus X'$ a normal crossings divisor such that the rational map $f\colon \overline{X}' \to \overline{\widetilde{\mathcal{A}}}_g$ is a representable morphism. Let $X'_m = X' \times_{\widetilde{\mathcal{A}}_g} \widetilde{\mathcal{A}}_g^{[m]}$, and let $\overline{X}'_m \to \overline{X}' \times_{\overline{\widetilde{\mathcal{A}}}_g} \overline{\mathcal{A}}_g^{[m]}$ be a resolution of singularities with representable projections $\pi_m^X\colon \overline{X}'_m \to \overline{X}'$ and $f_m\colon \overline{X}'_m \to \overline{\widetilde{\mathcal{A}}}_g^{[m]}$.

We now spread these objects over $\mathcal{O}_{K,S}$ for a suitable finite set of places S containing the archimedean places. Let $(\mathscr{X}, \mathscr{D})$ be a normal crossings model of $(\overline{X}', D)$ over $\operatorname{Spec} \mathcal{O}_{K,S}$. As above, write $\mathscr{D} = \sum_i \mathscr{D}_i$. Such a model exists, even for Deligne–Mumford stacks, by [7, Prop. 2.2]. To avoid clutter, in the special case when $\vec{\epsilon} = (\epsilon, \epsilon, \dots)$, an $\vec{\epsilon}$-Campana point of $\mathscr{X}$ shall be called an ϵ-Campana point, and we write $\mathscr{X}(\mathcal{O}_{K,S})_{\epsilon\mathscr{D}}$ for the set of ϵ-Campana points of $(\mathscr{X}, \mathscr{D})$.

2.5. Levels and Campana points. Let $X(K,S)_{[m]}$ be the set of K-rational points of X corresponding to principally polarized abelian varieties A/K with semistable reduction outside S, admitting full level-m structure. Define

$$X(K,S)_{\geq m_0} := \bigcup_{m \geq m_0} X(K,S)_{[m]}. \tag{2.1}$$

Proposition 2.2. *Fix $\epsilon > 0$. Then there exists m_0 such that $X(K,S)_{\geq m_0}$ is contained in the set $\mathscr{X}(\mathcal{O}_{K,S})_{\epsilon\mathscr{D}}$ of ϵ-Campana points of $(\mathscr{X}, \mathscr{D})$.*

Proof. Let $x_m \in X'_m(K)$, and write $\pi_m(x_m) =: x$ for its image in $\overline{X}'(K)$. Let $q \notin S$ be a finite place of K, and let $\mathcal{O}_{K,q}$ be the corresponding local ring. Let $\xi\colon \operatorname{Spec} \mathcal{O}_{K,q} \to \overline{X}'$ and $\xi_m\colon \operatorname{Spec} \mathcal{O}_{K,q} \to \overline{X}'_m$ be the extensions of x and x_m to $\operatorname{Spec} \mathcal{O}_{K,q}$, which exist by Proposition 2.1.

Write E for the boundary divisors of $\left(\overline{\widetilde{\mathcal{A}}}_g\right)_K$; on $\overline{X}'$ we have an equality of divisors

$$f^*E = \sum a_i D_i,$$

where each $a_i > 0$; see [3, (4.3)]. It follows from [2, Prop. 4.3] that there exists an integer M depending only on g such that

$$n_q(\mathscr{D}, x) \geq \frac{m}{M \cdot \max\{a_i\}}.$$

We note that in our case we can take $M = 1$ because x and x_m could be extended to $\mathcal{O}_{K,q}$-points, as the proof of [2, Prop. 4.3] shows. Thus, if $m \geq \max\{a_i\}/\epsilon$, the point $x \in \overline{X}'(K)$ is an ϵ-Campana point. □

Proof of Theorem 1.5. We proceed by Noetherian induction. For each integer $i \geq 1$, let

$$W_i = \overline{\widetilde{\mathcal{A}}_g(K)_{\geq i}}.$$

Note that W_i is a closed subset of $\mathcal{A}_g$, and that $W_i \supseteq W_{i+1}$ for every i. The chain of W_i must stabilize by the Noetherian property of the Zariski topology of $\mathcal{A}_g$. Say $W_n = W_{n+1} = \cdots$.

We claim that W_n has dimension ≤ 0. Suppose not, and let $X \subseteq W_n$ be an irreducible component of positive dimension. Fix $\epsilon > 0$ so that $K_{\overline{X}'} + (1-\epsilon)D$ is big: such an ϵ exists by [3, Cor. 1.7]. Hence the hypothesis of Conjecture 1.2 holds (with all ϵ_i equal to ϵ.) By Conjecture 1.2, the set $\mathscr{X}(\mathcal{O}_{K,S})_{\epsilon\mathscr{D}}$ of ϵ-Campana points is not Zariski-dense in X. On the other hand, Proposition 2.2 shows there is an integer m_0 such that $X(K,S)_{\geq m_0} \subseteq \mathscr{X}(\mathcal{O}_{K,S})_{\epsilon\mathscr{D}}$, from which it follows that $X(K,S)_{\geq m_0}$ is not Zariski-dense in X. Thus W_{m_0}, which equals W_n, does not contain X, and thus X is not an irreducible component after all. This proves that $\dim W_n \leq 0$.

Finally, if W_n is a finite set of points, then we can apply the Mordell–Weil theorem to conclude that $W_n(K)_{[m]} = \emptyset$ for all $m \gg 0$. □

3. Vojta's conjecture and Campana points

3.1. Counting functions for integral points. Let $(\mathscr{X}, \mathscr{D})$ be a normal crossings model. For $q \in \mathcal{O}_{K,S}$, we denote by $\kappa(q)$ the residue field of the associated local ring. Following Vojta [8, p. 1106], for $x \in \mathscr{X}(\mathcal{O}_{K,S})$, define the *counting function*

$$N(D, x) = \sum_{q \in \operatorname{Spec} \mathcal{O}_{K,S}} n_q(\mathscr{D}, x) \log |\kappa(q)|, \tag{3.1}$$

as well as the *truncated counting function*

$$N^{(1)}(D, x) = \sum_{\substack{q \in \operatorname{Spec} \mathcal{O}_{K,S} \\ n_q(\mathscr{D}, x) > 0}} \log |\kappa(q)|. \tag{3.2}$$

The quantities on the right in (3.1) and (3.2) depend on the model $(\mathscr{X}, \mathscr{D})$ and the finite set S only up to functions bounded on $X(\mathcal{O}_{K,S})$. Since we are interested in these quantities only up to such functions, the notation

$N(D,x)$ and $N^{(1)}(D,x)$ does not reflect the model $(\mathscr{X},\mathscr{D})$ or the finite set S.

3.2. Vojta's conjecture for integral points. For a smooth proper Deligne–Mumford stack $\mathscr{X} \to \operatorname{Spec}\mathcal{O}_{K,S}$ over the ring of S-integers $\mathcal{O}_{K,S}$ of a number field K, we write $X = \mathscr{X}_K$ for the generic fiber, which we assume is irreducible, and $\underline{X}$ for the coarse moduli space of X. Similarly, for a normal crossings divisor $\mathscr{D}$ of $\mathscr{X}$, we write D for its generic fiber.

For a divisor H on $\underline{X}$, we denote by $h_H(x)$ the Weil height of x with respect to H, which is well-defined up to a bounded function on $\underline{X}(\overline{K})$. If H is only a divisor on X, then some positive integer multiple rH descends to $\underline{X}$. Given a point $x \in X(\overline{K})$ we define $h_H(x) = \frac{1}{r}h_{rH}(\underline{x})$, where $\underline{x}$ is the image of x in $\underline{X}(\overline{K})$.

The following is a version of Vojta's conjecture for stacks, applied to integral points:

Conjecture 3.1. *Let $\mathscr{X} \to \operatorname{Spec}\mathcal{O}_{K,S}$, X, $\underline{X}$, and D be as above. Suppose that $\underline{X}$ is projective, and let H be a big line bundle on it. Fix $\delta > 0$. Then there is a proper Zariski-closed subset $Z \subset X$ containing D such that*

$$N^{(1)}(D,x) \geq h_{K_X+D}(x) - \delta h_H(x) - O(1)$$

for all $x \in \mathscr{X}(\mathcal{O}_{K,S}) \setminus Z(K)$.

In [2] we showed that a stronger conjecture, which applies to points $x \in X(\overline{K})$ with $[K(x) : K]$ bounded, follows from Vojta's original conjecture for schemes [8]. Here we have stated only its outcome for integral points.

3.3. From Vojta's conjecture to Conjecture 1.2.

Lemma 3.2. *If $x \in \mathscr{X}(\mathcal{O}_{K,S})$ is an $\vec{\epsilon}$-Campana point, then*

$$N^{(1)}(D,x) \leq N(D_{\vec{\epsilon}},x) \leq h_{D_{\vec{\epsilon}}}(x) + O(1).$$

Proof. If $x \in \mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ is an $\vec{\epsilon}$-Campana point, and $n_q(\mathscr{D},x) > 0$, then we have

$$\sum_i \epsilon_i \cdot n_q(\mathscr{D}_i,x) = n_q(\mathscr{D}_{\vec{\epsilon}},x) \geq 1.$$

The definition (3.2) of $N^{(1)}(D,x)$ gives

$$\begin{aligned} N^{(1)}(D,x) = \sum_{n_q(\mathscr{D},x)>0} \log|\kappa(q)| \leq \sum_{n_q(\mathscr{D},x)>0} n_q(\mathscr{D}_{\vec{\epsilon}},x)\log|\kappa(q)| \\ = N(D_{\vec{\epsilon}},x) \leq h_{D_{\vec{\epsilon}}}(x) + O(1), \end{aligned}$$

where the last inequality follows as in [8, p. 1113] or [6, Thm. B.8.1 (e)]. □

Corollary 3.3. *Vojta's conjecture 3.1 implies the $\vec{\epsilon}$-Campana conjecture 1.2.*

Proof. Since the divisor $K_X + \sum_i (1-\epsilon_i)D_i$ is big, we may choose an ample $\mathbb{Q}$-divisor H and $0 < \delta < 1$ such that $B := K_X + \sum_i (1-\epsilon_i)D_i - \delta H$ is big. Let $x \in \mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ be an $\vec{\epsilon}$-Campana point. Applying Conjecture 3.1, we obtain a proper Zariski closed subset $\mathscr{Z} \subset \mathscr{X}$ such that if $x \notin \mathscr{Z}$, the inequality

$$N^{(1)}(D,x) \geq h_{K_X+D}(x) - \delta h_H(x) - O(1)$$

holds. By Lemma 3.2, we may replace the left hand side with $h_{D_{\vec{\epsilon}}}(x) + O(1)$ to get

$$h_{D_{\vec{\epsilon}}}(x) + O(1) \geq h_{K_X+D}(x) - \delta h_H(x) - O(1).$$

This implies that

$$O(1) \geq h_{K_X + \left(\sum_i (1-\epsilon_i)D_i\right)}(x) - \delta h_H(x) = h_B(x).$$

Since B is big, the set of $x \in \mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ that avoid both its stable base locus $\mathscr{Z}'$ and $\mathscr{Z}$ is finite [6, Thm. B.3.2 (e,g)]. Since $\mathscr{Z} \cup \mathscr{Z}'$ is a proper Zariski-closed set, we conclude that the set $\mathscr{X}(\mathcal{O}_{K,S})_{\mathscr{D}_{\vec{\epsilon}}}$ of $\vec{\epsilon}$-Campana points is not Zariski-dense in X. □

References

[1] D. Abramovich, "Birational geometry for number theorists", in *Arithmetic geometry*, Clay Mathematics Proceedings, vol. 8, American Mathematical Society, 2009, p. 335-373.

[2] D. Abramovich & A. Várilly-Alvarado, "Level structures on abelian varieties and Vojta's conjecture", *Compos. Math.* **153** (2017), no. 2, p. 373-394.

[3] ———, "Level structures on abelian varieties, Kodaira dimensions, and Lang's conjecture", *Adv. Math.* **329** (2018), p. 523-540.

[4] F. Campana, "Fibres multiples sur les surfaces: aspects geométriques, hyperboliques et arithmétiques", *Manuscr. Math.* **117** (2005), no. 4, p. 429-461.

[5] G. Faltings & C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 22, Springer, 1990, xii+316 pages.

[6] M. Hindry & J. H. Silverman, *Diophantine geometry: An introduction*, Graduate Texts in Mathematics, vol. 201, Springer, 2000, xiii+558 pages.

[7] M. C. Olsson, "Hom-stacks and restriction of scalars", *Duke Math. J.* **134** (2006), no. 1, p. 139-164.

[8] P. Vojta, "A more general *abc* conjecture", *Int. Math. Res. Not.* **1998** (1998), no. 21, p. 1103-1116.

Dan Abramovich
Department of Mathematics, Box 1917
Brown University
Providence, RI, 02912, USA
E-mail: abrmovic@math.brown.edu
URL: http://www.math.brown.edu/~abrmovic/

Anthony Várilly-Alvarado
Department of Mathematics MS 136
Rice University
6100 S. Main St.
Houston, TX 77005, USA
E-mail: av15@rice.edu
URL: http://math.rice.edu/~av15/