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Products of primes in arithmetic progressions: a footnote in parity breaking

par OLIVIER RAMARÉ et ALED WALKER

RÉSUMÉ. Nous montrons que, étant donnés x et $q \leq x^{1/16}$, toute classe inversible a modulo q contient au moins un produit d'exactly trois nombres premiers, chacun étant inférieur ou égal à $x^{1/3}$.

ABSTRACT. We prove that, if x and $q \leq x^{1/16}$ are two parameters, then for any invertible residue class a modulo q there exists a product of exactly three primes, each one below $x^{1/3}$, that is congruent to a modulo q .

1. Introduction and results

Xylouris' version of Linnik's Theorem [15] tells us that, for every modulus q and every invertible residue class a modulo q , one can find a prime congruent to a modulo q that is below $q^{5.18}$ *provided* q be large enough. The proof relies on intricate techniques, and though the result is indeed effective, no one has been able to give any explicit version of it. The aim of this paper is to show that one can easily access a fully explicit result, with respectable constants, provided one replaces primes by products of three primes. Here is what we prove, by combining a simple sieve technique together with classical additive combinatorics.

Theorem 1.1. *Let x and $q \leq x^{1/16}$ be two parameters. Then for any invertible residue class a modulo q , there exists a product of three primes, all below $x^{1/3}$, that is congruent to a modulo q .*

We did not try to be optimal in our treatment but sought the simplest argument. The main surprise is that we use sieve techniques in the form of Brun–Titchmarsh inequality but we are not blocked by the parity principle. The reader may argue that we use a lower bound for $L(1, \chi)$, but the bound we employ is the weakest possible and does not rely on Siegel's Theorem.

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In particular, it is not strong enough to push a possible Siegel zero away from 1, a fact known to be equivalent to the parity phenomenon (see [8] and [11], or [12, Chapter 6] for a more complete discussion).

Our theorem is also linked with a conjecture of Erdős that says that every invertible congruence class should contain a product of two primes not more than q . This is discussed in work of the second author [14].

A numerically improved version is being prepared.

2. Lemmas

We begin with some crude bounds. Let us define

$$(2.1) \quad f_0(q) = \prod_{p|q} (1 - 1/\sqrt{p})^{-1}.$$

Lemma 2.1. *For $q \geq 2$ we have $f_0(q) \leq 3.32\sqrt{q}$*

Proof. For all primes p we have $(1 - 1/\sqrt{p})^{-1} \leq \alpha_p \sqrt{p}$, where

$$\alpha_p = \begin{cases} \frac{1}{\sqrt{2}-1} & p = 2 \\ \frac{1}{\sqrt{3}-1} & p = 3 \\ 1 & \text{otherwise,} \end{cases}$$

and since $\alpha_2 \leq 2.42$ and $\alpha_3 \leq 1.37$, we obtain the inequalities

$$f_0(q) \leq 2.42 \cdot 1.37 \cdot \sqrt{q} \leq 3.32 \cdot \sqrt{q}. \quad \square$$

We will also require a rudimentary estimate on $\phi(q)$.

Lemma 2.2. *If $q \geq 31$ then $\phi(q) > 8$.*

Proof. Recall that

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Therefore if $\phi(q) \leq 8$, the only prime factors of q are 2, 3, 5, 7. By performing an easy case analysis on which of these primes divides q , one sees that the only q for which $\phi(q) \leq 8$ are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 24, and 30. \square

We will use the elementary theory of Dirichlet characters, referring the reader to the excellent monograph [2] of Davenport for an introduction on the subject. In particular we note the following easy bound.

Lemma 2.3. *Let χ be a non-principal Dirichlet character modulo q . Let I be a subset of $\{1, \dots, q\}$. We have*

$$\left| \sum_{n \in I} \chi(n) \right| \leq \phi(q)/2.$$

The same bound holds true for any finite interval instead of I .

Proof. We know on the one hand that, we have $\sum_{1 \leq n \leq q} \chi(n) = 0$ by orthogonality, and on the other hand that $\chi(n)$ does not vanish only when n belongs to the multiplicative group, say \mathcal{U}_q , of $\mathbb{Z}/q\mathbb{Z}$. We can hence bound $|\sum_{n \in I} \chi(n)|$ by the cardinal of $I \cap \mathcal{U}_q$ and by the cardinal of $\mathcal{U}_q \setminus I \cap \mathcal{U}_q$. One of them is not more than $\phi(q)/2$, proving the first part of the lemma. When I is a finite interval, we note that the sum of the values of $\chi(n)$ on any q consecutive integers vanishes, reducing the problem to the first case. \square

We next modify an idea of Gel'fond from [4], which is maybe more easily read in [5].

Lemma 2.4. *Let χ be a non-principal quadratic character modulo q . We have*

$$L(1, \chi) \geq \frac{\pi}{4\phi(q)} - \frac{\pi}{\phi(q)^2}.$$

Proof. We consider the sum $S(\alpha) = \sum_{n \geq 1} (1 \star \chi)(n)e^{-n\alpha}$ for real positive α . Since $(1 \star \chi)(m^2) \geq 1$ for every integer m , and $(1 \star \chi)(n) \geq 0$ in general, a comparison with an integral gives us

$$1 + S(\alpha) \geq \sum_{m \geq 0} e^{-m^2\alpha} \geq \int_0^\infty e^{-\alpha t^2} dt = \frac{\Gamma(1/2)}{2\sqrt{\alpha}} = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}.$$

On the other hand we can expand $(1 \star \chi)(n) = \sum_{d|n} \chi(d)$ and get

$$S(\alpha) = \sum_{d \geq 1} \frac{\chi(d)}{e^{\alpha d} - 1} = \frac{L(1, \chi)}{\alpha} - \sum_{d \geq 1} \chi(d)g(\alpha d)$$

by using the non-negative non-increasing function $g(x) = \frac{1}{x} - \frac{1}{e^x - 1}$. We find that, by Lemma 2.3,

$$\begin{aligned} \sum_{d \geq 1} \chi(d)g(\alpha d) &= - \sum_{d \geq 1} \chi(d) \int_{\alpha d}^\infty g'(t) dt \\ &= - \int_0^\infty \sum_{d \leq t/\alpha} \chi(d)g'(t) dt \\ &\geq \frac{\phi(q)}{2} \int_0^\infty g'(t) dt \\ &= -\phi(q)/4 \end{aligned}$$

since $\lim_{x \rightarrow 0^+} g(x) = 1/2$ as x tends to 0 from above. By comparing both upper and lower estimate for $S(\alpha)$, we reach

$$L(1, \chi) \geq \frac{\sqrt{\pi\alpha}}{2} - \alpha - \frac{\alpha\phi(q)}{4}.$$

We select $\alpha = \pi/\phi(q)^2$. The lemma follows. \square

Lemma 2.5. *Let $q \geq 3$ be an integer and χ be a non-principal quadratic character modulo q . Then there is a prime p at most q^4 such that $\chi(p) = 1$.*

Proof. We adapt the proof of J. Pintz taken from [10]. Assume that no primes not more than a given real number x are in the kernel of χ . We use the notation $d|q^\infty$ to say that all the prime factors of d divides q . Then on the one side we have

$$\sum_{n \leq x} (1 \star \chi)(n) = \sum_{\substack{d|q^\infty \\ (m,q)=1}} \sum_{m^2 \leq x/d} 1 \leq \sum_{d|q^\infty} \sqrt{\frac{x}{d}} \leq \sqrt{x} f_0(q)$$

where f_0 is the function defined in (2.1), while on the other side we can approximate this sum by $L(1, \chi)$ as follows:

$$\sum_{n \leq x} (1 \star \chi)(n) = \sum_{d \leq x} \chi(d) \left[\frac{x}{d} \right] = x \sum_{d \leq x} \frac{\chi(d)}{d} - \sum_{d \leq x} \chi(d) \left\{ \frac{x}{d} \right\}.$$

The first summation over d is an approximation of $L(1, \chi)$ (recall Lemma 2.3):

$$\begin{aligned} L(1, \chi) &= \sum_{d \geq 1} \frac{\chi(d)}{d} = \sum_{d \leq x} \frac{\chi(d)}{d} + \int_x^\infty \sum_{x < d \leq t} \chi(d) dt/t^2 \\ &= \sum_{d \leq x} \frac{\chi(d)}{d} + \mathcal{O}^*\left(\frac{\phi(q)}{2x}\right). \end{aligned}$$

We treat the second summation in d above by Axer’s method from [1] (see also [7, Theorem 8.1]):

$$\left| \sum_{d \leq x} \chi(d) \left\{ \frac{x}{d} \right\} \right| \leq \sum_{d \leq y} 1 + \sum_{\substack{m \leq x/y \\ d: [x/d]=m}} \left| \sum \chi(d) \left\{ \frac{x}{d} \right\} \right| \leq y + \frac{\phi(q)x}{2y} \leq \sqrt{2\phi(q)x}$$

by selecting $y = \sqrt{\phi(q)x/2}$, the second inequality following by Abel summation. All of this implies that $\sqrt{x}L(1, \chi) \leq f_0(q) + \sqrt{2\phi(q)} + \phi(q)/(2\sqrt{x})$. However, the previous lemma gives us a lower bound for $L(1, \chi)$ and thus we should have

$$\frac{\pi}{4\phi(q)} - \frac{\pi}{\phi(q)^2} \leq \frac{f_0(q)}{\sqrt{x}} + \sqrt{\frac{2\phi(q)}{x}} + \frac{\phi(q)}{2x}.$$

We substitute $x = q^4$, using the upper bound for $f_0(q)$ provided by Lemma 2.1. Replacing the left hand side of the above inequality by $\pi/8\phi(q)$, which is permissible by Lemma 2.2, together with the bound $\phi(q) \leq q$, after a short calculation we derive a contradiction for all $q \geq 45$. Calculating using the exact expressions for $f_0(q)$ and $\phi(q)$ when $q \in \{15, \dots, 45\}$, we also derive a contraction. For the remaining q it is easy enough to find

primes $p \leq q^4$ such that $p \equiv 1$ modulo q . Indeed, for $q = 2, \dots, 14$ we may take $p = 3, 7, 5, 11, 7, 29, 17, 19, 11, 23, 37, 53, 29$ respectively. \square

We quote the following result from [6], which is a strong form of the Brun–Titchmarsh inequality.

Lemma 2.6. *When $1 \leq q < x$, we have*

$$\sum_{\substack{y < p \leq y+x, \\ p \equiv a[q]}} 1 \leq \frac{2x}{\phi(q) \log(x/q)}.$$

for any positive y .

Lemma 2.7. *We have $\pi(x) \geq x/(\log x - 1)$ when $x \geq 5\,393$. Furthermore the number of primes not more than x but prime to some fixed modulus q below x is at least $x/\log x$, again when $x \geq 5\,393$.*

Proof. The first inequality is taken from [3]. For the second, we simply note that the number of prime factors of q is at most $(\log x)/\log 2$ and that

$$\frac{x}{\log x - 1} - \frac{\log x}{\log 2} \geq \frac{x}{\log x}$$

when $x \geq 5\,000$. \square

The final ingredient in the argument will be Kneser’s Theorem, which we now recall (see [9, Theorem 4.3] or [13, Theorem 5.5]).

Lemma 2.8. *Let A and B be two subsets of the finite abelian group G . Let H be the subgroup of elements h of G that stabilizes $A + B$, i.e. that are such that $h + A + B = A + B$. We have*

$$|A + B| \geq |A + H| + |B + H| - |H|.$$

3. Proof of Theorem 1.1

Let us first treat the case $x \geq 10^{16}$.

Let $X = x^{1/3}$. Since this parameter is at least 10^5 , Lemma 2.7 tells us that the number $\pi_q(X)$ of primes below X which are coprime to q is at least $X/\log X$. The Brun–Titchmarsh inequality in the form given by Montgomery & Vaughan, recalled in Lemma 2.6, tells us that the number of primes less than X in any progression $a \pmod q$, for a prime to q , is at most $\frac{32}{13}X/(\phi(q) \log X)$. This implies, when compared to the total number of primes coprime to q given by Lemma 2.7, that at least $\frac{13}{32}\phi(q)$ such residue classes contain a prime. Let us call this set of classes \mathcal{A} and apply Kneser’s Theorem (Lemma 2.8) to the group G of invertible residues modulo q . Let H be the stabilizer of $\mathcal{A} \cdot \mathcal{A}$. We divide into cases according to the index of H .

If H is equal to G then, since $\mathcal{A} \cdot \mathcal{A} \cdot H = \mathcal{A} \cdot \mathcal{A}$, we have $\mathcal{A} \cdot \mathcal{A} = G$ and of course $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G$.

If H has index 2, then it is the kernel of some quadratic character χ . Because \mathcal{A} generates G multiplicatively, there is a point a in \mathcal{A} such that $\chi(a) = -1$. By Lemma 2.5, there is another one, say a' , such that $\chi(a') = 1$. Hence $\mathcal{A} \cdot \mathcal{A}$ also has a point b such that $\chi(b) = 1$ and one, say b' , such that $\chi(b') = -1$. This implies that $\mathcal{A} \cdot \mathcal{A} \cdot H = G$, i.e. $\mathcal{A} \cdot \mathcal{A} = G$.

When H is of index 3, then $\mathcal{A} \cdot H$ covers at least 2 H -cosets (since $\frac{13}{32} > \frac{1}{3}$) and is thus of cardinality at least $2\phi(q)/3$. Kneser's Theorem ensures that $|\mathcal{A} \cdot \mathcal{A}| \geq \phi(q)$, i.e. that again $\mathcal{A} \cdot \mathcal{A} = G$.

When H is of index 4, then $\mathcal{A} \cdot H$ covers at least 2 H -cosets (since $\frac{13}{32} > \frac{1}{4}$) and is thus of cardinality at least $\phi(q)/2$. By Kneser's Theorem,

$$|\mathcal{A} \cdot \mathcal{A}| \geq 2|\mathcal{A} \cdot H| - |H| \geq \frac{3}{4}\phi(q).$$

When H is of index Y say, with Y at least 5, let us write $|\mathcal{A}|/\phi(q) = 1/U$. The set $\mathcal{A} \cdot H$ is made out of at least $\lceil Y/U \rceil$ cosets modulo H . Using the same manipulation as above, Kneser's Theorem ensures that $|\mathcal{A} \cdot \mathcal{A}|/\phi(q) \geq (2\lceil Y/U \rceil - 1)/Y$. A quick computation shows that the minimum of $(2\lceil Y/U \rceil - 1)/Y$ when Y ranges $\{5, 6, 7, 8, 9\}$ is reached at $Y = 7$ and has value $5/7$. When Y is larger than 10, we directly check that $(2\lceil Y/U \rceil - 1)/Y \geq \frac{2}{U} - \frac{1}{Y} \geq \frac{13}{16} - \frac{1}{10} \geq \frac{7}{10}$.

Combining these final two cases, we have proved that $|\mathcal{A} \cdot \mathcal{A}| \geq \frac{7}{10}\phi(q)$. Let b be an arbitrary invertible residue class modulo q . The set b/\mathcal{A} is of cardinality at least $|\mathcal{A}|$ and, since $\frac{13}{32}$ is greater than $\frac{3}{10}$, this is strictly larger than the size of the complementary set of $\mathcal{A} \cdot \mathcal{A}$. Therefore these sets have a point in common: there exist a, a_1 and a_2 , all three in \mathcal{A} such that $b/a = a_1 a_2$, proving our theorem in this case.

It remains to deal with $x < 10^{16}$, which is done by explicit calculation. The inclusion of this addendum was kindly suggested to us by an anonymous referee. Indeed, when $x < 10^{16}$, the modulus q is restricted to be not more than 10, implying that only a limited number of congruence classes are to be looked at. We proceed by hand:

- when $q = 2$, we only need $x \geq 3^3$;
- when $q = 3$, we only need $x \geq 7^3$;
- when $q = 4$, we only need $x \geq 5^3$;
- when $q = 5$, we only need $x \geq 19^3$;
- when $q = 6$, we only need $x \geq 11^3$;
- when $q = 7$, we only need $x \geq 29^3$;
- when $q = 8$, we only need $x \geq 23^3$;
- when $q = 9$, we only need $x \geq 23^3$;
- when $q = 10$, we only need $x \geq 19^3$.

This takes care of the situation when $x \geq 29^3$. However, when x is below 29^3 , the bound $x^{1/16}$ is less than 2. This ends the proof of our theorem.

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