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ERRATUM TO "COMPLETED IWAHORI-HECKE ALGEBRA AND PARAHORIC HECKE ALGEBRAS FOR KAC-MOODY GROUPS OVER LOCAL FIELDS"

by Ramla Abdellatif & Auguste Hébert

Abstract. — We modify the definition of the completed Iwahori-Hecke algebra given in our previous article (J. Éc. Polytechnique **6**, 79–118) and explain why the construction we gave earlier is not correct as such.

Résumé (Erratum à «Algèbres d'Iwahori-Hecke complétées et algèbres de Hecke parahoriques pour les groupes de Kac-Moody sur les corps locaux»)

Nous modifions la définition de l'algèbre de Hecke complétée donnée dans notre précédent article (J. Éc. Polytechnique **6**, 79–118) et expliquons pourquoi la définition que nous avions donnée n'était pas correcte.

Contents

The goal of this erratum is to fix the construction of the *completed Iwahori-Hecke algebra* given in [AH19, §4.4], as the definition given there does not always provide an actual algebra (see Section 2 below). We define here an algebra that must be used instead. Consequently, the following modifications should be operated in [AH19] : the paragraph before [AH19, Prop. 4.33] is wrong and must be replaced by Section 3 below; Theorem 4.21, Corollary 4.23 and Theorem 4.30 of [AH19] are wrong as stated there and must respectively be replaced by Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.

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1. Introduction

Let G be a split Kac-Moody group (as defined by Tits in [Tit87]) over a non-Archimedean local field $\mathcal{K}$. Given a ring $\mathscr{R}$ containing $\mathbb{Z}$ and satisfying light technical conditions (as in [AH19, Rem. 4.1]), Braverman-Kazdhan-Patnaik [BK11, BKP16] and Bardy–Panse-Gaussent-Rousseau [GR14, BPGR16] associated to G a spherical Hecke algebra $\mathcal{H}_s$ and an Iwahori-Hecke algebra $\mathcal{H}$, both defined over $\mathscr{R}$. Fixing a maximal split torus T of G, and letting Y (resp. Y^+) be the cocharacter lattice (resp. its intersection with the Tits cone) and W^v be the Weyl group of (G,T), then these authors moreover proved the existence of a Satake isomorphism from $\mathcal{H}_s$ to $\mathscr{R}[\![Y]\!]^{W^v}$, where $\mathscr{R}[\![Y]\!]$ is the Looijenga algebra, which is a completion of the group algebra $\mathscr{R}[Y]$ of Y over $\mathscr{R}$ (see [AH19, Def. 4.6] for its definition). A striking difference with the classical case of reductive groups is that for G non-reductive, this spherical Hecke algebra is not isomorphic to the center of the Iwahori-Hecke algebra $\mathcal{H}$.

In [AH19], we announced the definition of a completed Iwahori-Hecke algebra $\widehat{\mathcal{H}}$ that contains $\mathcal{H}$ and whose center is isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$, hence to $\mathcal{H}_s$ [AH19, Th. 4.30]. Nevertheless, the construction of $\widehat{\mathcal{H}}$ we gave in [AH19, §4.4, p. 94–100] is not correct as stated, since as such, $\widehat{\mathcal{H}}$ is actually not stable in general under the convolution product. Indeed, the product of two elements of $\widehat{\mathcal{H}}$ can lead to infinite coefficients, as will be seen below in Sections 2.1 and 2.2. This erratum corrects this mistake by defining a slightly different algebra $\widetilde{\mathcal{H}}$, contained in the vector space $\widehat{\mathcal{H}}$, for a suitable convolution product (see Corollary 3.13). The main point is to use the correct notion of almost-finiteness in the definition of the support of the elements of the completed algebra. We check here that $\widetilde{\mathcal{H}}$ contains $\mathcal{H}$ and that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$, hence to $\mathcal{H}_s$ (see Theorem 3.14), as aimed at first. Moreover note that this modified definition of the completed Iwahori-Hecke algebra suppresses the aforementioned gap between the reductive and non-reductive cases, as for G reductive, $\widetilde{\mathcal{H}}$ is actually isomorphic to the classical Iwahori-Hecke algebra (see Proposition 3.17).

This erratum is organized as follows. In Section 2, we give two counter-examples to [AH19, Th. 4.21]: one in the reductive case and one in the non-reductive case. Then we introduce the required modifications in the definition of the completed algebra to build $\widetilde{\mathcal{H}}$ in Section 3. In particular, we explain in Section 3.3 how to adapt the content of [AH19, §4.4, p. 94–100] to prove that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$.

Acknowledgements. — We thank the referees for their valuable comments and suggestions.

2. Two counter-examples to [AH19, Th. 4.21]

We keep the notation of [AH19, §2]. Let us briefly recall that, as in [AH19, §2.1], given a root generating system $\mathcal{S} = (A, X, Y, (\alpha)_{i\in I}, (\alpha_i^\vee)_{i\in I})$, we set $\mathbb{A} := Y \otimes \mathbb{R}$, let W^v denote the Weyl group of $\mathcal{S}$, $Q^\vee := \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ denote its coroot lattice, $Q^\vee_{\mathbb{R},+} := \bigoplus_{i\in I} \mathbb{R}_+\alpha_i^\vee$ and $Q^\vee_{\mathbb{R}} = \bigoplus_{i\in I} \mathbb{R}\alpha_i^\vee$.

We then have $\mathcal{H} = \bigoplus_{\lambda \in Y^+, w \in W^v} \mathscr{R} Z^\lambda H_w$, where Z^λ and H_w are symbols that satisfy relations (BL1) to (BL4) of [AH19, §4.1, page 91]. The notion of support is defined in [AH19, Def. 4.11]. In this section, we give two examples of elements $(a_j)_{j\in J}, (b_k)_{k\in K}$ in $\mathcal{H}$ that are summable in $\widehat{\mathcal{H}}$ in the sense of [AH19, Def. 4.20], but such that $(a_j * b_k)_{j\in J, k\in K}$ is not summable: one is a reductive case (Section 2.1), the other one is an affine Kac-Moody case (Section 2.2). This prevents [AH19, Th. 4.21] from being true as stated, and we explain in Section 2.3 why the problem stands in the notion of almost-finiteness initially used in the definition of $\widehat{\mathcal{H}}$, and how to modify it to get a correct analogue of [AH19, Th. 4.21].

2.1. A counter-example in the reductive case. — Assume that the standard apartment $\mathbb{A}$ is associated with a Cartan matrix. Fix $\lambda \in Y = Y^+$ and $i \in I$. For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j\alpha_i^\vee} H_i$ and $b_j = Z^{\lambda - j\alpha_i^\vee}$. As $\lambda - \mathbb{N}\alpha_i^\vee$ is almost finite, $(a_j)_{j\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are summable in $\widehat{\mathcal{H}}$. Now let $j, k \in \mathbb{N}$. By (BL4) (see [AH19, §4.1 p. 91]), there exists $c_{j,k} \in \mathscr{R}[\![Y]\!]$ such that

$$a_j * b_k = Z^{\lambda + r_i(\lambda) + (k-j)\alpha_i^\vee} H_i + c_{j,k}.$$

This implies that $(\lambda + r_i(\lambda) + (k-j)\alpha_i^\vee, r_i) \in \operatorname{supp}(a_j * b_k)$, hence $(a_j * b_k)_{(j,k)\in\mathbb{N}^2}$ satisfies none of the conditions of [AH19, Def. 4.20], so it is not summable in $\widehat{\mathcal{H}}$.

2.2. A counter-example in the non-reductive case. — Assume now that $\mathbb{A}$ is associated with an indecomposable affine Kac-Moody matrix A. Let $\delta : \mathbb{A} \to \mathbb{R}$ be the smallest positive imaginary root associated with A. Fix $\lambda \in Y^+$ such that $\delta(\lambda) > 0$ and $i \in I$: then [AH19, §4.2.2, p. 92–93] ensures that $\lambda - \mathbb{N}\alpha_i^\vee$ is an almost finite subset of Y^+. For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j\alpha_i^\vee} H_i$ and $b_j = Z^{\lambda - j\alpha_i^\vee}$: then the same process as in the reductive case (Section 2.1 above) shows that $(a_j * b_k)_{(j,k)\in\mathbb{N}^2}$ satisfies none of the conditions of [AH19, Def. 4.20], hence is not summable in $\widehat{\mathcal{H}}$, although both (a_j) and (b_k) are summable in $\widehat{\mathcal{H}}$.

2.3. Position of the problem and modifications required. — The definition of $\widehat{\mathcal{H}}$ given in [AH19, p. 95] crucially relies on the notion of almost finiteness defined in [AH19, Def. 4.12]. The problem is that almost finiteness is not preserved by the action of W^v on Y^+: there can exist (depending on the Kac-Moody matrix A) an almost finite set E such that $w \cdot E$ is not almost finite, for some $w \in W^v$. To fix this problem, we introduce in the next section a refined notion of almost-finiteness, namely the notion of W^v-*almost finiteness*. Using this new notion, we define an algebra $\widetilde{\mathcal{H}}$ through an analogous construction to the one done for $\widehat{\mathcal{H}}$ in [AH19, §4.4, pages 94-100]. We then explain why the results and proofs stated for $\widehat{\mathcal{H}}$ in [AH19] are now valid for $\widetilde{\mathcal{H}}$.

Before going further, let us list precisely what modifications are actually done in this erratum.

– The notion of almost finiteness defined in [AH19, Def. 4.12] must be replaced by the notion of W^v-*almost finiteness* introduced in Definition 3.1 below to define $\widetilde{\mathcal{H}}$ as we defined $\widehat{H}$ but with the aforementioned replacement.

– The statement and proof of [AH19, Th. 4.21, Cor. 4.23 & Th. 4.30] must be respectively replaced by the statement and proof of Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.

– The content of the paragraph before [AH19, Prop. 4.33], which explains what happens in the reductive case, must be replaced by Section 3.4 below.

3. The completed Iwahori-Hecke algebra $\widetilde{\mathcal{H}}$

The goal of this section is to build an algebra $\widetilde{\mathcal{H}}$ that appears to be smaller than $\widehat{\mathcal{H}}$ (that is *not* always an algebra) whose center is (still) isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$ and that (still) contains $\mathcal{H}$ as the subalgebra of finitely supported elements. It actually boils down to defining the right notion of almost-finiteness and checking that what we did in [AH19, §4] transposes in this setting to define an actual algebra $\widetilde{\mathcal{H}}$ with the required properties.

3.1. W^v-almost finiteness and definition of $\widetilde{\mathcal{H}}$.

— The idea behind the use of the following refined notion of almost finiteness is that it is preserved by the action of W^v.

Definition 3.1. — Let $u \in W^v$.

– A subset E of Y^+ is *u-almost finite* if $u \cdot E$ is almost finite in the sense of [AH19, Def. 4.3].

– A subset E of $Y^+ \times W^v$ is called *u-almost finite* if its projection on W^v is finite and if its projection on Y^+ is u-almost finite (as a subset of Y^+).

– A subset of Y^+ or of $Y^+ \times W^v$ is *W^v-almost finite* if it is u-almost finite for any $u \in W^v$.

As in [AH19, p. 95], we set $\mathscr{B} = \prod_{\lambda \in Y^+, w \in W^v} \mathscr{R}$ and for $(\lambda, w) \in Y^+ \times W^v$, we let $Z^\lambda H_w$ denote the element whose coefficients are all equal to 0 apart from the coefficient indexed by (λ, w), which is equal to 1. This allows us to write $a = (a_{\lambda,w})_{(\lambda,w) \in Y^+ \times W^v} \in \mathscr{B}$ as the formal linear combination

$$a = \sum_{\substack{\lambda \in Y^+ \\ w \in W^v}} a_{\lambda,w} Z^\lambda H_w.$$

Also recall that any $(\nu, u) \in Y^+ \times W^v$ is associated to a projection map $\pi_{\nu,u} : \mathscr{B} \to \mathscr{R}$ defined by

$$\pi_{\nu,u}\Bigg(\sum_{\substack{u' \in W^v \\ \nu' \in Y^+}} c_{\nu',u'} Z^{\nu'} H_{u'}\Bigg) =: c_{\nu,u}$$

for any $\sum c_{\nu',u'} Z^{\nu'} H_{u'} \in \mathscr{B}$.

We can now define $\widetilde{\mathcal{H}}$ as the set of elements of $\mathscr{B}$ with W^v-almost finite support. To prove that $\widetilde{\mathcal{H}}$ can be endowed with a convolution product $*$ that turns it into an associative algebra containing $\mathcal{H}$, we will basically follow the same steps as in [AH19, §4.4], replacing the almost finiteness condition by the W^v-almost finiteness condition.

We let $\mathrm{conv}_{\mathbb{R}}(F)$ denote the convex hull of any part F of $\mathbb{A}$, and we set $\mathrm{conv}(E) := \mathrm{conv}_{\mathbb{R}}(E) \cap Y$ for any subset E of Y. Following [AH19, p. 95], recall that for any part E of Y and any $i \in I$, we let $R_i(E) = \mathrm{conv}(E \cup r_i(E)) \subset E + Q^\vee$ and that, for any pair $(\lambda, w) \in Y^+ \times W^v$, we set

$$R_w(\lambda) := \bigcup R_{i_1}(R_{i_2}(\dots(R_{i_k}(\{\lambda\})\dots)),$$

where the union is taken over all the reduced writings $r_{i_1} r_{i_2} \dots r_{i_k}$ of w. The next two results replace [AH19, Rem. 4.13] and act as preparation for the proof of Lemma 3.4 below, which replaces [AH19, Lem. 4.15].

Lemma 3.2. — *For any $(\lambda, w) \in Y^+ \times W^v$, we have*

$$R_w(\lambda) \subset \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w]\}),$$

where $[1, w] := \{u \in W^v \mid u \leqslant w\}$ *is defined as in* [AH19, bottom of p. 94].

Proof. — We prove this result by induction on $\ell(w)$. If $\ell(w) = 0$, there is nothing to prove, so let $w \in W^v$ be an element of length $\ell(w) \geqslant 1$ and assume by induction that the lemma holds for any element $w' \in W^v$ such that $\ell(w') < \ell(w)$. Let $\mu \in R_w(\lambda)$, then there exists $i \in I$ such that $w' := r_i w$ satisfies $w' < w$ and $\mu \in R_i(R_{w'}(\lambda)) = \mathrm{conv}(R_{w'}(\lambda), r_i R_{w'}(\lambda))$. As $\ell(w') < \ell(w)$, we have $R_{w'}(\lambda) \subset \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w']\})$ by induction hypothesis. Since [Kum02, Cor. 1.3.19] ensures that $\{1, r_i\} \cdot [1, w'] \subset [1, w]$, we obtain that

$$\mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w']\}) \cup r_i \cdot \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w']\}) \subset \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w]\}).$$

Consequently, we get that

$$\begin{aligned}\mu \in \mathrm{conv}(&R_{w'}(E), r_i \cdot R_{w'}(E)) \\ &\subset \mathrm{conv}(\mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w']\}) \cup r_i \cdot \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w']\})) \\ &\subset \mathrm{conv}(\mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w]\})) = \mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w]\}).\end{aligned}$$

This proves that $R_w(\lambda)$ is contained in $\mathrm{conv}(\{u \cdot \lambda \mid u \in [1, w]\})$, hence the lemma. □

Lemma 3.3. — *Let E be a W^v-almost finite subset of Y^+. Then, for any pair $(\nu, w) \in Y^+ \times W^v$, the set $\{\mu \in E \mid \nu \in R_w(\mu)\}$ is finite.*

Proof. — Let $E \subset Y^+$ and $(\nu, w) \in Y^+ \times W^v$ be as in the statement. Applying the definition of almost finiteness [AH19, Def. 4.3] to $u \cdot E$ for any $u \in [1, w]$ provides a finite set $F \subset Y^+$ such that :

$$\forall\, u \in [1, w],\ \forall\, \mu \in u \cdot E,\ \exists\, \lambda \in F \mid \mu \leqslant_{Q^\vee} \lambda.$$

Set $\mathcal{X} := \{\mu \in E \mid \nu \in R_w(\mu)\}$ and pick some $\mu \in \mathcal{X}$. As ν belongs to $R_w(\mu)$, Lemma 3.2 implies the existence of $(t_u)_{u \in [1,w]} \in [0, 1]^{[1,u]}$ such that

$$\sum_{u \in [1,w]} t_u = 1 \quad \text{and} \quad \nu = \sum_{u \in [1,w]} t_u u \cdot \mu.$$

For any $u \in [1,w]$, choose $\lambda(u) \in F$ such that $u \cdot \mu \leqslant_{Q^\vee} \lambda(u)$ and write $\lambda(u) - u \cdot \mu$ as $\sum_{i\in I} n_i(u)\alpha_i^\vee$ with $n_i(u) \in \mathbb{N}$ for all $i \in I$. Then we have:

$$\nu = \sum_{u\in[1,w]} t_u u \cdot \mu = \sum_{u\in[1,w]} t_u \lambda(u) - \sum_{\substack{u\in[1,w]\\ i\in I}} t_u n_i(u)\alpha_i^\vee.$$

Set $a(\mu) := \sum_{u\in[1,w]} t_u\lambda(u) \in \mathrm{conv}_{\mathbb{R}}(F)$ and $q(\mu) := \sum_{u\in[1,w],\ i\in I} t_u n_i(u)\alpha_i^\vee \in Q^\vee_{\mathbb{R},+}$. Since F is finite, $\mathrm{conv}_{\mathbb{R}}(F)$ is bounded. As $q(\mu) = a(\mu)-\nu$ lies in $\mathrm{conv}_{\mathbb{R}}(F)-\nu$, the set $\{q(\mu') \mid \mu' \in \mathfrak{X}\}$ is bounded too. Moreover, as $\sum_{u\in[1,w]} t_u = 1$, there exists $u' \in [1,w]$ such that $t_{u'} \geqslant 1/|[1,w]|$. Letting $f_j(x)$ denote the j-th coordinate of $x \in Q^\vee_{\mathbb{R}}$ in the basis $(\alpha_i^\vee)_{i\in I}$ for all $j \in J$, we have:

$$\forall\, i \in I, \quad f_i\left(q(\mu)\right) = \sum_{u\in[1,w]} t_u n_i(u) \geqslant t_{u'} n_i(u') \geqslant 0.$$

We hence obtain that:

$$\forall\, i \in I, \quad 0 \leqslant n_i(u') \leqslant \frac{\sup_{\mu'\in\mathfrak{X}} f_i\left(q(\mu')\right)}{t_{u'}} \leqslant \frac{\sup_{\mu'\in\mathfrak{X}} f_i\left(q(\mu')\right)}{|[1,u]|}.$$

Consequently, if we set $N := \max_{i\in I}(\sup_{\mu'\in\mathfrak{X}} f_i\left(q(\mu')\right)/|[1,u]|)$, then we have:

$$u' \cdot \mu \in \lambda(u') - \sum_{i\in I} [\![0,N]\!]\alpha_i^\vee \subset F - \sum_{i\in I} [\![0,N]\!]\alpha_i^\vee.$$

This proves that μ lies in $\bigcup_{u\in[1,w]} u^{-1}\cdot(F-\sum_{i\in I}[\![0,N]\!]\alpha_i^\vee)$, hence that $\mathfrak{X}$ is contained in the finite set $\bigcup_{u\in[1,w]} u^{-1}\cdot(F-\sum_{i\in I}[\![0,N]\!]\alpha_i^\vee)$, which proves that $\mathfrak{X}$ is finite too, as claimed. □

The next lemma replaces [AH19, Lem. 4.15]: the only modification consists in replacing $[1,w]w'$ by $[1,w']w$ in the aforementioned statement. In particular, the proof follows the exact same lines as [AH19, p. 96], hence we do not rewrite it here.

Lemma 3.4. — *For all $w, w' \in W^v$ and all $\lambda \in Y$, $H_{w'} * Z^\lambda H_w$ is in*

$$\bigoplus_{(\nu,t)\in R_{w'}(\lambda)\times[1,w']\cdot w} \mathscr{R}\cdot Z^\nu H_t.$$

Using the definitions of supp_{W^v} and supp_Y given by [AH19, Def. 4.11], one can straightforward deduce from Lemma 3.4 the following inclusions.

Lemma 3.5. — *For all $a, b \in \mathcal{H}$, we have:*

(1) $\mathrm{supp}_Y(a*b) \subset \mathrm{supp}_Y(a) + \bigcup_{\substack{w\in\mathrm{supp}_{W^v}(a)\\ \lambda\in\mathrm{supp}_Y(b)}} R_w(\lambda)$;

(2) $\mathrm{supp}_{W^v}(a*b) \subset \bigcup_{\substack{v\in\mathrm{supp}_{W^v}(a)\\ w\in\mathrm{supp}_{W^v}(b)}} [1,v]\cdot w$.

Before we give the definition of summable families in $\widetilde{\mathcal{H}}$, we prove two more statements related to W^v-almost finiteness in Y^+.

Lemma 3.6. — *For any almost finite set $E \subset Y^+$, $\mathrm{conv}(E)$ is also almost finite.*

Proof. — Let $E \subset Y^+$ be an almost finite set and let F be a finite set such that:

$$\forall \lambda \in E,\ \exists \mu \in F \mid \lambda \leqslant_{Q^\vee} \mu.$$

Given $\lambda \in \mathrm{conv}(E)$, there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [0,1]$ and $\lambda_1, \ldots, \lambda_n \in E$ such that

$$\sum_{i=1}^{n} t_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} t_i \lambda_i = \lambda.$$

For each index $i \in [\![1, n]\!]$, choose $\kappa_i \in F$ such that $\lambda_i \leqslant_{Q^\vee} \kappa_i$: then

$$\sum_{i=1}^{n} t_i \kappa_i - \sum_{i=1}^{n} t_i \lambda_i \in \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^\vee.$$

We can hence write $\sum_{i=1}^{n} t_i \kappa_i - \sum_{i=1}^{n} t_i \lambda_i = \sum_{i \in I} x_i \alpha_i^\vee$ for some nonnegative real numbers $(x_i)_{i \in I}$. Now let $(x_i') \in [0,1[^I$ be such that $x_i + x_i'$ lies in $\mathbb{N}$ for all $i \in I$ and set $\nu := \sum_{i \in I} (x_i + x_i') \alpha_i^\vee + \lambda$. Then we have

$$\nu \geqslant_{Q^\vee} \lambda \quad \text{and} \quad \nu \in (\mathrm{conv}_{\mathbb{R}}(F) + \textstyle\bigoplus_{i \in I} [0,1] \alpha_i^\vee) \cap Y.$$

Since F is finite, $(\mathrm{conv}_{\mathbb{R}}(F) + \bigoplus_{i \in I} [0,1] \alpha_i^\vee) \cap Y$ is a finite set that can be taken as J in [AH19, Def. 4.3] for $\mathrm{conv}(E)$, and the lemma is proved. □

Lemma 3.7. — *Let E be a W^v-almost finite subset of Y^+. Then, for any $w \in W^v$, the set $\bigcup_{\lambda \in E} R_w(\lambda)$ is W^v-almost finite.*

Proof. — Let $w \in W^v$. By Lemma 3.2, we have

$$\bigcup_{\lambda \in E} R_w(\lambda) \subset \bigcup_{\lambda \in E} \mathrm{conv}(\{u \cdot \lambda,\ u \in [1, w]\}) \subset \mathrm{conv}\Big(\bigcup_{u \in [1,w]} u \cdot E\Big).$$

Let $v \in W^v$. Since $[1, w]$ is finite, the set $v \cdot \bigcup_{u \in [1,w]} u \cdot E$ is almost finite, hence Lemma 3.6 implies that $v \cdot \mathrm{conv}(\bigcup_{u \in [1,w]} u \cdot E) = \mathrm{conv}(v \cdot \bigcup_{u \in [1,w]} u \cdot E)$ is almost finite. This proves that $\mathrm{conv}(\bigcup_{u \in [1,w]} u \cdot E)$ is v-almost finite for any $v \in W^v$, and the lemma is proved. □

3.2. $\widetilde{\mathcal{H}}$ is an associative algebra. — This subsection contains the main modification of the paper, as it aims to prove that $\widetilde{\mathcal{H}}$ is actually an associative algebra. To do this, we first need to introduce the correct definition of summable families, which is the counterpart of [AH19, Def. 4.20] for W^v-almost finite sets.

Definition 3.8. — A family $(a_j)_{j \in J} \in (\widetilde{\mathcal{H}})^J$ is *summable in* $\widetilde{\mathcal{H}}$ when the two following properties hold:

(i) for any $\lambda \in Y^+$, the set $\{j \in J \mid \exists w \in W^v,\ \pi_{\lambda,w}(a_j) \neq 0\}$ is finite;

(ii) the set $\bigcup_{j \in J} \mathrm{supp}(a_j) := \bigcup_{j \in J} \{(\lambda, w) \in Y^+ \times W^v \mid \pi_{\lambda,w}(a_j) \neq 0\}$ is W^v-almost finite.

Given a summable family $(a_j)_{j \in J} \in (\widetilde{\mathcal{H}})^J$, we define $\sum_{j \in J} a_j \in \widetilde{\mathcal{H}}$ by the following formula:

$$\sum_{j \in J} a_j := \sum_{(\lambda,w) \in Y^+ \times W^v} a_{\lambda,w} Z^\lambda H_w, \text{ with } a_{\lambda,w} := \sum_{j \in J} \pi_{\lambda,w}(a_j) \text{ for all } (\lambda, w) \in Y^+ \times W^v.$$

Lemma 3.9. — *For any almost finite subsets E, E' of Y^+ and any $\rho \in Y^+$, the set*

$$E'' := \{\lambda \in E \mid \exists \nu \in E',\ \lambda + \nu = \rho\}$$

is finite.

Proof. — By definition, there exists a finite set $F \subset Y^+$ such that:

$$\forall \mu \in E \cup E',\ \exists \kappa \in F \mid \mu \leqslant_{Q^\vee} \kappa.$$

Now let $\lambda \in E''$ and $\nu \in E'$ be such that $\lambda + \nu = \rho$. Then we have:

$$\exists (\kappa, \kappa') \in F^2 \mid \rho - \kappa' \leqslant_{Q^\vee} \lambda \leqslant_{Q^\vee} \kappa,$$

and the lemma follows. □

The next lemma is the cornerstone that ensures that the convolution product on $\widetilde{\mathcal{H}}$ is well-defined.

Lemma 3.10. — *Let $(a_\lambda), (b_\mu) \in \mathscr{R}^{Y^+}$ be such that $\{\lambda \in Y^+ \mid a_\lambda \neq 0$ or $b_\lambda \neq 0\}$ is W^v-almost finite. Then, for any $w \in W^v$, $(a_\lambda b_\mu Z^\lambda H_w * Z_\mu)_{(\lambda,\mu)\in(Y^+)^2}$ is summable in $\widetilde{\mathcal{H}}$. Said differently, if $\sum_{\lambda \in Y^+} a_\lambda Z^\lambda H_w, \sum_{\mu \in Y^+} b_\mu Z^\mu \in \widetilde{\mathcal{H}}$, then*

$$\Big(\sum_{\lambda \in Y^+} a_\lambda Z^\lambda H_w\Big) * \Big(\sum_{\mu \in Y^+} b_\mu Z^\mu\Big) := \sum_{\lambda,\mu \in Y^+} a_\lambda b_\mu Z^\lambda H_w * Z^\mu$$

is a well-defined element of $\widetilde{\mathcal{H}}$.

Proof. — Set

$$S_a := \{\lambda \in Y^+ \mid a_\lambda \neq 0\}, \quad S_b = \{\mu \in Y^+ \mid b_\mu \neq 0\} \quad \text{and} \quad E = \bigcup_{\mu \in S_b} R_w(\mu).$$

Note that E is almost finite by Lemma 3.7. Given $\mu \in Y^+$, Lemma 3.4 ensures the existence of $(z_\mu^{v,\nu})_{\nu \in R_w(\mu), v \in [1,w]} \in \mathscr{R}^{R_w(\mu)\times[1,w]}$ such that

$$H_w * Z^\mu = \sum_{\nu \in R_w(\mu), v \in [1,w]} z_\mu^{v,\nu} Z^\nu H_v.$$

Let us fix $v \in [1, w]$. Given any $\lambda, \rho \in Y^+$, we have

$$\pi_{\rho,v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) = \sum_{\nu \in R_w(\mu) \mid \lambda+\nu=\rho} a_\lambda b_\mu z_\mu^{v,\nu}. \tag{3.1}$$

Set $F_1 := \big\{\lambda \in Y^+ \mid \exists \mu \in Y^+, \pi_{\rho,v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\big\} \subset S_a$ and let $\lambda \in F_1$. By equality (3.1), there exists $\nu \in E$ such that $\lambda + \nu = \rho$. Since λ lies in S_a, applying Lemma 3.9 to E and S_a implies that F_1 is finite.

Fix now $\lambda \in F_1$ and set $F_2(\lambda) = \{\mu \in Y^+ \mid \pi_{\rho,v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\} \subset S_b$. Given $\mu \in F_2(\lambda)$, we know from equality (3.1) that $\rho - \lambda \in R_w(\mu)$. As S_b is W^v-almost finite, Lemma 3.3 yields the finiteness of $F_2(\lambda)$, hence the finiteness of

$$F_v := \{(\lambda, \mu) \in (Y^+)^2 \mid \pi_{\rho,v}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu) \neq 0\} = \bigcup_{\lambda \in F_1} F_2(\lambda).$$

Finally, we obtain that

$$\{(\lambda,\mu)\in(Y^+)^2 \mid \exists\, v\in W^v, \pi_{\rho,v}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)\neq 0\} = \bigcup_{v\in[1,w]} F_v$$

is finite, which proves that $(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)_{(\lambda,\mu)\in(Y^+)^2}$ satisfies condition (i) of Definition 3.8.

Now let $\lambda,\mu\in Y^+$. Then equality (3.1) ensures that:

$$\operatorname{supp}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)\subset(\lambda+R_w(\mu))\times[1,w]\subset(\lambda+E)\times[1,w].$$

In particular, we have

$$\bigcup_{(\lambda,\mu)\in(Y^+)^2}\operatorname{supp}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)\subset(S_a+E)\times[1,w].$$

As E is W^v-almost finite, and as the sum of two W^v-almost finite sets is W^v-almost finite, we obtain that $(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)_{(\lambda,\mu)\in(Y^+)^2}$ satisfies condition (ii) of Definition 3.8, which ends the proof. □

Lemma 3.11. — *For any summable family $(a_j)_{j\in J}\in(\widetilde{\mathcal{H}})^J$ in $\widetilde{\mathcal{H}}$ and any $i\in I$, the family $(a_j * H_i)_{j\in J}$ is summable in $\widetilde{\mathcal{H}}$.*

Proof. — By definition, there exist a W^v-almost finite subset E of Y^+ and a finite subset F of W^v such that: $\forall\, j\in J,\ \operatorname{supp}(a_j)\subset E\times F$. By (BL2) of [AH19, p. 91], we get that

$$\forall\, j\in J,\quad \operatorname{supp}(a_j * H_i)\subset E\times(F\cup F\cdot r_i),$$

which proves that $\bigcup_{j\in J}\operatorname{supp}(a_j * H_i)$ is W^v-almost finite. Now let $\lambda\in Y^+$, $w\in W^v$ and $j\in J$. By (BL2) again, we know that $\pi_{\lambda,w}(a_j * H_i)\neq 0$ implies $\pi_{\lambda,w}(a_j)\neq 0$ or $\pi_{\lambda,wr_i}(a_j)\neq 0$. Since there are finitely many such $j\in J$, this completes the proof of the lemma. □

The next statement replaces [AH19, Th. 4.21] and its proof basically follows the same lines as the proof of [AH19, Th. 4.21], replacing almost finiteness by W^v-almost finiteness. Recall that the elements of $\mathcal{H}$ correspond to the elements of $\widetilde{\mathcal{H}}$ with finite support.

Theorem 3.12. — *Let $(a_j)_{j\in J}\in(\mathcal{H})^J$ and $(b_k)_{k\in J}\in(\mathcal{H})^K$ be two families that are both summable in $\widetilde{\mathcal{H}}$. Then $(a_j * b_k)_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$ and $\sum_{(j,k)\in J\times K} a_j * b_k$ only depends on the two elements $\sum_{j\in J} a_j$ and $\sum_{k\in K} b_k$ of $\widetilde{\mathcal{H}}$.*

Proof. — For $j\in J$ and $k\in K$, write

$$a_j := \sum_{v\in W^v} a_{v,j} * H_v \quad\text{and}\quad b_k := \sum_{w\in W^v} b_{w,k} * H_w,$$

with $(a_{v,j})_{j\in J}\in\mathscr{R}[\![Y]\!]^J$ and $(b_{w,k})_{k\in K}\in\mathscr{R}[\![Y]\!]^K$ for any $v,w\in W^v$. Given $v,w\in W^v$, Lemma 3.10 ensures that $(a_{v,j} * H_v * b_{w,k})_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$. By induction on $\ell(w)$ and using Lemma 3.11, we get that $(a_{v,j} * H_v * b_{w,k} * H_w)_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$. Moreover, as (a_j) and (b_k) are summable in $\widetilde{\mathcal{H}}$, there are at most finitely many

$v,w\in W^v$ satisfying $(a_{v,j})_{j\in J}\neq 0$ and $(b_{w,k})_{k\in K}\neq 0$. Consequently, the family $(a_j*b_k)_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$.

Now, given any triple $(u,v,\mu)\in W^v\times W^v\times Y^+$, applying Lemma 3.4 to $H_u*Z^\mu H_v$ gives a family $(z^{u,v,\mu}_{\nu,t})_{(\nu,t)\in R_u(\mu)\times[1,u].v}$ of scalars that satisfy

$$H_u*Z^\mu H_v=\sum_{(\nu,t)\in R_u(\mu)\times[1,u]\cdot v} z^{u,v,\mu}_{\nu,t}Z^\nu H_t.$$

For any pair $(\rho,s)\in Y^+\times W^v$, we have

$$\begin{aligned}\pi_{\rho,s}\Big(\sum_{(j,k)\in J\times K} a_j*b_k\Big)&=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{\nu\in R_u(\mu)\mid\lambda+\nu=\rho}\ \sum_{(j,k)\in J\times K} a_{j,\lambda,u}b_{k,\mu,v}z^{u,v,\mu}_{\nu,s}\\&=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{\nu\in R_u(\mu)\mid\lambda+\nu=\rho} a_{\lambda,u}b_{\mu,v}z^{u,v,\mu}_{v,s},\end{aligned}$$

where we set

$$\sum_{j\in J}a_j=\sum_{(\lambda,u)\in Y^+\times W^v}a_{\lambda,u}Z^\lambda H_u\quad\text{and}\quad\sum_{k\in K}b_k=\sum_{(\mu,v)\in Y^+\times W^v}b_{\mu,v}Z^\mu H_v,$$

hence the theorem is proved. □

The mistake done in the former proof of [AH19, Th. 4.20] is to implicitly assume that $S_Y=\bigcup_{j\in J}\mathrm{supp}_Y(a_j)\cup\bigcup_{k\in K}\mathrm{supp}_Y(b_k)$ is such that $\{\lambda^{++}\mid\lambda\in S_Y\}$ is almost finite, which is not true in general, as shown by the counter-examples given in Section 2. As spotted by the referee, the same kind of subtlety underlies a mistake made by Looijenga in his seminal 1980 work [Loo80, (4.1), end of the first paragraph].

For $a=\sum_{(\lambda,v)\in Y^+\times W^v}a_{\lambda,v}Z^\lambda H_v\in\widetilde{\mathcal{H}}$ and $b=\sum_{(\mu,w)\in Y^+\times W^v}b_{\mu,w}Z^\mu H_w\in\widetilde{\mathcal{H}}$, we set

$$a*b=\sum_{(v,\lambda),(w,\mu)\in Y^+\times W^v}a_{\lambda,v}b_{\mu,w}Z^\lambda H_v*Z^\mu H_w,$$

which is well-defined by Theorem 3.12. We can now formulate the statement that replaces [AH19, Cor. 4.23], providing the required structure on $\widetilde{\mathcal{H}}$. Its proof is the same as [AH19, Cor. 4.23], replacing [AH19, Th. 4.21] by Theorem 3.12 above.

Corollary 3.13. — *The convolution product $*$ equips $\widetilde{\mathcal{H}}$ with the structure of an associative algebra over $\mathscr{R}$ that contains $\mathcal{H}$ as subspace of finitely supported elements.*

3.3. The center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathcal{H}_s$.

— Recall that the definition of the Looijenga algebra $\mathscr{R}[\![Y]\!]$ and its variants $\mathscr{R}[\![Y]\!]^{W^v}$ and $\mathscr{R}[\![Y^+]\!]$ is given by [AH19, Def. 4.6]. Also, we proved in [AH19, Prop. 4.9] that $\mathscr{R}[\![Y]\!]^{W^v}$ is a subspace of $\mathscr{R}[\![Y^+]\!]$. Now note that the latter can be seen as a subspace of $\mathscr{B}$, so it makes sense to compare these algebras with the algebra $\widetilde{\mathcal{H}}$ we built earlier.

Given $a\in\mathscr{R}[\![Y]\!]^{W^v}$, we have $\mathrm{supp}(a)=\mathrm{supp}_Y(a)\times\{1\}$, with $\mathrm{supp}_Y(a)$ being W^v-invariant and almost finite, hence $\mathrm{supp}(a)$ is W^v-almost finite. In particular, this implies that $\mathscr{R}[\![Y]\!]^{W^v}$ is contained in $\widetilde{\mathcal{H}}$. However, note that in general, $\mathscr{R}[\![Y]\!]$ may not be entirely contained in $\widetilde{\mathcal{H}}$, as can be seen for instance in Example 2.2.

Replacing $\widehat{\mathcal{H}}$ by $\widetilde{\mathcal{H}}$ in the proof of [AH19, Th. 4.30] provides the following theorem, which replaces [AH19, Th. 4.30].

Theorem 3.14. — *The center of the algebra $\widetilde{\mathcal{H}}$ is $\mathscr{Z}(\widetilde{\mathcal{H}}) = \mathscr{R}[\![Y]\!]^{W^v}$, hence is isomorphic to $\mathcal{H}_s$ via the Satake isomorphism.*

Remark 3.15. — By [AH19, Lem. 4.5], we know that if $\mathbb{A}$ is associated with an indefinite size 2 Kac-Moody matrix, then any subset of Y^+ is almost finite, hence any subset of Y^+ is W^v-almost finite.

3.4. The reductive case

Lemma 3.16. — *Assume that $\mathbb{A}$ is associated with a Cartan matrix A. Then a subset of $Y^+ = Y$ is W^v-almost finite if, and only if, it is finite.*

Proof. — Thanks to [AH19, Lem. 5.17], we may assume that $\bigcap_{i\in I} \ker \alpha_i = \{0\}$. Let $A_1, \ldots, A_r$ denote the indecomposable components of A: then $\mathbb{A} = \bigoplus_{i=1}^r \mathbb{A}_i$, where $\mathbb{A}_i$ is a realization of A_i (as defined in [AH19, 5.4.1]) for all $i \in [\![1, r]\!]$. For $i \in [\![1, r]\!]$, we denote by W_i^v the Weyl group of $\mathbb{A}_i$, by $Q_i^\vee$ its coroot lattice and by Y_i its cocharacter lattice, so that we have $W^v = W_1^v \times \cdots \times W_r^v$, $Q^\vee = \bigoplus_{i=1}^r Q_i^\vee$ and $Y = \bigoplus_{i=1}^r Y_i$.

Now let E be a W^v-almost finite subset of Y. For $w \in W^v$, set

$$E_w := E \cap w \cdot \overline{C_f^v}.$$

Since E is w^{-1}-almost finite, there exists a finite set $F \subset Y$ such that:

$$\forall\, \lambda \in E,\ \exists\, \mu \in F \mid \lambda \leqslant_{Q^\vee} \mu.$$

For $i \in [\![1, r]\!]$, set $Y_i^{++} := Y_i \cap \overline{C_{f,i}^v}$, where $C_{f,i}^v$ denotes the fundamental chamber of $\mathbb{A}_i$: then we know from [Kac90, Th. 4.3] that $Y_i^{++} \subset Q_{i,+}^\vee$. Given $\lambda = (\lambda_1, \ldots, \lambda_r) \in E_w$, let $\mu = (\mu_1, \ldots, \mu_r) \in F$ be such that $\lambda \leqslant_{Q^\vee} \mu$. Then we have $0 \leqslant_{Q_i^\vee} \lambda_i \leqslant_{Q_i^\vee} \mu_i$ for all $i \in [\![1, r]\!]$, hence E_w must be finite. As W^v is finite, we get that E is finite too, and the lemma is proved as the converse statement is straightforward. □

Since $\mathcal{H}$ corresponds to the subspace of elements of $\widetilde{\mathcal{H}}$ with finite support, we directly obtain the following result from Lemma 3.16, which states that $\widetilde{\mathcal{H}}$ is just the usual Iwahori-Hecke algebra in the reductive case. This replaces the first paragraph of [AH19, §4.6.1, p. 105].

Proposition 3.17. — *If $\mathbb{A}$ is associated with a Cartan matrix, then $\widetilde{\mathcal{H}} = \mathcal{H}$.*

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