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GLOBAL EXPONENTIAL STABILISATION FOR THE BURGERS EQUATION WITH LOCALISED CONTROL

by Armen Shirikyan

Abstract. — We consider the 1D viscous Burgers equation with a control localised in a finite interval. It is proved that, for any $\varepsilon > 0$, one can find a time T of order $\log \varepsilon^{-1}$ such that any initial state can be steered to the ε-neighbourhood of a given trajectory at time T. This property combined with an earlier result on local exact controllability shows that the Burgers equation is globally exactly controllable to trajectories in a finite time that does not depend on the initial conditions.

Résumé (Stabilisation exponentielle globale pour l'équation de Burgers avec contrôle localisé)

Nous considérons l'équation de Burgers visqueuse 1D avec un contrôle localisé dans un intervalle fini. Nous montrons que, pour tout $\varepsilon > 0$, on peut trouver un temps T d'ordre $\log \varepsilon^{-1}$ tel que tout état initial peut être amené dans un ε-voisinage d'une trajectoire donnée au temps T. Cette propriété, jointe à un résultat précédent de contrôle local exact, montre que l'équation de Burgers est globalement exactement contrôlable vers les trajectoires en un temps fini qui ne dépend pas des conditions initiales.

Contents

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0. Introduction

Let us consider the controlled Burgers equation on the interval $I=(0,1)$ with the Dirichlet boundary condition:

$$\partial_t u-\nu\partial_x^2 u+u\partial_x u=h(t,x)+\zeta(t,x), \tag{0.1}$$

$$u(t,0)=u(t,1)=0. \tag{0.2}$$

Here $u=u(t,x)$ is an unknown function, $\nu>0$ is a parameter, h is a fixed function, and ζ is a control that is assumed to be localised in an interval $[a,b]\subset I$. As is known, the initial-boundary value problem for (0.1) is well posed; see Proposition 2.1. Namely, if $h\in L^2_{\mathrm{loc}}(\mathbb{R}_+,L^2(I))$ and $\zeta\equiv 0$, then, for any $u_0\in L^2(I)$, Problem (0.1), (0.2) has a unique solution $u(t,x)$ that belongs to the space

$$\mathscr{X}=\{u\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^1_0(I)):\partial_t u\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1}(I))\}$$

and satisfies the initial condition

$$u(0,x)=u_0(x); \tag{0.3}$$

see the end of this Introduction for the definition of functional spaces. Let us denote by $\mathscr{R}_t(u_0,h)$ the mapping that takes the pair (u_0,h) to the solution $u(t)$ (with $\zeta\equiv 0$). We wish to study the problem of controllability for (0.1). This question received great deal of attention in the last twenty years, and we now recall some achievements related to our paper.

One of the first results was obtained by Fursikov and Imanuvilov [FI95, FI96]. They established the following two properties:

Local exact controllability. — *Let $\widehat{u}_0\in H^1_0(I)$ and $h\in L^2_{\mathrm{loc}}(\mathbb{R}_+,L^2(I))$ be some functions, let $\widehat{u}=\mathscr{R}_t(\widehat{u}_0,h)$ be the corresponding trajectory of Problem* (0.1), (0.2) *with $\zeta\equiv 0$, and let $T>0$. Then there is $\varepsilon>0$ such that, for any $u_0\in H^1_0(I)$ satisfying the inequality $\|u_0-\widehat{u}_0\|_{H^1}\leqslant\varepsilon$, one can find a control*(1) *$\zeta\in L^2(J_T\times I)$ supported in $J_T\times[a,b]$ for which $\mathscr{R}_T(u_0,h+\zeta)=\widehat{u}(T)$. Moreover, when T is fixed, the number ε can*(2) *be chosen to be the same for all $\widehat{u}_0$ and h varying in bounded subsets of the spaces $H^1_0(I)$ and $L^2(J_T\times I)$, respectively.*

Absence of approximate controllability. — *For any $u_0\in L^2(I)$ and any positive numbers T and R, one can find $\widehat{u}\in L^2(I)$ such that, for any control $\zeta\in L^2(J_T\times I)$ supported by $J_T\times[a,b]$, we have*

$$\|\mathscr{R}_T(u_0,h+\zeta)-\widehat{u}\|\geqslant R. \tag{0.4}$$

These results were extended and developed in many works. In particular, Diaz [Dia96] established some a priori bounds for solutions of Equation (0.1) and used them to prove the absence of approximate controllability in various functional classes.

(1) We denote by J_T the time interval $[0,T]$.

(2) This property is not explicitly mentioned in [FI95]. However, it is implied by the proof, and we discuss it in Section 1 when proving our result on exact controllability to trajectories.

His results show that the global approximate controllability does not hold, even if we allow infinite time of control. Glass and Guererro [GG07] and Léautaud [Léa12] proved global exact boundary controllability to constant states, Coron [Cor07b] and Fernández-Cara–Guererro [FCG07] established some estimates for the time and cost of control, and Chapouly [Cha09] (see also Marbach [Mar14]) proved global exact controllability to trajectories with two boundary and one distributed scalar controls, provided that $h \equiv 0$. Horsin [Hor08] proved the local exact controllability in the Lagrangian setting. Some non-controllability results for Equation (0.1) with $h \equiv \zeta \equiv 0$ and boundary controls are established by Guererro–Imanuvilov [GI07], who used the Cole–Hopf transform and some qualitative properties of solutions for the heat equation. The problem of stabilisation of the viscous Burgers equations was also studied in a number of papers. In particular, Thevenet–Buchot–Raymond [TBR10] constructed a nonlinear feedback law stabilising the 2D problem and Kröner–Rodrigues [KR15] studied the stabilisation to a non-stationary solution. A large number of works were devoted to the investigation of similar questions for other, more sophisticated equations of fluid mechanics; see the books [Fur00, Cor07a] and the review paper [Cor10], as well as the references therein. We do not discuss them here, because their methods are not likely to apply to the class of problems we deal with.

In view of the above-mentioned controllability properties for the viscous Burgers equation, a natural questions arises: does the *exact controllability to trajectories* hold for *arbitrary initial conditions and nonzero right-hand sides*? It turns out that the answer to this question is positive, provided that the time of control is sufficiently large. Namely, the main result of this paper combined with the above-mentioned property of local exact controllability to trajectories imply the following theorem.(3)

Main Theorem. — *Let $\nu > 0$ and $[a,b] \subset I$ be fixed. Then, for any $K > 0$, there is $T > 0$ such that the following property holds: given a function $h \in (H^1_{\mathrm{ul}} \cap L^\infty)(\mathbb{R}_+ \times I)$ whose norm does not exceed K and arbitrary initial conditions $u_0, \widehat{u}_0 \in L^2(I)$ one can find a control $\zeta \in L^2(J_T \times I)$ supported by $J_T \times [a,b]$ such that*

$$\mathscr{R}_T(u_0, h + \zeta) = \mathscr{R}_T(\widehat{u}_0, h). \tag{0.5}$$

We emphasise that the time of control T does not depend on the initial conditions, so that we have global exact controllability to trajectories at a fixed time, provided that ν, $[a,b]$, and h are fixed. To the best of my knowledge, the Main Theorem stated above provides a first global controllability result for Burgers-type equations with no further conditions on the data. It remains in fact valid for a much larger class of damped–driven scalar conservation laws in higher dimension, and this question will be addressed in a subsequent publication.

The rest of the paper is organised as follows. In Section 1, we formulate a result on exponential stabilisation to trajectories, outline the scheme of its proof, and derive the Main Theorem. Section 2 is devoted to some preliminaries about the Burgers

(3)See the Notation below for definition of the spaces used in the statement.

equation. In Section 3, we present the details of the proof of exponential stabilisation. Finally, the appendix gathers the proofs of some auxiliary results.

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Notation. — Let $I=(0,1)$, $J_T=[0,T]$, $\mathbb{R}_+=[0,+\infty)$, and $D_T=(T,T+1)\times I$. We use the following function spaces.

– $L^p(D)$ and $H^s(D)$ are the usual Lebesgue and Sobolev spaces, endowed with natural norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. In the case $p=2$ (or $s=0$), we write $\|\cdot\|$ and denote by $(\cdot,\cdot)$ the corresponding scalar product.

– $C^\gamma(D)$ denotes the space of Hölder-continuous functions with exponent $\gamma\in(0,1)$.

– $H^s_{\mathrm{loc}}(D)$ is the space of functions $f:D\to\mathbb{R}$ whose restriction to any bounded open subset $D'\subset D$ belongs to $H^s(D')$.

– $H_0^s=H_0^s(I)$ is the closure in $H^s(I)$ of the space of infinitely smooth functions with compact support, and we write $V=H_0^1(I)\cap H^2(I)$.

– $H^s_{\mathrm{ul}}(\mathbb{R}_+\times I)$ stands for the space of functions $u\in H^s_{\mathrm{loc}}(\mathbb{R}_+\times I)$ satisfying the condition

$$\|u\|_{H^s_{\mathrm{ul}}}:=\sup_{T\geqslant 0}\|u\|_{H^s(D_T)}<\infty.$$

Very often, the context implies the domain on which a functional space is defined, and in this case we omit it from the notation. For instance, we write L^2, H^s, etc.

– $L^p(J,X)$ is the space of Borel-measurable functions $f:J\to X$ (where $J\subset\mathbb{R}$ is a closed interval and X is a separable Banach space) such that

$$\|f\|_{L^p(J,X)}=\biggl(\int_J\|f(t)\|_X^p\,\mathrm{d}t\biggr)^{1/p}<\infty.$$

In the case $p=\infty$, this condition should be replaced by

$$\|f\|_{L^\infty(J,X)}=\operatorname{ess\,sup}_{t\in J}\|f(t)\|_X<\infty.$$

– $H^k(J,X)$ stands for the space of functions $f\in L^2(J,X)$ such that $\partial_t^j f\in L^2(J,X)$ for $1\leqslant j\leqslant k$, and if J is unbounded, then $H^k_{\mathrm{loc}}(J,X)$ is the space of functions whose restriction to any bounded interval $J'\subset J$ belongs to $H^k(J',X)$.

– $C(J,X)$ is the space of continuous functions $f:J\to X$.

– $B_X(a,R)$ denotes the closed ball in X of radius $R\geqslant 0$ centred at $a\in X$. In the case $a=0$, we write $B_X(R)$.

1. Exponential stabilisation to trajectories

1.1. Main result on stabilisation.

— Let us consider Problem (0.1), (0.2), in which $\nu>0$ is a fixed parameter, $h(t,x)$ is a given function belonging to $H^1_{\mathrm{ul}}\cap L^\infty$ on the domain $I\times\mathbb{R}_+$, and ζ is a control taking values in the space of functions in $L^2(I)$ with support in a given interval $[a,b]\subset I$. Recall that $\mathscr{R}_t(u_0,h+\zeta)$ stands for the value of the solution for (0.1)–(0.3) at time t. The following theorem is the main result of this paper.

Theorem 1.1. — *For any $K>0$ there exist positive numbers C and γ such that, given $h\in H^1_{\mathrm{ul}}\cap L^\infty$ with $\|h\|_{H^1_{\mathrm{ul}}\cap L^\infty}\leqslant K$ and arbitrary initial data $u_0,\widehat{u}_0\in L^2(I)$, one can find a piecewise continuous control $\zeta:\mathbb{R}_+\to H^1(I)$ supported in $\mathbb{R}_+\times[a,b]$ for which*

$$\|\mathscr{R}_t(u_0,h+\zeta)-\mathscr{R}_t(\widehat{u}_0,h)\|_{H^1}+\|\zeta(t)\|_{H^1}\leqslant Ce^{-\gamma t}\min\bigl(\|u_0-\widehat{u}_0\|_{L^1}^{2/5},1\bigr),\quad t\geqslant 1. \tag{1.1}$$

Moreover, the control ζ regarded as a function of time may have discontinuities only at positive integers.

Let us emphasise that the above theorem is trivial if $h\equiv 0$: in this case, all the solutions go to zero exponentially fast, and inequality (1.1) is valid with $\zeta\equiv 0$. On the other hand, when h is a time-independent function, and the corresponding stationary equation has two solutions, then one does need to apply a control to one of them to make them converge to each other exponentially fast. Furthermore, the fact that the right-hand side of (1.1) depends on the minimum of the initial distance between solutions and the number 1 is a manifestation of uniformity of stabilisation with respect to initial data: if $\|u_0-\widehat{u}_0\|_{L^1}$ is small, then the H^1-norm of the difference between the corresponding solutions remains small and decays exponentially with time, while for the initial data that are far from each other, the stabilisation takes place for $t\geqslant 1$ with a rate that is independent of them. This phenomenon is due to the strong nonlinear dissipation of the viscous Burgers equation.

Taking Theorem 1.1 for granted, let us prove the exact controllability result stated in the Introduction.

Proof of the Main Theorem. — We shall combine Theorem 1.1 with a version of the Fursikov–Imanuvilov result on local exact controllability to trajectories. Namely, suppose we know that there are positive functions $\varepsilon(\rho)$ and $C(\rho)$ defined for $\rho\geqslant 0$ such that, for any functions $\widehat{u}_0\in H^1_0(I)$ and $h\in(H^1\cap L^\infty)([0,1]\times I)$ whose norms are no greater than ρ, the following property holds: if $u_0\in H^1_0(I)$ and $\|u_0-\widehat{u}_0\|_{H^1}\leqslant\varepsilon(\rho)$, then there is a control $\eta\in L^2([0,1]\times I)$ such that

$$\mathscr{R}_1(u_0,h+\eta)=\mathscr{R}_1(\widehat{u}_0,h),\quad \|\eta\|_{L^2}\leqslant C(\rho)\|u_0-\widehat{u}_0\|_{H^1}. \tag{1.2}$$

Let us take any $\widehat{u}_0\in L^2(I)$. In view of Proposition 1.2 and Remark 3.1 (see below), the corresponding trajectory $\widehat{u}(t)=\mathscr{R}_t(\widehat{u}_0,h)$ is bounded in $H^1_0(I)$ for $t\geqslant 1$ by a number depending only on $\|h\|_{H^1_{\mathrm{ul}}\cap L^\infty}$ and ν. Thus, we can find $\rho>1$ such that

$$\|h\|_{H^1\cap L^\infty}\leqslant\rho,\quad \|\widehat{u}(T)\|_{H^1}\leqslant\rho\quad\text{for any } T\geqslant 1,$$

where the norms are taken on $[T,T+1]\times I$ and I, respectively. In view of the above-mentioned local exact controllability result applied to the interval $[T,T+1]$ (rather than to $[0,1]$), one can find $\varepsilon>0$ such that, if $v_0\in H^1_0(I)$ satisfies the inequality $\|v_0-\widehat{u}(T)\|_{H^1}\leqslant\varepsilon$ for some $T\geqslant 1$, then there is a control $\eta_T\in L^2(D_T)$ supported in $[T,T+1]\times[a,b]$ such that $v(T+1)=\widehat{u}(T+1)$, where $v(t,x)$ stands for the solution of Problem (0.1), (0.2) with $\zeta=\eta_T$ such that $v(T)=v_0$. Due to (1.1), there is a number(4) $T_\varepsilon > 0$ such that, for any $u_0 \in L^2(I)$, one can find a piecewise continuous control $\zeta : J_{T_\varepsilon} \to H^1(I)$ supported in $J_{T_\varepsilon} \times [a,b]$ for which

$$\|\mathscr{R}_{T_\varepsilon}(u_0, h+\zeta) - \widehat{u}(T_\varepsilon)\|_{H^1} \leqslant \varepsilon.$$

Extending ζ to $[T_\varepsilon, T_\varepsilon+1]$ by $\zeta(t) = \eta_{T_\varepsilon}(t)$, we see that (0.5) holds with $T = T_\varepsilon + 1$.

We now discuss briefly the proof of (1.2), which follows from the argument of [FI95, §5]. Let us set $\mathscr{Y} = L^2(J_1, V) \cap H^1(J_1, L^2)$ and seek $u \in \mathscr{Y}$ in the form $u = \widehat{u} + v$. Then $v \in \mathscr{Y}$ must be a solution of the problem

$$\partial_t v - \nu\partial_x^2 v + \partial_x\big((\widehat{u} + \tfrac{1}{2}v)v\big) = \eta, \quad v(t,0) = v(t,1) = 0, \quad v(0) = v_0 := u_0 - \widehat{u}_0. \tag{1.3}$$

Together with (1.3), let us consider the linear problem

$$\partial_t v - \nu\partial_x^2 v + \partial_x\big(a(t,x)v\big) = \eta, \quad v(t,0) = v(t,1) = 0, \quad v(0) = v_0. \tag{1.4}$$

By Theorem 4.3 in [FI95], for any function $a \in \mathscr{Y}$ there exists a linear operator $\mathscr{C}_a : H_0^1(I) \to L^2(J_1 \times I)$ such that the operator norm of $\mathscr{C}_a$ is bounded by a number N depending only on $\|a\|_{\mathscr{Y}}$, the solution $v \in \mathscr{Y}$ of (1.4) with $\eta = \mathscr{C}_a v_0$ vanishes at $t = 1$, and the mapping $\mathscr{S} : a \mapsto v$ acts continuously in $\mathscr{Y}$ and takes bounded subsets to relatively compact ones.

Let us fix a small number $r > 0$ and, given $v_0 \in B_{H_0^1}(r)$, consider a mapping $F : B_{\mathscr{Y}}(1) \to \mathscr{Y}$ that takes w to $v = \mathscr{S}(\widehat{u} + \frac{1}{2}w)$. Then v is a solution of Problem (1.4) with $a = \widehat{u} + \frac{1}{2}w$ and $\eta = \mathscr{C}_a v_0$, and F is a compact mapping that satisfies the inequality

$$\|F(v)\|_{\mathscr{Y}} \leqslant C_1(\|\widehat{u}\|_{\mathscr{Y}})\big(\|v_0\|_{H^1} + \|\eta\|_{L^2}\big) \leqslant C_1(\|\widehat{u}\|_{\mathscr{Y}})\big(1 + N(\|\widehat{u}\|_{\mathscr{Y}})\big)\|v_0\|.$$

It follows that F takes the ball $B_{\mathscr{Y}}(1)$ into itself, provided that r is sufficiently small. By the Leray–Schauder theorem, F has a fixed point $v \in \mathscr{Y}$. The function $\eta = \mathscr{C}_{\widehat{u}+v/2}(u_0 - \widehat{u}_0)$ is the required control for which (1.2) holds. This completes the proof of the exact controllability to trajectories. □

1.2. Description of the stabilisation scheme.

— We now outline the main steps of the proof of Theorem 1.1, which is given in Section 3. It is based on a comparison principle for nonlinear parabolic equations and the Harnack inequality.

Step A. Reduction to bounded regular initial data. — We first prove that it suffices to consider the case of H^2-smooth initial conditions with norm bounded by a fixed constant. Namely, let $V := H_0^1 \cap H^2$, and given a number $T > 0$, let us define the functional space

$$\mathscr{X}_T = L^2(J_T, H_0^1) \cap H^1(J_T, H^{-1}). \tag{1.5}$$

We have the following result providing a universal bound for solutions of (0.1), (0.2) at any positive time; see Section 3.1 for a proof.

(4)It is straightforward to see from (1.1) that $T_\varepsilon \leqslant C_1 \log \varepsilon^{-1}$, where $C_1 > 0$ depends only on ν, $[a,b]$, and the $H^1_{\mathrm{ul}} \cap L^\infty$ norm of h. This fact does not play any role in the argument.

Proposition 1.2. — *Let $h \in (H^1 \cap L^\infty)(J_T \times I)$ for some $T > 0$ and let $\nu > 0$. Then there is number $R > 0$, depending only on $\|h\|_{H^1 \cap L^\infty}$ and ν, such that any solution $u \in \mathscr{X}_T$ of (0.1) with $\zeta \equiv 0$ satisfies the inclusion $u(t) \in V$ for $0 < t \leqslant T$ and the inequality*

$$\|u(T)\|_{H^2} \leqslant R. \tag{1.6}$$

We emphasise that R does not depend on the solution u. Thus, if $h \in H^1_{\mathrm{ul}} \cap L^\infty$ is fixed, then, for any initial data $u_0, \widehat{u}_0 \in L^2(I)$, we have

$$\|\mathscr{R}_1(u_0, h)\|_{H^2} \leqslant R, \quad \|\mathscr{R}_1(\widehat{u}_0, h)\|_{H^2} \leqslant R,$$

where R is the constant in Proposition 1.2 with $T = 1$. Furthermore, in view of the contraction of the L^1-norm for the difference of two solutions (cf. Proposition 2.5 below), we have

$$\|\mathscr{R}_1(u_0, h) - \mathscr{R}_1(\widehat{u}_0, h)\|_{L^1} \leqslant \|u_0 - \widehat{u}_0\|_{L^1}.$$

Thus, applying zero control on the interval $[0, 1]$, we bring the solutions to some states u_1 and $\widehat{u}_1$ that belong to the ball in V of radius R centred at zero, and the L^1-distance between them does not exceed the initial distance. Hence, to prove Theorem 1.1, it suffices to establish the inequality in (1.1) for $t \geqslant 0$ and any initial data $u_0, \widehat{u}_0 \in B_V(R)$.

Step B. Interpolation. — Let us fix two initial conditions $u_0, \widehat{u}_0 \in B_V(R)$. Suppose we have constructed a control $\zeta(t, x)$ supported in $\mathbb{R}_+ \times [a, b]$ such that, for all $t \geqslant 0$,

$$\|\mathscr{R}_t(u_0, h + \zeta)\|_{H^2} + \|\mathscr{R}_t(\widehat{u}_0, h)\|_{H^2} \leqslant C_1, \tag{1.7}$$

$$\|\mathscr{R}_t(u_0, h + \zeta) - \mathscr{R}_t(\widehat{u}_0, h)\|_{L^1} \leqslant C_2 e^{-\alpha t} \|u_0 - \widehat{u}_0\|_{L^1}, \tag{1.8}$$

where C_1, C_2, and α are positive numbers not depending on u_0, $\widehat{u}_0$, and t. In this case, using the interpolation inequality (see Section 15.1 in [BIN79])

$$\|v\|_{H^1} \leqslant C_3 \|v\|_{L^1}^{2/5} \|v\|_{H^2}^{3/5}, \quad v \in H^2(I), \tag{1.9}$$

we can write

$$\|\mathscr{R}_t(u_0, h + \zeta) - \mathscr{R}_t(\widehat{u}_0, h)\|_{H^1} \leqslant C_4 e^{-\gamma t} \|u_0 - \widehat{u}_0\|_{L^1}^{2/5}, \tag{1.10}$$

where $\gamma = 2\alpha/5$, and $C_4 > 0$ does not depend on u_0, $\widehat{u}_0$, and t. This implies the required inequality for the first term on the left-hand side of (1.1). An estimate for the second term will follow from the construction; see relations (1.16) and (1.17) below.

Step C: Main auxiliary result. — Let us take two initial data $v_0, \widehat{u}_0 \in B_V(R)$ and consider the difference w between the corresponding solutions of Problem (0.1)–(0.3) with $\zeta \equiv 0$; that is, $w = v - \widehat{u}$, where $v(t) = \mathscr{R}_t(v_0, h)$ and $\widehat{u}(t) = \mathscr{R}_t(\widehat{u}_0, h)$. It is straightforward to check that w satisfies the linear equation

$$\partial_t w - \nu \partial_x^2 w + \partial_x \big(a(t, x) w\big) = 0, \tag{1.11}$$

where $a = \frac{1}{2}(v + \widehat{u})$. The following proposition is the key point of our construction.

Proposition 1.3. — *Let positive numbers ν, T, ρ, and $s<1$ be fixed, and let $a(t,x)$ be a function such that*

$$\|a\|_{C^s(J_T\times I)}+\|\partial_x a\|_{L^\infty(J_T\times I)}\leqslant\rho. \tag{1.12}$$

Then, for any closed interval $I'\subset I$, there are positive numbers ε and $q<1$, depending only on ν, T, ρ, s, and I', such that any solution $w\in\mathscr{X}_T$ of Equation (1.11) *satisfies one of the inequalities*

$$\|w(T)\|_{L^1}\leqslant q\,\|w(0)\|_{L^1}\quad\text{or}\quad\|w(T)\|_{L^1(I')}\geqslant\varepsilon\,\|w(0)\|_{L^1}. \tag{1.13}$$

This result can be described informally as follows. Let us consider the difference $w=v-\widehat{u}$ between two solutions of (0.1), (0.2). Then two cases are possible: either the L^1-norm of w at time $t=T$ is at least q^{-1} times smaller than at $t=0$, so that the distance between the two solutions decreases without any control, or the L^1-norm of the restriction of $w(T)$ to the subinterval I' is minorised by $\|w(0)\|_{L^1}$. In both cases, we can modify w in the neighbourhood of I' so that the function $u=v+w$ is a solution to Problem (0.1), (0.2) with a control ζ supported by $[a,b]$, and the L^1-norm of the difference at $t=T$ is at least θ^{-1} times smaller than the initial norm, where $\theta<1$ is a number. We now describe this idea in more detail.

Step D: Description of the controlled solution. — Let us fix a closed interval $I'\subset(a,b)$ and choose two functions $\chi_0\in C^\infty(\overline{I})$ and $\beta\in C^\infty(\mathbb{R})$ such that

$$0\leqslant\chi_0(x)\leqslant 1\text{ for }x\in I,\quad\chi_0(x)=0\text{ for }x\in I',\quad\chi_0(x)=1\text{ for }x\in I\smallsetminus[a,b], \tag{1.14}$$

$$0\leqslant\beta(t)\leqslant 1\text{ for }t\in\mathbb{R},\qquad\beta(t)=0\text{ for }t\leqslant\tfrac{1}{2},\qquad\beta(t)=1\text{ for }t\geqslant 1. \tag{1.15}$$

Let us set $\chi(t,x)=1-\beta(t)(1-\chi_0(x))$; see Figure 1.1. Given $u_0,\widehat{u}_0\in B_V(R)$, we denote by $\widehat{u}(t,x)$ the reference trajectory and define a controlled solution $u(t,x)$ of (0.1) consecutively on intervals $[k,k+1]$ with $k\in\mathbb{Z}_+$ by the following rules:

(a) *if $u(t)$ is constructed on $[0,k]$, then we denote by $v(t,x)$ the solution issued from $u(k)$ for Problem* (0.1), (0.2) *on $[k,k+1]$ with $\zeta\equiv 0$;*

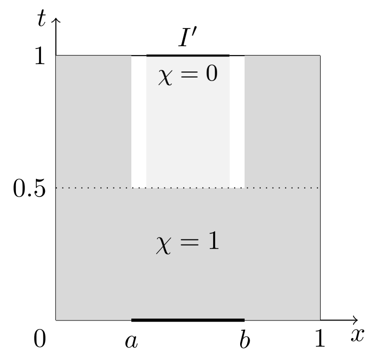

Figure 1.1. The function χ decreases to zero in I'

(b) *for any odd integer $k \in \mathbb{Z}_+$, we set*

$$u(t,x) = v(t,x) \quad \textit{for } (t,x) \in [k, k+1] \times I; \tag{1.16}$$

(c) *for any even integer $k \in \mathbb{Z}_+$, we set*

$$u(t,x) = \widehat{u}(t,x) + \chi(t-k,x)\big(v(t,x) - \widehat{u}(t,x)\big) \quad \textit{for } (t,x) \in [k, k+1] \times I. \tag{1.17}$$

It is not difficult to check that $u(t,x)$ is a solution of Problem (0.1), (0.2), in which ζ is supported by $\mathbb{R}_+ \times [a,b]$. Moreover, it will follow from Proposition 1.3 that, for any even integer $k \geqslant 0$, we have

$$\|u(k+1) - \widehat{u}(k+1)\|_{L^1} \leqslant \theta\, \|u(k) - \widehat{u}(k)\|_{L^1}, \tag{1.18}$$

where $\theta < 1$ does not depend on $\widehat{u}_0$, u_0, and k. On the other hand, the contraction of the L^1-norm between solutions of (0.1) implies that

$$\|u(t) - \widehat{u}(t)\|_{L^1} \leqslant \|u([t]) - \widehat{u}([t])\|_{L^1} \quad \text{for any } t \geqslant 0, \tag{1.19}$$

where $[t]$ stands for the largest integer not exceeding t. These two inequalities give (1.8). The uniform bound (1.7) for the H^2-norm will follow from regularity of solutions for Problem (0.1), (0.2).

2. Preliminaries on the Burgers equation

In this section, we establish some properties of the Burgers equation. They are well known, and their proofs can be found in the literature in more complicated situations. However, for the reader's convenience, we outline some of those proofs in the appendix to make the presentation self-contained. In this section, when talking about Equation (0.1), we always assume that $\zeta \equiv 0$.

2.1. Maximum principle and regularity of solutions.

— In this subsection, we discuss the well-posedness of the initial-boundary value problem for the Burgers equation. This type of results are very well known, and we only outline their proofs in the appendix. Recall that $V = H_0^1 \cap H^2$, and the space $\mathscr{X}$ was defined in the Introduction.

Proposition 2.1. — *Let $u_0 \in L^2(I)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(I))$. Then Problem* (0.1)–(0.3) *has a unique solution $u \in \mathscr{X}$. Moreover, the following two properties hold.*

- *L^∞ bound. If $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times I)$ and $u_0 \in L^\infty(I)$, then $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times I)$.*
- *Regularity. If, in addition, $u_0 \in V$ and $h \in H^1_{\mathrm{loc}}(\mathbb{R}_+ \times I)$, then*

$$u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}_+, H^1_0) \cap H^2_{\mathrm{loc}}(\mathbb{R}_+, H^{-1}). \tag{2.1}$$

Let us note that, if u_0 is only in the space $L^2(I)$, then the conclusions about the L^∞ bound and the regularity remain valid on the half-line $\mathbb{R}_\tau := [\tau, +\infty)$ for any $\tau > 0$. To see this, it suffices to remark that any solution $u \in \mathscr{X}$ of (0.1), (0.2) satisfies the inclusion $u(\tau) \in H^1_0 \cap H^2$ for almost every $\tau > 0$. For any such $\tau > 0$, one can apply Proposition 2.1 to the half-line $\mathbb{R}_\tau$ and conclude that the inclusions mentioned there are true with $\mathbb{R}_+$ replaced by $\mathbb{R}_\tau$.

2.2. Comparison principle. — The Burgers equation possesses a very strong dissipation property due to the nonlinear term. To state and prove the corresponding result, we need the concept of sub- and super-solution for Equation (0.1) with $\zeta \equiv 0$. Let us fix $T > 0$ and, given an interval $I' \subset I$, define[(5)]

$$\mathscr{X}_T(I') = L^2(J_T, H^1(I')) \cap H^1(J_T, H^{-1}(I')).$$

Definition 2.2. — A function $u \in \mathscr{X}_T(I')$ is called a *super-solution* for (0.1) if

$$\int_0^T \big((\partial_t u, \varphi) + (\nu \partial_x u - \tfrac{1}{2}u^2, \partial_x \varphi)\big)\, \mathrm{d}t \geqslant \int_0^T (h, \varphi)\, \mathrm{d}t, \tag{2.2}$$

where $\varphi \in L^\infty(J_T, L^2(I')) \cap L^2(J_T, H_0^1(I'))$ is an arbitrary non-negative function, and $(\cdot\,,\cdot)$ denotes the scalar product in $L^2(I')$. The concept of a *sub-solution* is defined similarly, replacing $\geqslant$ by $\leqslant$.

A proof of the following result can be found in Section 2.2 of [AL83] for a more general problem; for the reader's convenience, we outline it in the appendix.

Proposition 2.3. — *Let $h \in L^1(J_T, L^2)$, and let functions u^+ and u^- belonging to $\mathscr{X}_T(I')$ be, respectively, super- and sub-solutions for* (0.1) *such that*[(6)]

$$u^+(t,x) \geqslant u^-(t,x) \quad \textit{for } t = 0,\ x \in I' \textit{ and } t \in [0,T],\ x \in \partial I', \tag{2.3}$$

where the inequality holds almost everywhere. Then, for any $t \in J_T$, we have

$$u^+(t,x) \geqslant u^-(t,x) \quad \textit{for a.e. } x \in I'. \tag{2.4}$$

We now derive an a priori estimate for solutions of (0.1), (0.2).

Corollary 2.4. — *Let $u_0 \in L^\infty$ and $h \in L^\infty(J_T \times I)$ for some $T > 0$. Then the solution of Problem* (0.1)–(0.3) *with $\zeta \equiv 0$ satisfies the inequality*

$$\|u(T,\cdot)\|_{L^\infty} \leqslant C, \tag{2.5}$$

where $C > 0$ is a number continuously depending only on $\|h\|_{L^\infty}$ and T.

Proof. — We follow the argument used in the proof of Lemma 9 in [Cor07b, §2.1]. Given $\varepsilon > 0$ and $u_0 \in L^\infty(I)$, we set

$$B_\varepsilon = 1 + \|h\|_{L^\infty}^{1/3}(T+\varepsilon)^{2/3}, \quad L = \|u_0\|_{L^\infty}.$$

It is a matter of a simple calculation to check that the functions

$$u_\varepsilon^+(t,x) = \frac{B_\varepsilon(B_\varepsilon + x) + L\varepsilon}{t+\varepsilon}, \quad u_\varepsilon^-(t,x) = -\frac{B_\varepsilon(B_\varepsilon - x) + L\varepsilon}{t+\varepsilon}$$

are, respectively, super- and sub-solutions for (0.1) on $J_T \times I$ such that

$$u_\varepsilon^+(t,x) \geqslant u(t,x) \geqslant u_\varepsilon^-(t,x) \quad \text{for } t = 0,\ x \in I \text{ and } t \in [0,T],\ x = 0 \text{ or } 1.$$

[(5)] Note that, in contrast to $\mathscr{X}_T$, we do not require the elements of $\mathscr{X}_T(I')$ to vanish on $\partial I'$.

[(6)] It is not difficult to see that the restrictions of the elements of $\mathscr{X}_T(I')$ to the straight lines $t = t_0$ and $x = x_0$ are well defined.

Applying Proposition 2.3 with $I' = I$, we conclude that

$$u_\varepsilon^+(T,x) \geqslant u(T,x) \geqslant u_\varepsilon^-(T,x) \quad \text{for a.e. } x \in I.$$

Passing to the limit as $\varepsilon \to 0^+$, we arrive at (2.5) with $C = T^{-1}B_0(B_0+1)$. □

2.3. Contraction of the L^1-norm of the difference of solutions. — It is a well known fact that the resolving operator for (0.1), (0.2) regarded as a nonlinear mapping in the space $L^2(I)$ is locally Lipschitz. The following result shows that it is a contraction for the norm of $L^1(I)$.

Proposition 2.5. — *Let $u, v \in \mathscr{X}$ be two solutions of Equation* (0.1)*, in which $\zeta \equiv 0$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2)$. Then*

$$\|u(t) - v(t)\|_{L^1} \leqslant \|u(s) - v(s)\|_{L^1} \quad \text{for any } t \geqslant s \geqslant 0. \tag{2.6}$$

Inequality (2.6) follows from the maximum principle for linear parabolic PDE's, and more general results can be found in Sections 3.2 and 3.3 of [Hör97]. A simple proof of Proposition 2.5 is given in Section 4.3.

2.4. Harnack inequality. — Let us consider the linear homogeneous equation (1.11). The following result is a particular case of the Harnack inequality established in [KS80, Th. 1.1] (see also Section IV.2 in [Kry87]).

Proposition 2.6. — *Let a closed interval $K \subset I$ and positive numbers ν and T be fixed. Then, for any $\rho > 0$ and $T' \in (0,T)$, one can find $C > 0$ such that the following property holds: if $a(t,x)$ satisfies the inequality*

$$\|a\|_{L^\infty(J_T \times I)} + \|\partial_x a\|_{L^\infty(J_T \times I)} \leqslant \rho, \tag{2.7}$$

then for any non-negative solution $w \in L^2(J_T, H^3 \cap H_0^1) \cap H^1(J_T, H_0^1)$ of (1.11) *we have*

$$\sup_{x \in K} w(T', x) \leqslant C \inf_{x \in K} w(T, x). \tag{2.8}$$

3. Proof of the main result

In this section, we prove Theorem 1.1. Its scheme, together with some details, was presented in Section 1, and we now establish the claims that were not proved there.

3.1. Reduction to smooth initial data. — Let us prove Proposition 1.2. Fix arbitrary numbers $T_1 < T_2$ in the interval $(0,T)$. By Proposition 2.1 and the remark following it, for any $\tau > 0$ we have

$$u \in L^\infty(J_{\tau,T} \times I) \cap L^2(J_{\tau,T}, H^3) \cap H^1(J_{\tau,T}, H_0^1) \cap H^2(J_{\tau,T}, H^{-1}), \tag{3.1}$$

where $J_{\tau,T} = [\tau, T]$. Applying Corollary 2.4, we see that

$$\|u(t,\cdot)\|_{L^\infty} \leqslant C \quad \text{for } T_1 \leqslant t \leqslant T. \tag{3.2}$$

Furthermore, it follows from (3.1) that $u(t)$ is a continuous function of $t \in (0,T]$ with range in V. Thus, it remains to establish inequality (1.6) with a universal constant R. The proof of this fact can be carried out by a standard argument based on multipliers technique (e.g., see the proof of Theorem 2 in [BV92, §I.6] dealing with the 2D Navier–Stokes system). Therefore, we confine ourselves to outlining the main steps. Until the end of this subsection, we deal with Equation (0.1) in which $\zeta \equiv 0$ and denote by C_i unessential positive numbers not depending u.

Step 1: Mean H^1-norm. — Taking the scalar product of (0.1) with $2u$ and performing usual transformations, we derive

$$\partial_t \|u\|^2 + 2\nu \|\partial_x u\|^2 = 2(h,u) \leqslant \nu \|\partial_x u\|^2 + \nu^{-1}\|h\|^2.$$

Integrating in time and using (3.2) with $t = T_1$, we obtain

$$\int_{T_1}^T \|\partial_x u\|^2 \,\mathrm{d}t \leqslant \nu^{-1}\|u(T_1)\|^2 + \nu^{-2}\int_{T_1}^T \|h\|^2 \,\mathrm{d}t \leqslant C_1. \tag{3.3}$$

Step 2: H^1-norm and mean H^2-norm. — Let us take the scalar product of (0.1) with $-2(t-T_1)\partial_x^2 u$:

$$\begin{aligned}\partial_t\big((t-T_1)\|\partial_x u\|^2\big) - \|\partial_x u\|^2 + 2\nu(t-T_1)\|\partial_x^2 u\|^2 &= 2(t-T_1)(u\partial_x u - h, \partial_x^2 u)\\ &\leqslant 2(t-T_1)\big(\|u\|_{L^\infty}\|\partial_x u\| + \|h\|\big)\|\partial_x^2 u\|.\end{aligned}$$

Integrating in time and using (3.2) and (3.3), we obtain

$$\|u(t)\|_{H^1} + \int_{T_2}^t \|u(r)\|_{H^2}^2 \mathrm{d}r \leqslant C_2 \quad \text{for } T_2 \leqslant t \leqslant T. \tag{3.4}$$

Using (0.1), we also derive the following estimate for $\partial_t u$:

$$\int_{T_2}^T \|\partial_t u\|^2 \,\mathrm{d}t \leqslant C_3. \tag{3.5}$$

Step 3: L^2-norm of the time derivative. — Taking the time derivative of (0.1), we obtain the following equation for $v = \partial_t u$:

$$\partial_t v - \nu\partial_x^2 v + v\partial_x u + u\partial_x v = \partial_t h.$$

Taking the scalar product with $2(t-T_2)v$, we derive

$$\begin{aligned}\partial_t\big((t-T_2)\|v\|^2\big) - \|v\|^2 + 2\nu(t-T_2)\|\partial_x v\|^2 &= 2(t-T_2)(\partial_t h - u\partial_x v - v\partial_x u, v)\\ &\leqslant 2(t-T_2)\big(\|\partial_t h\| + 3\|u\|_{L^\infty}\|\partial_x v\|\big)\|v\|.\end{aligned}$$

Integrating in time and using (3.2) and (3.5), we obtain

$$\|v(T)\| \leqslant C_4. \tag{3.6}$$

Step 4: H^2-norm. — We now rewrite (0.1) in the form

$$\nu\partial_x^2 u = f(t) := v + u\partial_x u - h. \tag{3.7}$$

In view of (3.4) and (3.6), we have $\|f(T)\| \leqslant C_5$. Combining this with (3.7), we arrive at the required inequality (1.6).

Remark 3.1. — The argument given above shows that, under the hypotheses of Proposition 1.2, if $u_0 \in B_V(\rho)$, then $\|R_t(u_0,h)\|_{H^2} \leqslant R$ for all $t \geqslant 0$, where $R>0$ depends only on h, ν, and ρ. Moreover, similar calculations enable one to prove that, for any $t>0$, the resolving operator $\mathscr{R}_t(u_0,h)$ regarded as a function of u_0 is uniformly Lipschitz continuous from any ball of L^2 to H^2, and the corresponding Lipschitz constant can be chosen to be the same for $T^{-1} \leqslant t \leqslant T$, where $T>1$ is an arbitrary number.

3.2. Proof of the main auxiliary result. — In this subsection, we prove Proposition 1.3. In doing so, we fix parameter $\nu>0$ and do not follow the dependence of various quantities on it.

Step 1. — We begin with the case of non-negative solutions. Namely, we prove that, given $q \in (0,1)$, one can find $\delta = \delta(I', T, q, \rho) > 0$ such that, if $w \in \mathscr{X}_T$ is a non-negative solution of (1.11), then either the first inequality in (1.13) holds, or

$$\inf_{x\in I'} w(T,x) \geqslant \delta \|w(0)\|_{L^1}. \tag{3.8}$$

To this end, we shall need the following lemma, established at the end of this subsection.

Lemma 3.2. — *For any $0<\tau<T$ and $\rho>0$, there is $M>0$ such that, if $w \in \mathscr{X}_T$ is a solution of Equation (1.11) with a function $a(t,x)$ satisfying (1.12), then*

$$\sup_{(t,x)\in[\tau,T]\times I} |w(t,x)| \leqslant M \|w(0)\|_{L^1}. \tag{3.9}$$

In view of linearity, we can assume without loss of generality that $\|w(0)\|_{L^1} = 1$. Let us choose a closed interval $K \subset I$ containing I' such that

$$|I \smallsetminus K| \leqslant \frac{q}{2M}, \tag{3.10}$$

where $|\Gamma|$ denotes the Lebesgue measure of a set $\Gamma \subset \mathbb{R}$, and $M>0$ is the constant in (3.9) with $\tau = 2T/3$. By Proposition 2.1 and the remark following it, the function w satisfies the hypotheses of Proposition 2.6. Therefore, by the Harnack inequality (2.8), we have

$$\sup_{x\in K} w(2T/3,x) \leqslant C \inf_{x\in K} w(T,x), \tag{3.11}$$

where $C>0$ depends only on T, K, and ρ. Let us set $\delta = q/2C|K|$ and suppose that (3.8) is not satisfied. In this case, using (3.9)–(3.11) and the contraction of the L^1-norm of solutions for (1.11) (see Remark 4.2), we derive

$$\begin{aligned}\|w(T)\|_{L^1} \leqslant \|w(2T/3)\|_{L^1} &= \int_{I\smallsetminus K} w(2T/3,x)\,\mathrm{d}x + \int_K w(2T/3,x)\,\mathrm{d}x \\ &\leqslant M\,|I\smallsetminus K| + C\delta|K| \leqslant q.\end{aligned}$$

This is the first inequality in (1.13) with $\|w(0)\|_{L^1} = 1$.

Step 2. — We now consider the case of arbitrary solutions $w \in \mathscr{X}_T$, assuming again that $\|w(0)\|_{L^1} = 1$. Let us denote by w_0^+ and w_0^- the positive and negative parts of $w_0 := w(0)$, and let w^+ and w^- be the solutions of (1.11) issued from w_0^+ and w_0^-, respectively. Thus, we have

$$w_0 = w_0^+ - w_0^-, \quad \|w_0^+\|_{L^1} + \|w_0^-\|_{L^1} = 1, \quad w = w^+ - w^-.$$

Let us set $r := \|w_0^+\|_{L^1}$ and assume without loss of generality that $r \geqslant 1/2$. In view of the maximum principle for linear parabolic equations (see Section 2 in [Lan98, Chap. 3]), the functions w^+ and w^- are non-negative, and therefore the property established in Step 1 is true for them. If $\|w^+(T)\|_{L^1} \leqslant r/2$, then the contraction of the L^1-norm of solutions of (1.11) implies that

$$\|w(T)\|_{L^1} \leqslant \|w^+(T)\|_{L^1} + \|w^-(T)\|_{L^1} \leqslant r/2 + (1-r) \leqslant 3/4.$$

This coincides with the first inequality in (1.13) with $\|w(0)\|_{L^1} = 1$.

Suppose now that $\|w^+(T)\|_{L^1} > r/2$. Using the property of Step 1 with $q = 1/2$, we find $\delta_1 > 0$ such that

$$\inf_{x \in Q} w^+(T, x) \geqslant \delta_1 r. \tag{3.12}$$

Set $\varepsilon = \frac{1}{4}\delta_1 |I'|$ and assume that $\|w(T)\|_{L^1(I')} < \varepsilon$ (in the opposite case, the second inequality in (1.13) holds), so that

$$\|w^+(T)\|_{L^1(I')} - \|w^-(T)\|_{L^1(I')} < \varepsilon.$$

It follows that

$$\|w^-(T)\|_{L^1} \geqslant \|w^-(T)\|_{L^1(I')} \geqslant \|w^+(T)\|_{L^1(I')} - \varepsilon \geqslant \delta_1 r|I'| - \frac{\delta_1}{4}|I'| \geqslant \varepsilon.$$

By the L^1-contraction for w^-, we see that $\|w_0^-\|_{L^1} = 1 - r \geqslant \varepsilon$. Repeating the argument applied above to w^+, we can prove that if

$$\|w^-(T)\|_{L^1} \leqslant \frac{1}{2}(1-r), \tag{3.13}$$

then $\|w(T)\|_{L^1} \leqslant 1 - \varepsilon/2$, so that the first inequality in (1.13) holds with $q = 1 - \varepsilon/2$. Thus, it remains to consider the case when (3.13) does not hold. Applying the property of Step 1 to w^-, we find $\delta_2 > 0$ such that

$$\inf_{x \in I'} w^-(T, x) \geqslant \delta_2 (1-r). \tag{3.14}$$

Since $1/2 \leqslant r \leqslant 1 - \varepsilon$, the right-hand sides in (3.12) and (3.14) are minorised by $\theta = \min\{\frac{1}{2}\delta_1, \varepsilon\delta_2\}$. Denoting by $\chi_{I'}$ the indicator function of I', we write

$$\begin{aligned}
\|w(T)\|_{L^1} &= \int_I |w^+(T,x) - w^-(T,x)|\,\mathrm{d}x \\
&= \int_I \big|(w^+(T,x) - \theta\chi_{I'}(x)) - (w^-(T,x) - \theta\chi_{I'}(x))\big|\,\mathrm{d}x \\
&\leqslant \int_I \big(w^+(T,x) - \theta\chi_{I'}(x)\big)\,\mathrm{d}x + \int_I \big(w^-(T,x) - \theta\chi_{I'}(x))\big)\,\mathrm{d}x \\
&= \|w^+(T)\|_{L^1} + \|w^-(T)\|_{L^1} - 2\theta|I'|.
\end{aligned}$$

In view of the L^1-contraction for w^+ and w^-, the right-hand side of this inequality does not exceed

$$\|w_0^+\|_{L^1} + \|w_0^-\|_{L^1} - 2\theta|I'| = 1 - 2\theta|I'|.$$

Setting $q = \max\{3/4, 1-\varepsilon/2, 1-2\theta|I'|\}$, we conclude that one of the inequalities (1.13) holds for w. Thus, to complete the proof of Proposition 1.3, it only remains to establish Lemma 3.2.

Proof of Lemma 3.2. — By the maximum principle and regularity of solutions for linear parabolic equations, it suffices to prove that

$$\|w(\tau)\|_{L^\infty(I)} \leqslant C_1\|w(0)\|_{L^1(I)}, \tag{3.15}$$

where $C_1 > 0$ does not depend on w. To this end, along with (1.11), let us consider the dual equation

$$\partial_t z + \nu\partial_x^2 z + a(t,x)\partial_x z = 0, \tag{3.16}$$

supplemented with the initial condition

$$z(T,x) = z_0(x). \tag{3.17}$$

Let us denote by $G(t,x,y)$ the Green function of the Dirichlet problem for (3.16), (3.17). By Theorem 16.3 in [LSU68, Chap. IV], one can find positive numbers C_2 and C_3 depending only on ρ, s, and T such that

$$|G(t,x,y)| \leqslant C_2(T-t)^{-1/2}\exp\bigl(-C_3(x-y)^2/(T-t)\bigr) \quad \text{for } x,y\in I,\ t\in[0,T).$$

It follows that, for $z_0 \in L^2(I)$, the solution $z \in \mathscr{X}_T$ of Problem (3.16), (3.17) satisfies the inequality

$$\|z(0)\|_{L^\infty} \leqslant C_4\|z_0\|_{L^1}, \tag{3.18}$$

where $C_4 > 0$ does not depend on z_0.

Now let $w \in \mathscr{X}_T$ be a solution of (1.11). Taking any $z_0 \in L^2(I)$ and denoting by $z \in \mathscr{X}_T$ the solution of (3.16), (3.17), we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(w(t), z(t)\bigr) = (\partial_t w, z) + (w, \partial_t z) = 0. \tag{3.19}$$

Integrating in time and using (3.18), we obtain

$$\int_I w(\tau)z_0\,\mathrm{d}x = \int_I w(0)z(0)\,\mathrm{d}x \leqslant \|w(0)\|_{L^1}\|z(0)\|_{L^\infty} \leqslant C_4\|w(0)\|_{L^1}\|z_0\|_{L^1}.$$

Taking the supremum over all $z_0 \in L^2$ with $\|z_0\|_{L^1} \leqslant 1$, we arrive at the required inequality (3.15). □

3.3. Completion of the proof. — We need to prove inequalities (1.7) and (1.8), as well as the piecewise continuity of $\zeta : \mathbb{R}_+ \to H^1(I)$ and the estimate

$$\|\zeta(t)\|_{H^1} \leqslant C_1 e^{-\gamma t}\min\bigl(\|u_0 - \widehat{u}_0\|_{L^1}^{2/5}, 1\bigr), \quad t \geqslant 0. \tag{3.20}$$

Proof of (1.7). — The estimate for $\widehat{u}(t)=\mathscr{R}_t(\widehat{u}_0,h)$ follows from Remark 3.1. Setting $t_k=2k$, we now use induction on $k\geqslant 0$ to prove that $u(t)=\mathscr{R}_t(u_0,h+\zeta)$ is bounded on $[t_k,t_{k+1}]$ by a universal constant and that $u(t_{k+1})\in B_V(R)$, provided that $u(t_k)\in B_V(R)$. Indeed, it follows from (1.17) that

$$\sup_{t_k\leqslant t\leqslant s_k}\|u(t)\|_{H^2}\leqslant C_2\sup_{t_k\leqslant t\leqslant s_k}\bigl(\|\widehat{u}(t)\|_{H^2}+\|v(t)\|_{H^2}\bigr),$$

where $s_k=2k+1$. In view of Remark 3.1, the right-hand side of this inequality does not exceed a constant $C_3(R)$. Furthermore, recalling (1.16) and using Remark 3.1 and inequality (1.6) with $T=1$, we see that

$$\sup_{s_k\leqslant t\leqslant t_{k+1}}\|u(t)\|_{H^2}\leqslant C_3(R),\quad \|u(t_{k+1})\|_{H^2}\leqslant R.$$

This completes the induction step.

Proof of (1.8). — In view of (1.19), it suffices to establish (1.18) for any even integer $k\geqslant 0$. It follows from (1.17), (1.15), and the definition of χ that

$$\|u(k+1)-\widehat{u}(k+1)\|_{L^1}=\int_I\chi_0(x)|v(k+1)-\widehat{u}(k+1)|\,\mathrm{d}x. \tag{3.21}$$

We know that the norms of the functions v and $\widehat{u}$ are bounded in $L^\infty([k,k+1],H^2)$ by a constant depending only on R. Since they satisfy Equation (0.1) with $\zeta\equiv 0$, we see that $\partial_t v$ and $\partial_t\widehat{u}$ are bounded in $L^\infty([k,k+1],L^2)$ by a number depending on R. By interpolation and the continuous embedding $H^1(I)\subset C^{1/2}(I)$, we see that

$$\|v\|_{C^{1/2}([k,k+1]\times I)}+\|\widehat{u}\|_{C^{1/2}([k,k+1]\times I)}\leqslant C_4(R).$$

Since the difference $w=v-\widehat{u}$ satisfies Equation (1.11) with $a=\frac{1}{2}(v+\widehat{u})$, we conclude that Proposition 1.3 is applicable to w. Thus, we have one of the inequalities (1.13). If the first of them is true, then it follows from (3.21) that (1.18) holds with $\theta=q$. If the second inequality is true, then using (3.21), the contraction of the L^1-norm for w, and relations (1.14), we derive

$$\|u(k+1)-\widehat{u}(k+1)\|_{L^1}\leqslant\|w(k+1)\|_{L^1}-\|w(k+1)\|_{L^1(I')}\leqslant(1-\varepsilon)\|w(0)\|_{L^1},$$

and, hence, we obtain (1.18) with $\theta=1-\varepsilon$.

Proof of the properties of ζ. — In view of (1.16), on any interval $[k,k+1]$ with odd $k\geqslant 0$, the function u satisfies (0.1) with $\zeta\equiv 0$, and the required properties of ζ are trivial. Let us consider the case of an even $k\geqslant 0$. A direct calculation show that

$$\begin{aligned}\zeta(t,x)&=\partial_t u-\nu\partial_x^2u+u\partial_x u-h\\&=-\bigl(\chi_k(1-\chi_k)w+2\nu\partial_x\chi_k\bigr)\partial_x w+\bigl(\partial_t\chi_k-\nu\partial_x^2\chi_k+\widehat{u}\partial_x\chi_k+\chi_k w\partial_x\chi_k\bigr)w,\end{aligned}$$

where $\chi_k(t,x)=\chi(t-k,x)$. Since $\chi(t,x)=1$ for $x\notin[a,b]$ and for $t\leqslant\frac{1}{2}$, we have $\operatorname{supp}\zeta\subset[k+\frac{1}{2},k+1]\times[a,b]$. By Proposition 2.1, v and $\widehat{u}$ are V-valued continuous functions, whence we conclude that ζ is continuous in time with range in H_0^1. Moreover, since the H^2-norms of v and $\widehat{u}$ are bounded by a number depending only on R, for $t\in[k,k+1]$ we have

$$\|\zeta(t)\|_{H^1}\leqslant C_5(R)I_{[k+1/2,k]}(t)\|w(t)\|_{H^2}\leqslant C_6(R)\|v(k)-\widehat{u}(k)\|_{H^1}, \tag{3.22}$$

where $I_{[k+1/2,k]}(t)$ is the indicator function of the interval $[k+1/2,k]$, and we used the fact that the resolving operator for the Burgers equation is uniformly Lipschitz continuous from any ball of H_0^1 to H^2 for positive times; see Remark 3.1. Since $v(k)-\widehat{u}(k)=u(k)-\widehat{u}(k)$, it follows from (1.8) and (3.22) that (3.20) holds. This completes the proof of Theorem 1.1.

4. Appendix: proofs of some auxiliary assertions

4.1. Proof of Proposition 2.1. — The existence and uniqueness of a solution $u\in\mathscr{X}$ is well known in more complicated situations; see Chapter 15 in [Tay97]. We thus confine ourselves to outlining the proofs of the L^∞ bound and regularity.

The solution $u(t,x)$ of (0.1), (0.2) can be regarded as the solution of the linear parabolic equation

$$\partial_t u-\nu\partial_x^2u+b(t,x)\partial_x u=h(t,x), \tag{4.1}$$

where $b\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H_0^1)$ coincides with u. If b, h, and u_0 were regular functions, then the classical maximum principle would imply that (see Section 2 in [Lan98, Chap. 3])

$$|u(t,x)|\leqslant\|u_0\|_{L^\infty}+t\,\|h\|_{L^\infty(J_t\times I)}\quad\text{for all }(t,x)\in\mathbb{R}_+\times I. \tag{4.2}$$

To deal with the general case, it suffices to approximate u_0 and h by smooth functions and to pass to the (weak) limit in inequality (4.2) written for approximate solutions. This argument shows that the inequality in (4.2) is valid almost everywhere for any solution u.

We now turn to the regularity of solutions. The function $u\in\mathscr{X}$ is the solution of the linear equation

$$\partial_t u-\nu\partial_x^2u=f(t,x),$$

where the right-hand side $f=h-u\partial_x u$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}_+,L^2)$. By standard estimates for the heat equation, we see that

$$u\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^2)\cap H^1_{\mathrm{loc}}(\mathbb{R}_+,L^2). \tag{4.3}$$

Differentiating (0.1) with respect to time and setting $v=\partial_t u$, we see that v satisfies the equations

$$\partial_t v-\nu\partial_x^2v+v\partial_x u+u\partial_x v=\partial_t h,\quad v(0)=v_0, \tag{4.4}$$

where $v_0=h(0)-u_0\partial_x u_0+\nu\partial_x^2u_0\in L^2$. Taking the scalar product of the first equation in (4.4) with v and carrying out some simple transformations, we conclude that $v\in\mathscr{X}$. On the other hand, it follows from (0.1) that

$$\partial_x^2u=v+u\partial_x u-h\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^1),$$

whence we see that $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^3)$. Combining this with the inclusion $\partial_t u \in \mathscr{X}$, we obtain (2.1).

4.2. Proof of Proposition 2.3. — Without loss of generality, we can assume that $t = T$. Define

$$u = u^- - u^+, \quad \psi_\delta(z) = 1 \wedge \big((z/\delta) \vee 0\big),$$

where $\delta > 0$ is a small parameter, and $a \wedge b$ ($a \vee b$) denotes the minimum (respectively, maximum) of the real numbers a and b. In view of inequality (2.2) and its analogue for sub-solutions (in which $(\cdot,\cdot)$ denotes the scalar product in $L^2(I')$), the function u is non-positive almost everywhere for $t = 0$ and satisfies the inequality

$$\int_0^T (\partial_t u, \varphi)\,\mathrm{d}t + \nu \int_0^T (\partial_x u, \partial_x \varphi)\,\mathrm{d}t - \frac{1}{2}\int_0^T (w, \partial_x \varphi)\,\mathrm{d}t \leqslant 0, \tag{4.5}$$

where $w = (u^-)^2 - (u^+)^2$, and $\varphi \in L^\infty(J_T, L^2(I')) \cap L^2(J_T, H^1_0(I'))$ is an arbitrary non-negative function. Let us take $\varphi(t,x) = \psi_\delta(u(t,x))$ in (4.5). It is easy to check that

$$\begin{aligned}
\int_0^T (\partial_t u, \varphi)\,\mathrm{d}t &= \int_{I'} \Psi_\delta(u(T))\,\mathrm{d}x,\\
\int_0^T (\partial_x u, \partial_x \varphi)\,\mathrm{d}t &= \int_0^T\int_{I'} |\partial_x u|^2 \psi_\delta'(u)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0,\\
\left|\int_0^T (w, \partial_x\varphi)\,\mathrm{d}t\right| &\leqslant \int_0^T\int_{I'} |u|\,|u^+ + u^-|\,|\partial_x u|\psi_\delta'(u)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \int_0^T\int_{I'} \Big(\nu|\partial_x u|^2 + \frac{1}{4\nu}|u|^2\,|u^+ + u^-|^2\Big)\psi_\delta'(u)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $\Psi_\delta(z) = \int_0^z \psi_\delta(r)\mathrm{d}r$. Substituting these relations into (4.5), we derive

$$\begin{aligned}
\int_{I'} \Psi_\delta(u(T))\,\mathrm{d}x &\leqslant \frac{1}{8\nu}\int_0^T\int_{I'} |u|^2\,|u^+ + u^-|^2 \psi_\delta'(u)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \frac{\delta}{8\nu}\int_0^T\int_{I'} |u^+ + u^-|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{\delta}{8\nu}\left\|u^+ + u^-\right\|^2_{L^2(J_T\times I)},
\end{aligned}$$

where we used the fact that $0 \leqslant u \leqslant \delta$ on the support of $\psi_\delta'(u)$. Passing to the limit as $\delta \to 0^+$ and using the Fatou lemma, we derive

$$\int_{I'} \big(u(T) \vee 0\big)\,\mathrm{d}x = 0.$$

This inequality implies that the Lebesgue measure of the set of points $x \in I'$ for which $u(T,x) > 0$ is equal to zero. We thus obtain (2.4).

4.3. Proof of Proposition 2.5. — We apply an argument similar to that used in the proof of Lemma 3.2; see Section 3.2. Let us note that the difference $w = u - v \in \mathscr{X}$ satisfies the linear equation (1.11), in which $a = \frac{1}{2}(u+v)$. Along with (1.11), let us consider the dual equation (3.16). The following result is a particular case of the classical maximum principle. Its proof is given in Section III.2 of [Lan98] for regular functions $a(t,x)$ and can be obtained by a simple approximation argument in the general case.

Lemma 4.1. — *Let $a \in L^2(J_T, H^1)$ for some $T > 0$. Then, for any $z_0 \in L^2(I)$, Problem (3.16), (3.17) has a unique solution $z \in \mathscr{X}_T$. Moreover, if $z_0 \in L^\infty(I)$, then $z(t)$ belongs to $L^\infty(I)$ for any $t \in J_T$ and satisfies the inequality*

$$\|z(t)\|_{L^\infty} \leqslant \|z_0\|_{L^\infty} \quad \text{for } t \in J_T. \tag{4.6}$$

To prove (2.6), we fix $t = T$ and assume without loss of generality that $s = 0$. By duality, it suffices to show that, for any $z_0 \in L^\infty(I)$ with norm $\|z_0\|_{L^\infty} \leqslant 1$, we have

$$\int_I w(T) z_0 \,\mathrm{d}x \leqslant \|w(0)\|_{L^1}. \tag{4.7}$$

Let $z \in \mathscr{X}_T$ be the solution of (3.16), (3.17). Such solution exists in view of Lemma 4.1 and the inclusion $a \in L^2(J_T, H_0^1)$, which is ensured by the regularity hypothesis for u and v. It follows from (1.11) and (3.16) that relation (3.19) holds. Integrating it in time, we see that

$$\int_I w(T) z_0 \,\mathrm{d}x = \int_I w(0) z(0) \,\mathrm{d}x \leqslant \|w(0)\|_{L^1} \|z(0)\|_{L^\infty}.$$

Using (4.6) with $t = 0$, we arrive at the required inequality (4.7).

Remark 4.2. — We have proved in fact that if $w \in \mathscr{X}_T$ is a solution of the linear equation (1.11), in which the coefficient a belongs $L^2(J_T, H^1)$, then $\|w(t)\|_{L^1} \leqslant \|w(s)\|_{L^1}$ for $0 \leqslant s \leqslant t \leqslant T$.

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