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QUANTITATIVE DE GIORGI METHODS IN KINETIC THEORY

by Jessica Guerand & Clément Mouhot

Abstract. — We consider hypoelliptic equations of kinetic Fokker-Planck type, also known as Kolmogorov or ultraparabolic equations, with rough coefficients in the drift-diffusion operator. We give novel short quantitative proofs of the De Giorgi intermediate-value Lemma as well as weak Harnack and Harnack inequalities. This implies Hölder continuity with quantitative estimates. The paper is self-contained.

Résumé (Méthodes à la De Giorgi quantitatives en théorie cinétique). — Nous considérons des équations hypoelliptiques de type Fokker-Planck cinétique, également appelées équations de Kolmogorov ou ultraparaboliques, avec des coefficients sans régularité dans l'opérateur de dérive-diffusion. Nous donnons de nouvelles preuves quantitatives du lemme des valeurs intermédiaires de De Giorgi ainsi que des inégalités de Harnack faibles et fortes. Cela implique la continuité höldérienne avec bornes explicites. L'article ne fait pas appel à des résultats précédents.

Contents

1. Introduction

1.1. The problem studied. — This paper is concerned with local regularity properties, namely boundedness, Harnack inequalities and Hölder continuity, of solutions $f = f(t, x, v)$ to the following class of hypoelliptic partial differential equations in divergence form

$$(1.1)\qquad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d,$$

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where $A = A(t, x, v)$, $B = B(t, x, v)$ and $S = S(t, x, v)$ satisfy (for some constants $0 < \lambda < \Lambda$):

$$\begin{cases} A \text{ is a measurable symmetric real matrix field with eigenvalues in } [\lambda, \Lambda], \\ B \text{ is a measurable vector field such that } |B| \leqslant \Lambda, \\ S \text{ is a real scalar field in } L^\infty. \end{cases} \tag{1.2}$$

This equation naturally appears in kinetic theory where it is refereed to as the *kinetic Fokker-Planck equation*; it is included in the class considered by Kolmogorov [Kol34] (with constant A and linear B) that inspired the theory of hypoellipticity of Hörmander [Hör67] (see [AP20]). The coefficients are called "rough" because A, B and S in the drift-diffusion operator on the v variable are merely measurable with no further regularity.

Our class (1.1)–(1.2) is invariant under translations in t, x and under *Galilean translations*, i.e., under $z \mapsto z_0 \circ z$ where $z_0 = (t_0, x_0, v_0)$, $z = (t, x, v)$ and with the non-commutative group operation

$$z_0 \circ z = (t_0 + t, x_0 + x + tv_0, v_0 + v).$$

Finally for any $r > 0$ it is invariant under the scaling $z = (t, x, v) \mapsto rz := (r^2 t, r^3 x, rv)$. Using the invariances of the equation, we define for $z_0 \in \mathbb{R}^{1+2d}$ and $r > 0$:

$$Q_r(z_0) := z_0 \circ [rQ_1] := \left\{ -r^2 < t - t_0 \leqslant 0,\ |x - x_0 - (t - t_0)v_0| < r^3,\ |v - v_0| < r \right\}$$

and we simply write $Q_r(0) = Q_r$ when $z_0 = 0$. We denote $|E|$ the Lebesgue measure of a Lebesgue set E. We write $a \lesssim b$ (resp. $a \gtrsim b$) when $a \leqslant Cb$ (resp. $a \geqslant Cb$) for some constant $C > 0$ whose only relevant dependency, if any, is specified in the index, as in $\lesssim_{\text{parameter}}$. We write $a \sim b$ if $a \lesssim b$ and $a \gtrsim b$. We write $\fint$ for integrals normalized by the volume of the integration domain, and $\mathscr{T} := \partial_t + v \cdot \nabla_x$.

Definition 1 (Weak solution, sub-solution, super-solution)
Let $\mathscr{U} = (a, b) \times \Omega_x \times \Omega_v$ with $-\infty < a < b \leqslant +\infty$ and Ω_x and Ω_v two open sets of $\mathbb{R}^d$. A function $f : \mathscr{U} \to \mathbb{R}$ is a *weak solution* of (1.1) on $\mathscr{U}$ if

$$f \in L^\infty((a, b); L^2(\Omega_x \times \Omega_v)) \cap L^2((a, b) \times \Omega_x; H^1(\Omega_v))$$

and (1.1) is satisfied in the sense of distributions in $\mathscr{U}$. A function f is a *weak sub-solution* of (1.1) if

$$f \in L^\infty((a, b); L^2(\Omega_x \times \Omega_v)) \cap L^2((a, b) \times \Omega_x; H^1(\Omega_v))$$

and for all $\beta : \mathbb{R} \to \mathbb{R}$ in C^2 with $\beta' \geqslant 0$ and $\beta'' \geqslant 0$ both bounded, and any non-negative $\varphi \in C_c^\infty(\mathscr{U})$,

$$-\int_{\mathscr{U}} \beta(f) \mathscr{T} \varphi \, \mathrm{d}z \leqslant -\int_{\mathscr{U}} A \nabla_v \beta(f) \cdot \nabla_v \varphi \, \mathrm{d}z + \int_{\mathscr{U}} \left[B \cdot \nabla_v \beta(f) + S \beta'(f) \right] \varphi \, \mathrm{d}z.$$

It is a *weak super-solution* of (1.1) if $-f$ is a weak sub-solution.

Remark 2. — This definition is equivalent to those in [PP04] and [GIMV19] in the case of solutions, but is weaker than them in the case of sub- and super-solutions. Indeed [PP04, GIMV19] make respectively the extra regularity assumption $\mathscr{T}f \in L^2((a,b)\times\Omega_x\times\Omega_v)$ or $\mathscr{T}f \in L^2((a,b)\times\Omega_x;H^{-1}(\Omega_v))$. These assumptions were introduced to justify the energy estimates. It is however enough to assume the renormalization formulation above, and it allows to include important sub-solutions such as for instance $f=f(t)=\mathbf{1}_{t\leqslant 0}$ (when $S=0$) which were excluded by the definition in [PP04, GIMV19]. Our definition is equivalent to that of De Giorgi in the elliptic case (and reminiscent of the definition of solutions in [GV15]).

1.2. Main contributions. — Given the invariances, we only state results in unit centered cylinders.

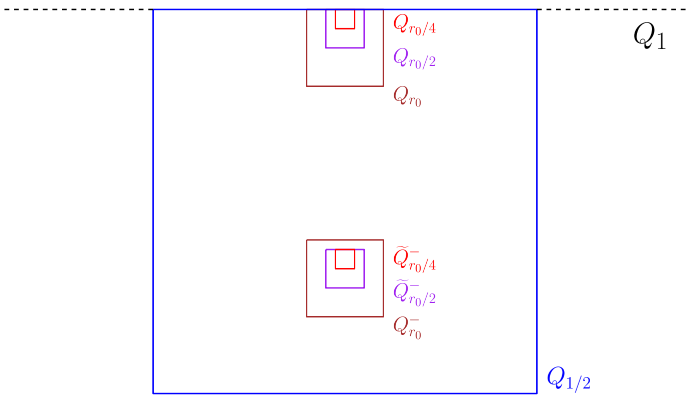

Figure 1. The different cylinders in the Intermediate-Value Lemma and Harnack inequalities.

Theorem 3 (Intermediate-Value Lemma). — *Given* $\delta_1,\delta_2\in(0,1)$, *there are explicit constants* $r_0\sim\sqrt{\delta_1}/\sqrt{1+\|S\|_{L^\infty(Q_1)}}$ *in* $(0,1/20)$ *if* $S\neq 0$ *and* $r_0=1/20$ *if* $S=0$, *and* $\theta\sim(\delta_1\delta_2)^{10d+15}/(1+\|S\|_{L^\infty(Q_1)})^{4d+3}$ *and* $\nu\gtrsim(\delta_1\delta_2)^{10d+16}/(1+\|S\|_{L^\infty(Q_1)})^{2d+1}$ *both in* $(0,1)$, *such that any sub-solution* $f:Q_1\to\mathbb{R}$ *to* (1.1)–(1.2) *so that* $f\leqslant 1$ *in* $Q_{1/2}$ *and*

$$|\{f\leqslant 0\}\cap Q^-_{r_0}|\geqslant\delta_1|Q^-_{r_0}|\quad\text{and}\quad|\{f\geqslant 1-\theta\}\cap Q_{r_0}|\geqslant\delta_2|Q_{r_0}|, \tag{1.3}$$

i.e., we control the measure of where f *is below* 0 *and above* $(1-\theta)$, *satisfies*

$$\big|\{0<f<1-\theta\}\cap Q_{1/2}\big|\geqslant\nu|Q_{1/2}|, \tag{1.4}$$

where $Q^-_{r_0}:=Q_{r_0}(-2r_0^2,0,0)=(-3r_0^2,-2r_0^2]\times B_{r_0^3}\times B_{r_0}$ *(see Figure 1).*

Remark 4. — This lemma is the kinetic quantitative counterpart of the quantitative elliptic [DG56, DG57, Vas16] and parabolic [Gue20] intermediate value lemma. As in the parabolic case, past and a future cylinders $Q^-_{r_0}$ and $Q^+_{r_0}$ are required to be disjoint but contrary to the parabolic case, a gap in time between the two cylinders is also required. This gap is also mentioned in [GIMV19, AP20]. Let us explain why it cannot be removed. Consider for instance $S=0$ and velocities bounded by $|v| \leqslant V_m$ in the cylinder. Then $\mathbf{1}_{x+ct<a}$ is a sub-solution for any $a \in \mathbb{R}$ and $|c| > V_m$. If $Q^-_{r_0}$ and Q_{r_0} were too close, a line of discontinuity of the form $x+ct=a$ could cross both and the previous sub-solution would contradict the conclusion of Theorem 3.

Theorem 5 (Harnack inequalities). — *There is $\zeta > 0$ depending only λ, Λ such that any non-negative weak super-solution f to* (1.1)–(1.2) *in Q_1 satisfies the weak Harnack inequality*

$$\left(\int_{\widetilde{Q}^-_{r_0/2}} f^\zeta(z)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\right)^{1/\zeta} \lesssim_{\lambda,\Lambda} \inf_{Q_{r_0/2}} f + \|S\|_{L^\infty(Q_1)}, \tag{1.5}$$

where $r_0 = 1/20$ and $\widetilde{Q}^-_{r_0/2} := Q_{r_0/2}((-\frac{19}{8}\,r_0^2, 0, 0))$ (see Figure 1*), and any non-negative weak solution f to* (1.1)–(1.2) *satisfies the following Harnack inequality (with $\widetilde{Q}^-_{r_0/4} := Q_{r_0/4}((-\frac{19}{8}\,r_0^2, 0, 0))$)*

$$\sup_{\widetilde{Q}^-_{r_0/4}} f \lesssim_{\lambda,\Lambda} \inf_{Q_{r_0/4}} f + \|S\|_{L^\infty(Q_1)}. \tag{1.6}$$

Remarks 6

(1) The "weak" Harnack inequality, in spite of its name, is not weaker than Harnack inequality since it holds for super-solutions. Combined with the $L^\zeta \to L^\infty$ gain of integrability in Proposition 12, it implies the Harnack inequality for solutions. Super-solutions of the form $\mathbf{1}_{x+ct\geqslant a}$ for $a \in \mathbb{R}$ and $|c| > V_m$ (included in our definition) show that the gap in time is required in (1.5).

(2) The Harnack inequality for equation (1.1) was first proved in [GIMV19] by a non-constructive argument. The present paper provides a new constructive De Giorgi approach. Another constructive proof by the Moser-Kružkov approach is proposed in [GI21]. The weak Harnack inequality was obtained for the long-range Boltzmann equation in [IS20], and was proved for the kinetic Fokker-Planck equations considered in this paper in [GI21] by the Moser-Kružkov approach.

(3) As compared to that in [GI21], our approach is based on trajectorial arguments and does not require working on the logarithm of the solution or the so-called inkspot lemma. Our Poincaré inequality (Proposition 13) and measure-to-pointwise estimate (Lemma 16) take into account a gap in time which removes the requirement for the sub-solution to be considered in a large domain. Our Poincaré inequality also holds without an information in measure around the center of the cylinder as in [GI21].

Theorem 7 (Hölder continuity). — *There is $\alpha \in (0,1)$, computable from the proof and only depending on λ, Λ and $\|S\|_{L^\infty}$, such that any weak solution f of* (1.1)–(1.2) *in Q_2 satisfies*

$$\begin{aligned}[f]_{C^\alpha(Q_1)} &:= \sup_{\substack{z_1,z_2\in Q_1\\ z_1\neq z_2}} \frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha} \\ &\lesssim_{\lambda,\Lambda} \big(1+\|S\|_{L^\infty(Q_2)}\big)\big(\|f\|_{L^2(Q_2)} + \|S\|_{L^\infty(Q_2)}\big).\end{aligned}$$

Remark 8. — The Hölder continuity was first proved in [WZ09, WZ11] (with constructive method) and this proof is revisited and simplified in [GI21], including ideas and methods from [Mos64, Kru63, Kru64]. An alternative non-constructive proof was proposed in [GIMV19] following the De Giorgi method [DG56]: the non-constructive part was the intermediate-value lemma and we provide here a new constructive argument.

1.3. Structure of the method. — The core of our proof is, given f sub-solution to (1.1)–(1.2) with $S = 0$:

$$\begin{aligned} f \in L^{\zeta},\ \zeta > 0 \quad &\xrightarrow{(1)} \quad f \in L^\infty \cap L^1_{t,v} W^{1/3-0,1}_x \\ &\xrightarrow{(2)} \quad \text{Weak Poincaré inequality in } L^1 \\ &\xrightarrow{(3)} \quad \text{Intermediate-Value Lemma (Theorem 3)} \\ &\xrightarrow{(4)} \quad \text{Measure-to-pointwise estimate} \\ &\xrightarrow{(5)} \quad \text{Weak log-Harnack estimate} \\ &\xrightarrow{(6)} \quad \text{Weak Harnack estimate.} \end{aligned}$$

Once these steps are proved, it is immediate to prove the Harnack inequality for solutions by combining the weak Harnack inequality for super-solutions and step (1) for sub-solutions. The Hölder continuity follows classically (see Subsection 4.2) from either the measure-to-pointwise estimate applied to both sub-solutions f and $-f$, or from the Harnack inequality. Step (1) (Section 2) is semi-novel: it elaborates upon ideas in [PP04] to prove the first Lemma of De Giorgi as well as a gain of Sobolev regularity with the help of Kolmogorov fundamental solutions. Step (2) (Proposition 13) is the most novel step and introduces an argument based on trajectories and the previous Sobolev regularity to "noise" the x-dependency of the trajectories. Step (3) (proof in Subsection 3.2) is novel and based on simple energy estimates. Step (4) (Lemma 16 in Subsection 3.3) is standard and sketched for the sake of obtaining quantitative constants. Step (5) (in Section 4) is semi-novel but immediate when constants are quantified properly in the previous steps. Step (6) (in Section 4) is novel in the context of hypoelliptic equations but inspired from elliptic equations [LZ17]; it uses an induction, Vitali's covering lemma and Step (5) at every scale.

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2. Integral estimates revisited

In this section, we briefly revisit estimates from [PP04, GIMV19] on the gain of integrability for sub-solutions (the kinetic counterpart to the first lemma of De Giorgi) and the low-order Sobolev regularity estimate for sub-solutions, first mentioned in [GIMV19]. We provide new proofs based on fundamental solutions which, albeit variants of existing ones, seem simpler and optimal.

2.1. The energy estimate

Proposition 9 (Energy estimate). — *Let f be a non-negative weak sub-solution to* (1.1)–(1.2) *in an open set $\mathscr{U} \in \mathbb{R}^{1+2d}$. Given any $Q_r(z_0) \subset Q_R(z_0) \subset \mathscr{U}$ with $0 < r < R$, one has*

$$\int_{Q_r(z_0)} |\nabla_v f|^2 \lesssim_{\lambda,\Lambda} \mathscr{C}(r,R,v_0)\Big(\int_{Q_R(z_0)} f^2 + \int_{Q_R(z_0)} f|S|\Big),$$

where $z_0 = (t_0, x_0, v_0)$, $Q_r^\tau(z_0) = \{(x,v) \in \mathbb{R}^{2d} : (\tau, x, v) \in Q_r(z_0)\}$, and

$$\mathscr{C}(r,R,v_0) := \Big(1 + \frac{1}{(R-r)^2} + \frac{|v_0| + R}{(R-r)r^2} + \frac{1}{(R-r)r}\Big). \tag{2.1}$$

Proof of Proposition 9. — Consider φ a smooth function valued in $[0,1]$ that is equal to 1 on $Q_r(z_0)$ and 0 outside $Q_R(z_0)$. In order to use $f\varphi^2$ as a test function, we argue by density. Introduce

$$\psi_n * [f\varphi]\varphi(z) := \int_{z' \in \mathbb{R}^{2d+1}} \psi_n\left(t - t', x - tv - (x' - t'v'), v - v'\right) f(z')\varphi(z')\varphi(z),$$

where $\psi_n(t,x,v) = n^{4d+2}\psi(n^2 t, n^3 x, nv)$ and $\psi(t,x,v) := \pi^{-d-1/2} e^{-t^2-|x|^2-|v|^2}$. Then

$$\begin{aligned} I_n := \big\langle \mathscr{T} f, \psi_n * [f\varphi]\varphi \big\rangle &= \big\langle f, \psi_n * [(\mathscr{T} f)\varphi]\varphi \big\rangle \\ &= \frac{1}{2}\left[\big\langle \mathscr{T} f, \psi_n * [f\varphi]\varphi \big\rangle + \big\langle f, \psi_n * [(\mathscr{T} f)\varphi]\varphi \big\rangle\right] \\ &= \frac{1}{2}\left[-\big\langle f, \psi_n * [f\varphi](\mathscr{T}\varphi) \big\rangle - \big\langle f, \psi_n * [f(\mathscr{T}\varphi)]\varphi \big\rangle\right], \end{aligned}$$

which converges to $-\frac{1}{2}\big\langle f^2, \mathscr{T}\varphi^2 \big\rangle$ as $n \to \infty$. The other terms in the inequation converge thanks to the bound $f \in L^\infty((a,b); L^2(\Omega_x \times \Omega_v)) \cap L^2((a,b) \times \Omega_x; H^1(\Omega_v))$. We deduce

$$\begin{aligned} \lambda \int_{Q_R(z_0)} |\nabla_v f|^2 \varphi^2 \, \mathrm{d}z \leqslant \int_{Q_R(z_0)} & f^2 \Big(|\partial_t \varphi|\varphi + (|v_0| + R)|\nabla_x \varphi|\varphi\Big) \, \mathrm{d}z \\ & + \Lambda \int_{Q_R(z_0)} |\nabla_v f||\nabla_v \varphi| f\varphi \, \mathrm{d}z \\ & + \Lambda \int_{Q_R(z_0)} |\nabla_v f| f\varphi^2 \, \mathrm{d}z + \int_{Q_R(z_0)} f|S|\varphi^2 \, \mathrm{d}z. \end{aligned}$$

The result follows from Cauchy-Schwarz' inequality and

$$|\partial_t \varphi| \lesssim \frac{1}{(R-r)r}, \quad |\nabla_x \varphi| \lesssim \frac{1}{(R-r)r^2}, \quad |\nabla_v \varphi| \lesssim \frac{1}{(R-r)}. \qquad \square$$

2.2. Integral estimates on Kolmogorov fundamental solutions. — We denote $\mathscr{K} := \mathscr{T} - \Delta_v$.

Lemma 10 (Estimates on the fundamental solution with constant coefficients)

Consider $f \geqslant 0$ locally integrable so that $\mathscr{K} f = \nabla_v \cdot F_1 + F_2 - m$ with $F_1, F_2 \in L^1 \cap L^2(\mathbb{R}_- \times \mathbb{R}^{2d})$ and $0 \leqslant m \in M^1(\mathbb{R}_- \times \mathbb{R}^{2d})$ (a non-negative measure with finite mass) and where F_1, F_2 and m have compact support in time included in some $(-\tau, 0]$. Then there for any $p \in [2, 2+1/d)$ and $\sigma \in [0, 1/3)$

$$\|f\|_{L^p(\mathbb{R}_- \times \mathbb{R}^{2d})} \lesssim_{\tau,\lambda,\Lambda} (2 + (1/d) - p)^{-1} \left[\|F_1\|_{L^2(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|F_2\|_{L^2(\mathbb{R}_- \times \mathbb{R}^{2d})}\right], \tag{2.2}$$

$$\begin{aligned} &\|f\|_{L^1_{t,v} W^{\sigma,1}_x(\mathbb{R}_- \times \mathbb{R}^{2d})} \\ &\qquad \lesssim_{\tau,\lambda,\Lambda} ((1/3) - \sigma)^{-1} \Big[\|F_1\|_{L^1(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|F_2\|_{L^1(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|m\|_{M^1(\mathbb{R}_- \times \mathbb{R}^{2d})}\Big]. \end{aligned} \tag{2.3}$$

Proof of Lemma 10. — We use the fundamental solution computed by Kolmogorov in [Kol34] (see for instance [BDM+20, App. A] for details):

$$\begin{aligned} &\forall\, t \in \mathbb{R}_-,\ x, v \in \mathbb{R}^d, \\ &\qquad f(t,x,v) = \int_{(t',x',v') \in \mathbb{R}^{2d+1}} G(t-t', x-x'-(t-t')v', v-v')(\mathscr{K} f)(t',x',v'), \\ &\quad \forall\, t \geqslant 0,\ x, v \in \mathbb{R}^d, \\ &\qquad G(t,x,v) := \begin{cases} \Big(\dfrac{3}{4\pi^2 t^4}\Big)^{d/2} \exp\Big[-\dfrac{3\,|x-(t/2)v|^2}{t^3} - \dfrac{|v|^2}{4t}\Big] & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases} \end{aligned}$$

Since f and G are non-negative, we deduce that

$$\begin{aligned} 0 \leqslant f(t,x,v) \leqslant \int_{(t',x',v') \in \mathbb{R}^{2d+1}} & G(t-t', x-x'-(t-t')v', v-v') \\ & \cdot \big[(\nabla_{v'} \cdot F_1)(t',x',v') + F_2(t',x',v')\big] \end{aligned}$$

and since

$$\begin{aligned} &\forall\, t \geqslant 0,\ x, v \in \mathbb{R}^d, \\ &\qquad |\nabla_v G(t,x,v)| + t|\nabla_x G(t,x,v)| \lesssim t^{-2d-1/2} \exp\Big[-\frac{3\,|x-(t/2)v|^2}{2t^3} - \frac{|v|^2}{8t}\Big] \end{aligned}$$

we have $\nabla_v G, t\nabla_x G \in L^{\frac{2d+1}{2d+1/2}-0}((0,\tau) \times \mathbb{R}^{2d})$ and therefore by integration by parts

$$\begin{aligned} f(t,x,v) \leqslant & \int_{(t',x',v') \in \mathbb{R}^{2d+1}} \nabla_{v'} G(t-t', x-x'-(t-t')v', v-v') F_1(t',x',v') \\ & + \int_{(t',x',v') \in \mathbb{R}^{2d+1}} G(t-t', x-x'-(t-t')v', v-v') F_2(t',x',v') \end{aligned}$$

and Young's convolution inequality (which works in unimodular spaces like $(\mathbb{R}^{2d+1}, \circ)$ with the Lebesgue measure), we deduce, by tracking down the dependency in p of the constant

$$\forall\, p \in [2, 2+1/d)\,,$$
$$\|f\|_{L^p(\mathbb{R}_-\times\mathbb{R}^{2d})} \lesssim_\tau (2+(1/d)-p)^{-1}\left[\|F_1\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})}\right].$$

(The threshold $2+1/d$ is likely to be optimal.) This proves (2.2). To prove (2.3) split

$$G = G_\varepsilon + G_\varepsilon^\perp \quad \text{with} \quad G_\varepsilon(t,x,v) := \chi\,(t/\varepsilon)\,G(t,x,v),$$

where $\varepsilon > 0$ and χ is a smooth function on $\mathbb{R}_+$ valued in $[0,1]$ equal to 1 in $[0,1]$ and 0 on $[2,+\infty)$. We have the following simple estimates for every $\ell \in \mathbb{N}$

$$\begin{aligned}\left|\nabla_x^\ell G_\varepsilon^\perp(t,x,v)\right| &\lesssim_\ell \varepsilon^{-(3/2)\ell} t^{-2d} \exp\Big[-\frac{3\,|x-(t)/2v|^2}{2t^3} - \frac{|v|^2}{8t}\Big]\\ \left|\nabla_v\nabla_x^\ell G_\varepsilon^\perp(t,x,v)\right| + t\left|\nabla_x\nabla_x^\ell G_\varepsilon^\perp(t,x,v)\right|&\\ &\lesssim_\ell \varepsilon^{-(3/2)\ell-1/2} t^{-2d} \exp\Big[-\frac{3\,|x-(t/2)v|^2}{2t^3} - \frac{|v|^2}{8t}\Big]\end{aligned}$$

which straightforwardly implies (assuming $\tau \geqslant 1$ and $\varepsilon < 1$ wlog)

$$\begin{aligned}\left\|G_\varepsilon^\perp\right\|_{L^1_{t,v}((0,\tau)\times\mathbb{R}^d;W^{\ell,1}_x(\mathbb{R}^d)))} + \left\|\nabla_v G_\varepsilon^\perp\right\|_{L^1_{t,v}((0,\tau)\times\mathbb{R}^d;W^{\ell,1}_x(\mathbb{R}^d))}&\\ + \left\|t\nabla_x G_\varepsilon^\perp\right\|_{L^1_{t,v}((0,\tau)\times\mathbb{R}^d;W^{\ell,1}_x(\mathbb{R}^d))} &\lesssim_\ell \tau\varepsilon^{-(3/2)\ell-1/2}\\ \|G_\varepsilon\|_{L^1((0,\tau)\times\mathbb{R}^{2d})} + \|\nabla_v G_\varepsilon\|_{L^1((0,\tau)\times\mathbb{R}^{2d})} + \|t\nabla_x G_\varepsilon\|_{L^1((0,\tau)\times\mathbb{R}^{2d})} &\lesssim \tau\varepsilon^{1/2}.\end{aligned}$$

The splitting $G = G_\varepsilon + G_\varepsilon^\perp$ yields $f = f_\varepsilon + f_\varepsilon^\perp$, and the convolution inequality $M^1 * L^1 \to L^1$ implies

$$\begin{aligned}\|f_\varepsilon\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} &\lesssim \tau\varepsilon^{1/2}\big(\|F_1\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|m\|_{M^1(\mathbb{R}_-\times\mathbb{R}^{2d})}\big)\\ \left\|f_\varepsilon^\perp\right\|_{L^1_{t,v}W^{\ell,1}_x(\mathbb{R}_-\times\mathbb{R}^{2d})}&\\ &\lesssim \tau\varepsilon^{-(3/2)\ell-1/2}\left[\|F_1\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|m\|_{M^1(\mathbb{R}_-\times\mathbb{R}^{2d})}\right].\end{aligned}$$

Since this decomposition holds for all $\varepsilon > 0$, it implies by standard interpolation the estimate (2.3) for any $\sigma \in [0,1/3)$ (again the exponent is likely to be optimal but in any case our constant degenerates as $\sigma \to 1/3$). In order to be self-contained let us a give a short proof. Given $\sigma \in [0,1/3)$, we Fourier-transform and decompose dyadically, defining $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$

$$\begin{aligned}&(1-\Delta_x)^{\sigma/2} f(t,x,v)\\ &\qquad = \int_{\xi,y\in\mathbb{R}^d} e^{i\xi\cdot(x-y)}\langle\xi\rangle^\sigma f(t,y,v) = \sum_{k\geqslant -1}\int_{\xi,y\in\mathbb{R}^d} e^{i\xi\cdot(x-y)} a_k(\xi) f(t,y,v)\\ &\qquad = \sum_{k\geqslant -1}\int_{\xi,y\in\mathbb{R}^d} e^{i\xi\cdot(x-y)} B_k(\xi)(1-\Delta_y)^{\ell/2} f(t,y,v),\end{aligned} \tag{2.4}$$

where $a_k(\xi) := \langle\xi\rangle^\sigma\varphi_k$ and $B_k(\xi) := \langle\xi\rangle^{\sigma-\ell}\varphi_k$, and where we have defined in the standard way $\varphi_k(\xi) := [\chi(2^{-k}\xi) - \chi(2^{-k+1}\xi)]$ for $k \geqslant 0$ with χ a smooth function valued in $[0,1]$ and equal to 1 in $B(0,1)$ and 0 outside $B(0,2)$, and $\varphi_{-1}(\xi)=\sum_{k\leqslant -1}[\chi(2^{-k}\xi)-\chi(2^{-k+1}\xi)]$. For a given $F=F(y)$ one has

$$\int_{x\in\mathbb{R}^d}\Big|\int_{\xi,y\in\mathbb{R}^d} e^{i\xi\cdot(x-y)}a_k(\xi)F(y)\Big|\lesssim 2^{k\sigma}\|F\|_{L^1},$$
$$\int_{x\in\mathbb{R}^d}\Big|\int_{\xi,y\in\mathbb{R}^d} e^{i\xi\cdot(x-y)}B_k(\xi)(1-\Delta_y)F(y)\Big|\lesssim 2^{k(\sigma-\ell)}\|F\|_{W^{\ell,1}}$$

by splitting the integrand into $|x-y|\leqslant 2^{-k}$ and $|x-y|>2^{-k}$ and integrating by parts the operator $\Delta_\xi^{\ell/2}$ with ℓ even and strictly greater than d. We then use the decomposition (2.4) in the "a_k" form on f_ε and in the "B_k" form on $f_\varepsilon^\perp$, and with a $\varepsilon=\varepsilon_k$ depending on k defined below:

$$\begin{aligned}\big\|(1-\Delta_x)^{\sigma/2}f\big\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})}&\lesssim \tau\sum_{k\geqslant -1}\big(\varepsilon_k^{1/2}2^{k\sigma}+\varepsilon_k^{-(3/2)\ell-1/2}2^{k(\sigma-\ell)}\big)\\&\qquad\times\big[\|F_1\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})}+\|F_2\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})}+\|m\|_{M^1(\mathbb{R}_-\times\mathbb{R}^{2d})}\big]\\&\lesssim\frac{\tau}{\delta}\big[\|F_1\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})}+\|F_2\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})}+\|m\|_{M^1(\mathbb{R}_-\times\mathbb{R}^{2d})}\big]\end{aligned}$$

with the choice $\sigma=1/3-\delta\in[0,1/3)$ and $\varepsilon_k:=2^{-2k(1/3-\delta/2)}$ and $\ell>1+4/9\delta$. This concludes the proof. □

2.3. Integral estimates for sub-solutions. — We combine the previous lemma with a localization argument and the energy estimate to get the

Proposition 11 (Integral regularization estimates for non-negative sub-solutions)
Let f be a non-negative weak sub-solution to (1.1)–(1.2) *in an open set* $\mathscr{U}\in\mathbb{R}^{1+2d}$. *Given any* $Q_r(z_0)\subset Q_R(z_0)\subset\mathscr{U}$ *with* $0<r<R\leqslant 1$, *and any* $p\in[2,2+1/d)$ *and* $\sigma\in[0,1/3)$, f *satisfies*

$$\|f\|_{L^p(Q_r(z_0))}\lesssim(2+1/d-p)^{-1}\mathscr{C}'(r,R,v_0)\big[\|f\|_{L^2(Q_R(z_0))}+\|S\|_{L^2(Q_R(z_0))}\big],\tag{2.5}$$
$$\|f\|_{L^1_{t,v}W^{\sigma,1}_x(Q_r(z_0))}\lesssim(1/3-\sigma)^{-1}\,\mathscr{C}''(r,R,v_0)\big[\|f\|_{L^2(Q_R(z_0))}+\|S\|_{L^2(Q_R(z_0))}\big],\tag{2.6}$$

where $\mathscr{C}$ *was defined in* (2.1) *and*

$$\mathscr{C}'(r,R,v_0):=\Big(1+\frac{1}{R-r}\Big)\mathscr{C}(r,R,v_0),$$
$$\mathscr{C}''(r,R,v_0):=R^{1+2d}\Big(1+\frac{1}{R-r}\Big)\mathscr{C}(r,R,v_0).$$

Proof of Proposition 11. — Since f is a sub-solution to (1.1), there is a non-negative measure $\overline{m}\geqslant 0$ so that

$$\mathscr{T}f=\nabla_v\cdot(A\nabla_vf)+B\cdot\nabla_vf+S-\overline{m}.$$

Consider φ_1 smooth valued in $[0,1]$ and equal to 1 on $Q_r(z_0)$ and 0 outside $Q_{r+(R-r)/2}(z_0)$ and $g_1 := \varphi_1 f$. The latter satisfies

$$\mathscr{K} g_1 = \nabla_v \cdot F_1 + F_2 - m \tag{2.7}$$
$$\text{with } \begin{cases} m := \overline{m}\varphi_1, \\ F_1 := (A\nabla_v f)\varphi_1 - (\nabla_v f)\varphi_1 - f\nabla_v\varphi_1, \\ F_2 := -A\nabla_v f \cdot \nabla_v\varphi_1 + (B\cdot\nabla_v f)\,\varphi_1 + S\varphi_1 + f\mathscr{T}\varphi_1. \end{cases}$$

The energy estimate in Proposition 9 implies

$$\begin{aligned} &\|F_1\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})} \\ &\quad\lesssim \Big(1+\frac{1}{R-r}\Big)\mathscr{C}\Big(r+\frac{R-r}{2},R,v_0\Big)\big(\|f\|_{L^2(Q_R(z_0))}+\|S\|_{L^2(Q_R(z_0))}\big) \\ &\quad\lesssim \mathscr{C}'(r,R,z_0)\big(\|f\|_{L^2(Q_R(z_0))}+\|S\|_{L^2(Q_R(z_0))}\big), \end{aligned}$$

which, combined with (2.2), shows (2.5).

Consider then φ_2 smooth valued in $[0,1]$ and equal to 1 on $Q_{r+(R-r)/2}(z_0)$ and 0 outside $Q_R(z_0)$ and $g_2 := \varphi_2 f$. The function g_2 satisfies a similar equation as g_1 in (2.7), with φ_2 replacing φ_1. Integrating this equation simply against 1 yields (thanks to the cancellation of divergence terms)

$$\begin{aligned} \|\overline{m}\|_{M^1(Q_{r+(R-r)/2}(z_0))} &\lesssim \|\varphi_2 m\|_{M^1(\mathbb{R}_-\times\mathbb{R}^{2d})} \\ &\lesssim \int_{Q_{r+(R-r)/2}(z_0)} [-A\nabla_v f\cdot\nabla_v\varphi_2 + (B\cdot\nabla_v f)\,\varphi_2 + S\varphi_2 + f\mathscr{T}\varphi_2] \\ &\lesssim \mathscr{C}\Big(r+\frac{R-r}{2},R,v_0\Big)\|f\|_{L^1(Q_R(z_0))} + \|S\|_{L^1(Q_R(z_0))} \\ &\lesssim \mathscr{C}\,(r,R,v_0)\,\big[\|f\|_{L^2(Q_R(z_0))}+\|S\|_{L^2(Q_R(z_0))}\big]. \end{aligned}$$

Combined with (2.3) and (thanks to the localization)

$$\|F_1\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^1(\mathbb{R}_-\times\mathbb{R}^{2d})} \lesssim \|F_1\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})} + \|F_2\|_{L^2(\mathbb{R}_-\times\mathbb{R}^{2d})},$$

it implies (2.6). ☐

2.4. Iterated gain of integrability for sub-solutions

— We give a short proof of this result first obtained in [PP04, Th. 1.2] and then proved differently [GIMV19, Th. 12]. This is the counterpart of the "first lemma of De Giorgi" for elliptic equations, in the context of kinetic hypoelliptic equations. We allow for an initial integrability L^ζ with exponent $\zeta \in (0,2)$ (such extension is well-known for elliptic equations).

Proposition 12 (Upper bound for sub-solutions). — *Let f be a non-negative weak sub-solution to (1.1)–(1.2) in an open set $\mathscr{U} \Subset \mathbb{R}^{1+2d}$. Given any $Q_r(z_0) \subset Q_R(z_0) \subset \mathscr{U}$ with $0 < r < R \leqslant 1$, and $\zeta > 0$, f satisfies*

$$\|f\|_{L^\infty(Q_r(z_0))} \lesssim_{\lambda,\Lambda} \Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{(1+4d)/\zeta}\big[\|f\|_{L^\zeta(Q_R(z_0))}+\|S\|_{L^\infty(Q_R(z_0))}\big].$$

Proof of Proposition 12. — Fix $p_0 := 2+1/2d$ and define $q := p_0/2$ and $q_n := q^n$. Consider $\beta_{n,k}$ on $\mathbb{R}_+$ with $\beta'_{n,k} \geqslant 0$ and $\beta''_{n,k} \geqslant 0$ both bounded and so that $\beta_{n,k}(z) \to z^{q_n}$ as $k \to \infty$ and $\beta_{n,k}(z) \lesssim z^{q_n}$ and $\beta'_{n,k}(z) \lesssim z^{q_n-1}$ uniformly in $k \in \mathbb{N}^*$. Definition 1 implies that $\beta_{n,k}(f)$ is a weak sub-solution with source term $S_{n,k} := \beta'_{n,k}(f)S$. Define $r_0 = R$ and $r_n := r_{n-1} - \delta n^{-2}$ with $\delta = \frac{1}{2}(\sum_{k\geqslant 1} k^{-2})^{-1}(R-r)$. Since $p_0 \in [2, 2+1/d)$, the estimate (2.5) implies for all $n \geqslant 1$

$$\begin{aligned}\|\beta_{n,k}(f)\|_{L^{p_0}(Q_{r_n}(z_0))} &\lesssim \mathscr{C}'(r_n, r_{n-1}, v_0)\big[\|\beta_{n,k}(f)\|_{L^2(Q_{r_{n-1}}(z_0))} + \|S_{n,k}\|_{L^2(Q_{r_{n-1}}(z_0))}\big]\\ &\lesssim \frac{(1+|v_0|)n^6}{r^2(R-r)^3}\big[\|\beta_{n,k}(f)\|_{L^2(Q_{r_{n-1}}(z_0))} + \|S_{n,k}\|_{L^2(Q_{r_{n-1}}(z_0))}\big]\end{aligned}$$

for $n \geqslant 1$, which means by taking $k \to \infty$ and coming back to f

$$\begin{aligned}&\|f\|_{L^{2q_{n+1}}(Q_{r_n}(z_0))}\\ &\lesssim \Big(\frac{(1+|v_0|)n^6}{r^2(R-r)^3}\Big)^{1/q^n} 2^{-1+1/q_n}\Big[\|f\|_{L^{2q_n}(Q_{r_{n-1}}(z_0))} + \|f\|^{1-1/q_n}_{L^{2q_n}(Q_{r_{n-1}}(z_0))}\|S\|^{1/q_n}_{L^\infty(Q_R(z_0))}\Big]\\ &\qquad\lesssim \Big(\frac{(1+|v_0|)n^6}{r^2(R-r)^3}\Big)^{1/q^n}\Big[\Big(1+\frac{1}{q_n}\Big)\|f\|_{L^{2q_n}(Q_{r_{n-1}}(z_0))} + \frac{1}{q_n}\|S\|_{L^\infty(Q_R(z_0))}\Big],\end{aligned}$$

assuming by induction $\|f\|_{L^{2q_n}(Q_{r_{n-1}}(z_0))} < +\infty$. The convergence of the infinite product then implies

$$\|f\|_{L^\infty(Q_r(z_0))} \lesssim \Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{1+4d}\big[\|f\|_{L^2(Q_R(z_0))} + \|S\|_{L^\infty(Q_R(z_0))}\big].$$

This proves the claim when $\zeta \geqslant 2$. To prove it when $\zeta \in (0,2)$, we deduce from the previous estimate

$$\begin{aligned}&\|f\|_{L^\infty(Q_r(z_0))} + \|S\|_{L^\infty(Q_r(z_0))}\\ &\qquad\lesssim \Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{1+4d}\big[\|f\|^{1-\zeta}_{L^\infty(Q_R(z_0))}\|f\|^{\zeta}_{L^\zeta(Q_R(z_0))} + \|S\|_{L^\infty(Q_R(z_0))}\big]\end{aligned}$$

and thus by Young inequality, the quantity $A(r) := \|f\|_{L^\infty(Q_r(z_0))} + \|S\|_{L^\infty(Q_r(z_0))}$ satisfies, for some $C > 0$,

$$A(r) \leqslant \frac{1}{2}A(R) + C\Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{(1+4d)/\zeta}\big[\|f\|_{L^\zeta(Q_R(z_0))} + \|S\|_{L^\infty(Q_R(z_0))}\big].$$

Introducing an (increasing this time) sequence of radii $r_n := r_{n-1} + \delta n^{-2}$ we obtain by induction

$$\begin{aligned}A(r_n) &\leqslant (1/2)A(r_{n+1})\\ &\qquad + Cn^{(2+8d)/\zeta}\Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{(1+4d)/\zeta}\big[\|f\|_{L^\zeta(Q_R(z_0))} + \|S\|_{L^\infty(Q_R(z_0))}\big],\\ A(r_0) &\leqslant (1/2)^n A(r_{n+1})\\ &\qquad + C\Big(\sum_{k=1}^n \frac{k^{(2+8d)/\zeta}}{2^k}\Big)\Big(\frac{1+|v_0|}{r^2(R-r)^3}\Big)^{(1+4d)/\zeta}\big[\|f\|_{L^\zeta(Q_R(z_0))} + \|S\|_{L^\infty(Q_R(z_0))}\big],\end{aligned}$$

which yields the result by taking $n \to \infty$ in the right hand side. □

3. Intermediate-Value Lemma and oscillations

3.1. Weak Poincaré inequality. — The adjective 'weak' refers to the small additional L^2 error term below.

Proposition 13 (Hypoelliptic Poincaré inequality with error). — *Given any $\varepsilon \in (0,1)$ and $\sigma \in (0,1/3)$, any non-negative sub-solution f to (1.1)–(1.2) on Q_5 satisfies*

$$\begin{aligned}(3.1)\quad \big\|(f - \langle f\rangle_{Q_1^-})_+\big\|_{L^1(Q_1)} \\ \lesssim_{\lambda,\Lambda} \frac{1}{\varepsilon^{d+2}}\|\nabla_v f\|_{L^1(Q_5)} + \varepsilon^{\sigma}\,(1/3-\sigma)^{-1}\,\|f\|_{L^2(Q_5)} + \|S\|_{L^2(Q_5)},\end{aligned}$$

where $Q_1^- := Q_1(-2,0,0) = (-3,-2]\times B_1\times B_1$ and $\langle f\rangle_{Q_1^-} := \fint_{Q_1^-} f := \frac{1}{|Q_1^-|}\int_{Q_1^-} f$.

Remark 14. — The motivation for the following argument was [Vas16, Lem. 10, p. 11], where a simple quantitative proof of the intermediate value lemma of De Giorgi (also sometimes called De Giorgi's isoperimetric inequality) is sketched in the elliptic case, based on introducing the trajectory between two points of the domain and using the vector field ∇_v to connect them. We have to deal here with the hypoelliptic structure.

Proof. — Consider, for $\varepsilon \in (0,1)$, a smooth function $\varphi_\varepsilon = \varphi_\varepsilon(y,w)$ which satisfies $0 \leqslant \varphi_\varepsilon \leqslant 1$ and has compact support in B_1^2 and such that $\varphi_\varepsilon = 1$ in $B_{(1-\varepsilon)} \times B_{(1-\varepsilon)}$ and with $|\nabla_y \varphi_\varepsilon| \lesssim \varepsilon^{-1}$ and $|\nabla_w \varphi_\varepsilon| \lesssim \varepsilon^{-1}$. We then split the integral to be estimated as follows

$$\begin{aligned}\big\|(f - \langle f\rangle_{Q_1^-})_+\big\|_{L^1(Q_1)} &\lesssim \big\|(f - \langle f\varphi_\varepsilon\rangle_{Q_1^-})_+\big\|_{L^1(Q_1)} \\ &\lesssim \int_{(t,x,v)\in Q_1}\Big\{\fint_{(s,y,w)\in Q_1^-}[f(t,x,v)-f(s,y,w)]\,\varphi_\varepsilon(y,w)\Big\}_+ \\ &\qquad\qquad + \|f\|_{L^1(Q_1)}\fint_{Q_1^-}(1-\varphi_\varepsilon(y,w)) \\ &\lesssim \int_{(t,x,v)\in Q_1}\Big\{\fint_{(s,y,w)\in Q_1^-}[f(t,x,v)-f(s,y,w)]\,\varphi_\varepsilon(y,w)\Big\}_+ + \varepsilon^{2d}\|f\|_{L^2(Q_1)},\end{aligned}$$

where we have used $\langle f\varphi_\varepsilon\rangle_{Q_1^-} \leqslant \langle f\rangle_{Q_1^-}$ and the Cauchy-Schwarz inequality.

Let us estimate the first term of the previous inequality. Given t,x,v fixed, we decompose the trajectory $(t,x,v)\to(s,y,w)$ into four sub-trajectories in Q_5: a trajectory of length $O(\varepsilon)$ along ∇_x, two trajectories of length $O(1)$ along ∇_v, and finally one trajectory of length $O(1)$ along $\mathscr{T} := \partial_t + v\cdot\nabla_x$:

$$\begin{aligned}(t,x,v) \underset{\nabla_x}{\longrightarrow} (t,x+\varepsilon w,v) \underset{\nabla_v}{\longrightarrow} \Big(t,x+\varepsilon w,\frac{x+\varepsilon w-y}{t-s}\Big) \\ \underset{\mathscr{T}}{\longrightarrow} \Big(s,y,\frac{x+\varepsilon w-y}{t-s}\Big) \underset{\nabla_v}{\longrightarrow} (s,y,w).\end{aligned}$$

The first sub-trajectory is estimated by the *integral* regularity $L^1_{t,v}W^{\sigma,1}_x$ proved in (2.6). The other trajectories are estimated directly by the vector fields in the equation. The position $x+\varepsilon w \in Q_2$ since $x,w\in B_1$ and $\varepsilon\in(0,1)$. The velocity

$(x+\varepsilon w-y)/(t-s)\in Q_3$ since $x,w,y\in B_1$ and $t-s\geqslant 1$ due to the definitions of Q_1^+ and Q_1^-, and this velocity yields a transport line from $(t,x+\varepsilon w)$ to (s,y). Note that we are implicitly using the Hörmander commutator condition: $\nabla_v,\mathscr{T},[\nabla_v,\mathscr{T}]$ span all the vector fields on $\mathbb{R}^{2d+1}$.

Decompose along the previous trajectories

$$\begin{aligned}
&f(t,x,v)-f(s,y,w)\\
&\quad=\big[f(t,x,v)-f(t,x+\varepsilon w,v)\big]+\Big[f(t,x+\varepsilon w,v)-f\Big(t,x+\varepsilon w,\frac{x+\varepsilon w-y}{t-s}\Big)\Big]\\
&\qquad+\Big[f\Big(t,x+\varepsilon w,\frac{x+\varepsilon w-y}{t-s}\Big)-f\Big(s,y,\frac{x+\varepsilon w-y}{t-s}\Big)\Big]\\
&\qquad+\Big[f\Big(s,y,\frac{x+\varepsilon w-y}{t-s}\Big)-f(s,y,w)\Big]
\end{aligned}$$

and integrate against $\varphi_\varepsilon(y,w)$ on $(s,y,w)\in Q_1^-$, which gives the four terms

$$\begin{aligned}
I_1(t,x,v)&:=\int_{(s,y,w)\in Q_1^-}\big[f(t,x,v)-f(t,x+\varepsilon w,v)\big]\varphi_\varepsilon(y,w),\\
I_2(t,x,v)&:=\int_{(s,y,w)\in Q_1^-}\Big[f(t,x+\varepsilon w,v)-f\Big(t,x+\varepsilon w,\frac{x+\varepsilon w-y}{t-s}\Big)\Big]\varphi_\varepsilon(y,w),\\
I_3(t,x,v)&:=\int_{(s,y,w)\in Q_1^-}\Big[f\Big(t,x+\varepsilon w,\frac{x+\varepsilon w-y}{t-s}\Big)-f\Big(s,y,\frac{x+\varepsilon w-y}{t-s}\Big)\Big]\varphi_\varepsilon(y,w),\\
I_4(t,x,v)&:=\int_{(s,y,w)\in Q_1^-}\Big[f\Big(s,y,\frac{x+\varepsilon w-y}{t-s}\Big)-f(s,y,w)\Big]\varphi_\varepsilon(y,w).
\end{aligned}$$

Regarding the term I_2, we use Taylor's formula and $0\leqslant\varphi_\varepsilon\leqslant 1$ to deduce

$$\begin{aligned}
I_2(t,x,v)&\leqslant\int_{(s,y,w)\in Q_1^-}\int_{\tau\in[0,1]}\Big(v-\frac{x+\varepsilon w-y}{t-s}\Big)\cdot\\
&\qquad\qquad\times\nabla_v f\Big(t,x+\varepsilon w,\tau v+(1-\tau)\frac{x+\varepsilon w-y}{t-s}\Big)\varphi_\varepsilon(y,w)\\
&\lesssim\int_{(s,y,w)\in Q_1^-}\int_{\tau\in[0,1]}|\nabla_v f|\Big(t,x+\varepsilon w,\tau v+(1-\tau)\frac{x+\varepsilon w-y}{t-s}\Big).
\end{aligned}$$

Integrate then on $(t,x,v)\in Q_1^+$ to get

$$\begin{aligned}
&\fint_{(t,x,v)\in Q_1} I_2\\
&\lesssim\int_{(t,X,v)\in(-1,0)\times B_2\times B_1}\int_{(s,Y,w)\in(-3,-2)\times B_4\times B_1}\int_{\tau\in(0,1)}|\nabla_v f|\,(t,X,v+(1-\tau)Y)\\
&\lesssim\int_{(t,X,V)\in(-1,0)\times B_2\times B_5}\int_{(s,Y,w)\in(-3,-2)\times B_4\times B_1}\int_{\tau\in(0,1)}|\nabla_v f|\,(t,X,V)\lesssim\int_{Q_5}|\nabla_v f|,
\end{aligned}\tag{3.2}$$

where we have used successively the following changes of variables with bounded Jacobians:

$$x\longrightarrow X=x+\varepsilon w\in B_2,\quad y\longrightarrow Y=\frac{X-y}{t-s}-v\in B_4,\quad v\longrightarrow V=v+(1-\tau)Y\in B_5.$$

The term I_4 is treated like I_2:

$$\fint_{(t,x,v)\in Q_1} I_4 \lesssim \int_{Q_5} |\nabla_v f|. \tag{3.3}$$

Regarding the term I_1, we perform the change of variable $w \in B_1 \to x' = x + \varepsilon w \in B_\varepsilon(x)$ with Jacobian ε^{-d} and use the $L^1_{t,v} W^{\sigma,1}_x$ regularity of non-negative sub-solutions proved in (2.6):

$$\begin{aligned}
\fint_{(t,x,v)\in Q_1} I_1 &\lesssim \int_{(t,x,v)\in Q_1,\,(s,y,w)\in Q_1^-} |f(t,x,v) - f(t,x+\varepsilon w,v)| \\
&\lesssim \int_{(t,x,v)\in Q_1,\,w\in B_1} \frac{|f(t,x,v) - f(t,x+\varepsilon w,v)|}{|\varepsilon w|^{d+\sigma}} |\varepsilon w|^{d+\sigma} \\
&\lesssim \varepsilon^\sigma \int_{(t,x,v)\in Q_1,\,x'\in B_2} \frac{|f(t,x,v) - f(t,x',v)|}{|x-x'|^{d+\sigma}} \\
&\lesssim \varepsilon^\sigma \|f\|_{L^1_{t,v}W^{\sigma,1}_x(Q_2)} \lesssim \varepsilon^\sigma (1/3-\sigma)^{-1} \left[\|f\|_{L^2(Q_3)} + \|S\|_{L^2(Q_3)}\right].
\end{aligned} \tag{3.4}$$

Regarding the term I_3, we note first that $\mathscr{T} f \in L^2_{t,x} H^{-1}_v + M^1_{t,x,v}$ with finite norm in $Q_R(z_0)$ (arguing as in proof of Proposition 11). The Taylor formula between $(t, x+\varepsilon w)$ and (s,y) along $\mathscr{T}$ thus holds in weak form against φ_ε thanks to the latter bounds and the non-singular change of variable (3.7) discussed below:

$$\begin{aligned}
&I_3(t,x,v) \\
&\quad= \int_{(s,y,w)\in Q_1^-} \left[f\Big(t, x+\varepsilon w, \frac{x+\varepsilon w - y}{t-s}\Big) - f\Big(s,y,\frac{x+\varepsilon w - y}{t-s}\Big)\right] \varphi_\varepsilon(y,w) \\
&\quad\lesssim \int_{(s,y,w)\in Q_1^-} \int_{\tau\in[0,1]} (t-s) \\
&\qquad\qquad \times \mathscr{T} f\Big(\tau t + (1-\tau)s, \tau(x+\varepsilon w) + (1-\tau) y, \frac{x+\varepsilon w - y}{t-s}\Big) \varphi_\varepsilon(y,w).
\end{aligned} \tag{3.5}$$

We then use the fact that f is a sub-solution to (1.1) in the distributional sense:

$$\begin{aligned}
&I_3(t,x,v) \\
&\quad\lesssim \int_{(s,y,w)\in Q_1^-} \int_{\tau\in[0,1]} (t-s) \\
&\qquad\qquad \times \nabla_v \cdot (A \nabla_v f)\Big(\tau t + (1-\tau)s, \tau(x+\varepsilon w) + (1-\tau) y, \frac{x+\varepsilon w - y}{t-s}\Big) \varphi_\varepsilon(y,w) \\
&\qquad + \int_{(s,y,w)\in Q_1^-} \int_{\tau\in[0,1]} (t-s) \\
&\qquad\qquad \times B \cdot \nabla_v f\Big(\tau t + (1-\tau)s, \tau(x+\varepsilon w) + (1-\tau) y, \frac{x+\varepsilon w - y}{t-s}\Big) \varphi_\varepsilon(y,w) \\
&\qquad + \int_{(s,y,w)\in Q_1^-} \int_{\tau\in[0,1]} (t-s) \\
&\qquad\qquad \times S\Big(\tau t + (1-\tau)s, \tau(x+\varepsilon w) + (1-\tau) y, \frac{x+\varepsilon w - y}{t-s}\Big) \varphi_\varepsilon(y,w) \\
&\quad := I_{31} + I_{32} + I_{33}.
\end{aligned}$$

Arguing as for I_2 and I_4, we have

$$\fint_{(t,x,v)\in Q_1} I_{32} \lesssim \int_{Q_5} |\nabla_v f| \quad \text{and} \quad \fint_{(t,x,v)\in Q_1} I_{33} \lesssim \int_{Q_5} |S|, \tag{3.6}$$

where we performed consecutively the changes of variable

$$y \longrightarrow V = \frac{x+\varepsilon w - y}{t-s}, \quad x \longrightarrow X = x + \varepsilon w - (1-\tau)(t-s)V,$$
$$s \longrightarrow s' = t-s \quad \text{and} \quad t' \longrightarrow t-(1-\tau)s'.$$

To estimate the remaining term I_{31}, we use the change of variable

$$(y,w) \longmapsto (Y,W) \quad \text{with } Y := \tau(x+\varepsilon w) + (1-\tau)y \text{ and } W := \frac{x+\varepsilon w - y}{t-s} \tag{3.7}$$

such that $(y,w) \mapsto (Y,W)$ is a bijection from the set $(B_1)^2$ to the (diamond-shaped) set

$$E := E(\tau,\varepsilon,t,s,x) \subset B\left(\tau x, (1-\tau)+\tau\varepsilon\right) \times B\Big(\frac{x}{t-s}, \frac{1+\varepsilon}{t-s}\Big) \subset B_2 \times B_3$$

with Jacobian $(\varepsilon/(t-s))^d$ and which maps respective boundaries (to compute the Jacobian easily use the formula $\det\big(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big) = \det(A - BD^{-1}C)\det D$). We deduce

$$I_{31} = \frac{1}{\varepsilon^d} \int_{\tau\in[0,1]} \int_{s\in(-3,-2),\,(Y,W)\in E} (t-s)^{d+1} \nabla_v \cdot (A\nabla_v f)\,(\tau t + (1-\tau)s, Y, W)$$
$$\times \varphi_\varepsilon\Big(Y - \tau(t-s)W, \frac{Y - x + (1-\tau)(t-s)W}{\varepsilon}\Big)$$

and we integrate by parts in W, using that $\varphi_\varepsilon = 0$ on the boundary of $E(\tau,\varepsilon,t,s,x)$:

$$I_{31} = \frac{1}{\varepsilon^d} \int_{\tau\in[0,1]} \int_{s\in(-3,-2),(Y,W)\in E} (t-s)^{d+1} (A\nabla_v f)\,(\tau t + (1-\tau)s, Y, W)$$
$$\times \Big[\tau(t-s)\nabla_y \varphi_\varepsilon\Big(Y - \tau(t-s)W, \frac{Y - x + (1-\tau)(t-s)W}{\varepsilon}\Big)$$
$$- \frac{(1-\tau)(t-s)}{\varepsilon} \nabla_w \varphi_\varepsilon\Big(Y - \tau(t-s)W, \frac{Y - x + (1-\tau)(t-s)W}{\varepsilon}\Big)\Big].$$

Using the bounds on the derivatives of φ_ε then yields

$$I_{31}(t,x,v) \lesssim \frac{1}{\varepsilon^{d+2}} \int_{\tau\in[0,1]} \int_{s\in(-3,-2),(Y,W)\in E} |\nabla_v f|\,(\tau t + (1-\tau)s, Y, W) \tag{3.8}$$
$$\Longrightarrow \fint_{Q_1} I_{31} \lesssim \frac{1}{\varepsilon^{d+2}} \int_{Q_3} |\nabla_v f|.$$

The result follows from combining (3.2), (3.3), (3.4), (3.6) and (3.8). □

Remark 15. — Note that the regularity $W^{\sigma,1}_x$ is only used over a small trajectory that "noises" the position variable x in Q_1 with the velocity w in Q_1^-, hence allowing to integrate by parts the diffusion operator using *only* the variables in Q_1^-. Note also that it is possible to get some $W^{\sigma',1}_{t,x,v}$ regularity in all variable with $\sigma' \in (0,\sigma)$ small by the same method as in Lemma 10, however such regularity is too weak to yield any intermediate value estimate alone. Note also that the gap in time between Q_1^-

and Q_1 is used to make sure the intermediate velocity $(x+\varepsilon w-y)/(t-s)$ remains bounded and the various domains of integration remain bounded along the velocity variable. In fact, the result is false without such gap, see Remark 4.

3.2. Proof of the Intermediate-Value Lemma. — In this subsection, we prove that Proposition 13 implies Theorem 3. Take f a sub-solution to (1.1)–(1.2) on Q_1 and satisfying (1.3) for some given $\delta_1, \delta_2 > 0$:

$$|\{f \leqslant 0\} \cap Q^-_{r_0}| \geqslant \delta_1 |Q^-_{r_0}| \quad \text{and} \quad |\{f \geqslant 1-\theta\} \cap Q_{r_0}| \geqslant \delta_2 |Q_{r_0}|. \tag{3.9}$$

Define $g := f - (t + 25 r_0^2)\|S\|_{L^\infty(Q_1)}$. Then its positive part g_+ is a sub-solution to (1.1)–(1.2) in Q_{5r_0} with zero source term and with $g_+ \in [0,1]$ since $f \leqslant 1$ in $Q_{1/2}$. We set

$$r_0 = \begin{cases} \Big(\dfrac{\delta_1}{400(1+\|S\|_{L^\infty(Q_1)})}\Big)^{1/2} \leqslant \dfrac{1}{20} & \text{if } S \text{ non-zero}, \\ \dfrac{1}{20} & \text{if } S = 0, \end{cases}$$

and we apply (3.1) to g_+ at scale r_0, for some $\varepsilon > 0$ to be chosen later:

$$\begin{aligned} \fint_{Q_{r_0}} \big(g_+ - \langle g_+ \rangle_{Q^-_{r_0}}\big)_+ &\lesssim \frac{r_0}{\varepsilon^{d+2}} \fint_{Q_{5r_0}} |\nabla_v g_+| + \varepsilon^\sigma \Big(\fint_{Q_{5r_0}} g_+^2\Big)^{1/2} \\ &\lesssim \frac{1}{r_0^{4d+1}\varepsilon^{d+2}} \int_{Q_{5r_0}} |\nabla_v g_+| + \varepsilon^\sigma, \end{aligned} \tag{3.10}$$

where we have used the bound $g_+ \in [0,1]$ to control the L^2 norm. Then (3.9) implies

$$\begin{aligned} \langle g_+ \rangle_{Q^-_{r_0}} &= \fint_{(s,y,w)\in Q^-_{r_0}} \big[f(s,y,w) - (s+25r_0^2)\|S\|_{L^\infty(Q_1)}\big]_+ \\ &\leqslant \frac{\big|\{f>0\} \cap Q^-_{r_0}\big|}{|Q^-_{r_0}|} \leqslant 1-\delta_1 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\fint_{Q_{r_0}} \big(g_+ - \langle g_+ \rangle_{Q^-_{r_0}}\big)_+ \\ &\qquad \geqslant \frac{1}{|Q_{r_0}|} \int_{(t,x,v)\in Q_{r_0}} \big[f(t,x,v) - (t+25r_0^2)\|S\|_{L^\infty(Q_1)} - (1-\delta_1)\big]_+ \\ &\qquad \geqslant \frac{1}{|Q_{r_0}|} \int_{(t,x,v)\in Q_{r_0}} \big[f(t,x,v) - 25r_0^2\|S\|_{L^\infty(Q_1)} - (1-\delta_1)\big]_+ \\ &\qquad \geqslant \frac{1}{|Q_{r_0}|} \int_{\{f\geqslant 1-\theta\}\cap Q_{r_0}} \Big(\frac{\delta_1}{2} - \theta\Big)_+ \geqslant \delta_2 \Big(\frac{\delta_1}{2} - \theta\Big). \end{aligned} \tag{3.12}$$

We then estimate from above the right hand side of the Poincaré inequality (3.10):

$$\begin{aligned} \int_{Q_{5r_0}} |\nabla_v g_+| &\leqslant \int_{Q_{5r_0}} |\nabla_v f_+| \\ &\leqslant \int_{\{f=0\}\cap Q_{5r_0}} \cdots + \int_{\{0<f<1-\theta\}\cap Q_{5r_0}} \cdots + \int_{\{f\geqslant 1-\theta\}\cap Q_{5r_0}} \cdots \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

The first term $I_1 = 0$ since $\nabla_v f_+ = 0$ almost everywhere on $\{f_+ = 0\}$ (see [EG15, §4.2.2]). Combining the Cauchy-Schwarz inequality, Proposition 9 and the fact that $f \leqslant 1$, we get

$$I_2 \leqslant |\{0 < f < 1-\theta\} \cap Q_{5r_0}|^{1/2} \Big(\fint_{Q_{5r_0}} |\nabla_v f_+|^2 \Big)^{1/2}$$
$$\lesssim |\{0 < f < 1-\theta\} \cap Q_{1/2}|^{1/2} \Big(\fint_{Q_{1/2}} f_+^2 \Big)^{1/2} \lesssim |\{0 < f < 1-\theta\} \cap Q_{1/2}|^{1/2}$$

and (using that $\nabla_v f$ is zero almost everywhere on $\{f = \mathrm{cst}\}$, see again [EG15, §4.2.2])

$$\begin{aligned} I_3 &= \int_{Q_{5r_0}} \big|\nabla_v \big[(f-(1-\theta))_+ + (1-\theta)\big]\big| = \int_{Q_{5r_0}} \big|\nabla_v \big[f-(1-\theta)\big]_+\big| \\ &\lesssim \Big(\int_{Q_{5r_0}} \big|\nabla_v \big[f-(1-\theta)\big]_+\big|^2 \Big)^{1/2} \\ &\lesssim \Big(\int_{Q_{1/2}} \big[f-(1-\theta)\big]_+^2 + \int_{Q_{1/2}} \big[f-(1-\theta)\big]_+ |S| \Big)^{1/2} \\ &\lesssim \theta + \theta^{1/2} \|S\|_{L^\infty(Q_1)} \lesssim \theta^{1/2} \big(1 + \|S\|_{L^\infty(Q_1)}\big), \end{aligned}$$

where we have used the energy estimate in Proposition 9 on $[f-(1-\theta)]_+$.

The last two estimates on I_2 and I_3 yield the following control on the right hand side of (3.10):

$$\begin{aligned} (3.13)\qquad & \frac{1}{r_0^{4d+1} \varepsilon^{d+2}} \int_{Q_{5r_0}} |\nabla_v g_+| + \varepsilon^\sigma \\ &\qquad \lesssim \frac{\theta^{1/2}\left(1 + \|S\|_{L^\infty(Q_1)}\right)}{r_0^{4d+1}\varepsilon^{d+2}} + \frac{|\{0 < f < 1-\theta\} \cap Q_{1/2}|^{1/2}}{r_0^{4d+1}\varepsilon^{d+2}} + \varepsilon^\sigma. \end{aligned}$$

Combining (3.12) and (3.13) gives, for some universal constant $C \geqslant 1$:

$$(3.14)\quad \frac{\delta_1\delta_2}{2} \leqslant \delta_2\theta + \frac{C\left(1 + \|S\|_{L^\infty(Q_1)}\right)\theta^{1/2}}{r_0^{4d+1}\varepsilon^{d+2}} + \frac{C|\{0 < f < 1-\theta\} \cap Q_{1/2}|^{1/2}}{r_0^{4d+1}\varepsilon^{d+2}} + C\varepsilon^\sigma.$$

We choose ε such that $C\varepsilon^\sigma \leqslant \delta_1\delta_2/8$ and θ such that

$$\delta_2\theta + \frac{C\left(1 + \|S\|_{L^\infty(Q_1)}\right)\theta^{1/2}}{r_0^{4d+1}\varepsilon^{d+2}} \leqslant \frac{\delta_1\delta_2}{8},$$

e.g.

$$(3.15)\qquad \varepsilon = \Big(\frac{\delta_1\delta_2}{8C}\Big)^{1/\sigma}, \qquad \theta = \delta_1^2\delta_2^2 \bigg[8\Big(\delta_2 + \frac{C\left(1 + \|S\|_{L^\infty(Q_1)}\right)}{r_0^{4d+1}\left(\delta_1\delta_2/8C\right)^{(d+2)/\sigma}} \Big) \bigg]^{-2},$$

which finally implies the result with

$$(3.16)\qquad \nu := \frac{1}{|Q_{1/2}|} \bigg(\frac{\delta_1\delta_2}{4C} \Big(\frac{\delta_1\delta_2}{8C}\Big)^{(d+2)/\sigma} r_0^{4d+1} \bigg)^2 \gtrsim \frac{(\delta_1\delta_2)^{10d+16}}{\left(1 + \|S\|_{L^\infty(Q_1)}\right)^{4d+2}}.$$

3.3. Measure-to-pointwise estimate. — In this subsection, we combine Proposition 12 and Theorem 3 to prove a measure-to-pointwise estimate of "lowering of the maximum" à la De Giorgi.

Lemma 16 (Measure-to-pointwise upper bound). — *Given $\delta \in (0,1)$, define $r_0 = (\delta/800)^{1/2}$ if S non-zero and $r_0 = 1/20$ if $S = 0$. There is $\mu := \mu(\delta) \sim \delta^{2(1+\delta^{-10d-16})} > 0$ such that any sub-solution f to (1.1)–(1.2) in Q_1 with S such that $\|S\|_{L^\infty(Q_1)} \leqslant \mu$ and so that $f \leqslant 1$ in $Q_{1/2}$ and*

$$\left|\{f \leqslant 0\} \cap Q^-_{r_0}\right| \geqslant \delta \left|Q^-_{r_0}\right| \tag{3.17}$$

satisfies $f \leqslant 1 - \mu$ in $Q_{r_0/2}$, with $Q^-_{r_0} := Q_{r_0}(-2r_0^2, 0, 0) = (-3r_0^2, -2r_0^2] \times B_{r_0^3} \times B_{r_0}$ (see Figure 1).

Proof. — In view of Proposition 12 and the scaling invariance, there is $\delta' > 0$ depending only on λ and Λ such that for any $r > 0$, any sub-solution f on Q_{2r} so that $\int_{Q_r} f_+^2 \leqslant \delta' |Q_r|$ satisfies $f \leqslant 1/2$ in $Q_{r/2}$ (imposing $C\mu \leqslant 1/4$ with C the universal constant in the estimate of Proposition 12 used here). Define then $\nu, \theta > 0$ as in (3.15)–(3.16) with $\delta_1 = \delta$ and $\delta_2 = \delta'$ and a source term bounded in L^∞ by 1.

Define $f_k := \theta^{-k}[f - (1 - \theta^k)]$ for $k \geqslant 0$. The functions f_k are sub-solutions to (1.1)–(1.2) for all $k \geqslant 0$ with a source term of L^∞ norm less than 1 as long as $k \leqslant 1 + 1/\nu$ (assuming $\|S\|_{L^\infty(Q_1)} \leqslant \mu$ so that $\|S\|_{L^\infty(Q_1)} \leqslant \theta^{1+1/\nu}$). The sets $\{0 < f_k < 1 - \theta\} = \{1 - \theta^k < f < 1 - \theta^{k+1}\}$ are disjoints and each f_k satisfies (3.17). If $\int_{Q_{r_0}} (f_k)_+^2 \leqslant \delta' |Q_{r_0}|$ then $f_k \leqslant 1/2$ in $Q_{r_0/2}$ so $f \leqslant 1 - \mu$ with $\mu = \theta^k/2$ which concludes the proof. Consider $1 \leqslant k_0 \leqslant 1 + \nu^{-1}$ such that $\int_{Q_{r_0}} (f_k)_+^2 > \delta' |Q_{r_0}|$ for any k such that $0 \leqslant k \leqslant k_0$. Then for k such that $0 \leqslant k \leqslant k_0 - 1$

$$\begin{aligned}
&|\{f_k \geqslant 1 - \theta\} \cap Q_{r_0}| = |\{f_{k+1} \geqslant 0\} \cap Q_{r_0}| \geqslant \int_{Q_{r_0}} (f_{k+1})_+^2 > \delta' |Q_{r_0}|, \\
&\left|\{f_k \leqslant 0\} \cap Q^-_{r_0}\right| \geqslant \left|\{f \leqslant 0\} \cap Q^-_{r_0}\right| \geqslant \delta |Q^-_{r_0}|.
\end{aligned}$$

Theorem 3 for sub-solutions with source term of norm L^∞ less than 1 then implies, choosing $r_0 = (\delta/800)^{1/2}$,

$$\left|\{0 < f_k < 1 - \theta\} \cap Q_{1/2}\right| \geqslant \nu |Q_{1/2}|.$$

Summing these estimates and using the fact that the sets are disjoints we have

$$|Q_{1/2}| \geqslant \sum_{k=0}^{k_0-1} \left|\{0 < f_k < 1 - \theta\} \cap Q_{1/2}\right| \geqslant k_0 \nu |Q_{1/2}|.$$

So $k_0 \leqslant \nu^{-1}$ which ensures that source terms remain indeed less than one along the iteration, and we deduce

$$f \leqslant 1 - \frac{\theta^{k_0+1}}{2} \leqslant 1 - \frac{\theta^{(1+\nu)/\nu}}{2} \quad \text{in } Q_{r_0/2},$$

which yields $\mu(\delta) := \theta^{1+1/\nu}/2 \sim \delta^{2(1+\delta^{-10d-16})}$. □

4. Harnack inequalities and Hölder continuity

4.1. The Harnack inequalities. — To prove the weak Harnack inequality, we first assume $S=0$, and re-introduce S in the end. Without source term, $r_0 = 1/20$ can be taken constant in the measure-to-pointwise estimate. Consider then h non-negative super-solution to (1.1)–(1.2) on Q_1 with $S=0$. The contraposition of Lemma 16 on the sub-solution $g := 1 - h/M$ then implies for any $\delta \in (0,1)$ and $M \sim \delta^{-2(1+\delta^{-10d-16})}$ that

$$\text{(4.1)}\quad \forall\, Q_r(z) \subset Q_1 \text{ with } Q^+_{r/2}(z) \subset Q_1,\ \frac{|\{h > M\} \cap Q_r(z)|}{|Q_r(z)|} > \delta \implies \inf_{Q^+_{r/2}(z)} h \geqslant 1,$$

where $Q^+_{r/2}(z) = Q_{r/2}(z + (2r^2, 2r^2 v, 0))$, for $z = (t,x,v)$, is obtained by inverting the operation $Q_{r/2}(z) \to Q^-_r(z)$ in Lemma 16 (noting that $Q^-_r(z) = Q_r(z - (2r^2, 2r^2 v, 0)))$. It implies (inverting the relation $\delta \to M$ and using the layer-cake representation) that if $\inf_{Q_{r_0/2}} h < 1$,

$$\text{(4.2)}\quad \forall\, M \geqslant 1,\quad \frac{\big|\{h \geqslant M\} \cap Q^-_{r_0}\big|}{\big|Q^-_{r_0}\big|} \lesssim \delta(M) = \Big(\frac{1}{\ln(1+M)}\Big)^{1/(10d+17)} \implies \int_{Q^-_{r_0}} [\ln(1+h)]^{1/(10d+18)} \lesssim 1.$$

This "point-to-measure" estimate controls the decay of the upper level set in the manner of a *weak Harnack inequality*, although with a "logarithmic" rather than power-law integrability. We shall now improve the integrability to a power-law by going back to (4.1) and performing an inductive argument inspired from the elliptic theory [LZ17]. Note that the logarithmic integrability in (4.2) is reminiscent of Moser's approach.

We improve inductively the control of upper level sets in the following decreasing sequence of cylinders

$$\mathscr{Q}^k := Q_{(r_0/2)+\alpha_k}\left(-\frac{5}{2} r_0^2 + \frac{1}{2}\left(\frac{r_0}{2} + \alpha_k\right)^2, 0, 0\right) \subset Q^-_{r_0} \quad \text{with} \quad \alpha_k := \frac{r_0}{2 \times 7^{k-1}}.$$

These cylinders satisfy $\widetilde{Q}^-_{r_0/2} \subset \mathscr{Q}^k \subset \overline{\mathscr{Q}}^k \subset \mathring{\mathscr{Q}}^{k-1} \subset Q^-_{r_0}$ for all $k \geqslant 1$. We now claim that for $\delta_0 > 0$ small enough (to be chosen later), for any non-negative super-solution h with $\inf_{Q_{r_0/2}} h < 1$ we have

$$\text{(4.3)}\qquad \forall\, k \geqslant 1,\quad \frac{\big|\{h \geqslant M^k\} \cap \mathscr{Q}^k\big|}{|\mathscr{Q}^k|} \leqslant \frac{\delta_0}{210^{(4d+2)k}},$$

where $M \sim \delta^{-2(1+\delta^{-10d-16})}$ with $\delta := \delta_0/210^{4d+2}$ as in (4.1). Admitting first (4.3) we deduce by layer-cake representation that there is an explicit $\zeta \gtrsim \delta_0^{10d+17} > 0$ such that $\int_{\widetilde{Q}^-_{r_0/2}} h^\zeta \,\mathrm{d}z \lesssim 1$, which implies by linearity

$$\left(\int_{\widetilde{Q}^-_{r_0/2}} h(z)^\zeta \,\mathrm{d}z\right)^{1/\zeta} \lesssim \inf_{Q_{r_0/2}} h.$$

This implies the weak Harnack inequality (1.5) on any f non-negative super-solution to (1.1)–(1.2) by applying the previous estimate to $h := f + (1+t)\|S\|_{L^\infty(Q_1)}$. To deduce the Harnack inequality (1.6) we consider f a non-negative solution to (1.1)–(1.2) and combine the previous control with Proposition 12 to get

$$\sup_{\widetilde{Q}^-_{r_0/4}} f \lesssim \left(\int_{\widetilde{Q}^-_{r_0/2}} f(z)^\zeta \,\mathrm{d}z \right)^{1/\zeta} + \|S\|_{L^\infty(Q_1)}$$
$$\lesssim \inf_{Q_{r_0/2}} f + \|S\|_{L^\infty(Q_1)} \lesssim \inf_{Q_{r_0/4}} f + \|S\|_{L^\infty(Q_1)}.$$

Let us now prove the claim (4.3) to conclude the proof. The initialization $k = 1$ is proved in (4.2). Then define $A_{k+1} := \{h > M^{k+1}\} \cap \mathscr{Q}^{k+1}$ and denote the following translated centered cylinders

$$\mathfrak{C}_r[z] := z \circ Q_{2r}((2r^2, 0, 0)) = z \circ \left(-2r^2, 2r^2\right] \times B_{(2r)^3} \times B_{2r}.$$

Let us construct $z_\ell = (t_\ell, x_\ell, v_\ell) \in \mathscr{Q}^{k+1}$ and $r_\ell > 0$, $\ell \geqslant 1$, so that:

(1) $\forall\, \ell \geqslant 1$, $r_\ell \in (0, \alpha_{k+1}/15)$,
(2) $\forall\, \ell \geqslant 1$, $|A_{k+1} \cap \mathfrak{C}_{15 r_\ell}[z_\ell]| \leqslant \delta_0 |\mathfrak{C}_{15 r_\ell}[z_\ell]|$,
(3) $\forall\, \ell \geqslant 1$, $|A_{k+1} \cap \mathfrak{C}_{r_\ell}[z_\ell]| > \delta_0 |\mathfrak{C}_{r_\ell}[z_\ell]|$,
(4) the cylinders $\mathfrak{C}_{3r_\ell}[z_\ell]$, $\ell \geqslant 1$, are disjoint,
(5) A_{k+1} is covered by the family $\mathfrak{C}_{15r_\ell}[z_\ell]$, $\ell \geqslant 1$.

Note that inverting the operation $Q_{r/2}(z) \to Q^-_r(z)$ in Lemma 16 yields, when starting from $\mathfrak{C}_{r_\ell}[z_\ell]$, the cylinder $\mathfrak{C}_{r_\ell}[z_\ell]^+ := z_\ell \circ Q_{r_\ell}((10 r_\ell^2, 0, 0)) = z_\ell \circ \left(9r_\ell^2, 10 r_\ell^2\right] \times B_{r_\ell^3} \times B_{r_\ell}$. Note also that $\mathfrak{C}_{r_\ell}[z_\ell]^+ \subset \mathfrak{C}_{3r_\ell}[z_\ell]$ and that property (1) combined with $z_\ell \in \mathscr{Q}^{k+1}$ imply $\mathfrak{C}_{15r_\ell}[z_\ell] \subset \mathscr{Q}^k$. Let us prove that the family $\mathscr{F}$ of cylinders $\mathfrak{C}_r[z]$ with $z \in \mathscr{Q}^{k+1}$, $r \in (0, \alpha_{k+1}/15)$ and so that $|A_{k+1} \cap \mathfrak{C}_{15r}[z]| \leqslant \delta_0 |\mathfrak{C}_{15r}[z]|$ and $|A_{k+1} \cap \mathfrak{C}_r[z]| > \delta_0 |\mathfrak{C}_r[z]|$ cover A_{k+1}. We have, using (4.3) at the previous step k,

$$\forall\, r \in \left(\frac{\alpha_{k+1}}{15}, \alpha_{k+1} \right), \quad |A_{k+1} \cap \mathfrak{C}_r[z]| \leqslant |A_k \cap \mathfrak{C}_r[z]| \leqslant |A_k \cap \mathscr{Q}^k| \leqslant \frac{\delta_0}{210^{(4d+2)k}} |\mathscr{Q}^k| \leqslant \delta_0 |\mathfrak{C}_r[z]|. \tag{4.4}$$

If $z \in A_{k+1}$ is not covered by $\mathscr{F}$ it means that the continuous positive function $\varphi(r) = |A_{k+1} \cap \mathfrak{C}_r[z]| / |\mathfrak{C}_r[z]|$ on $(0, +\infty)$ satisfies $\varphi(r) \leqslant \delta_0$ or $\varphi(15r) > \delta_0$ for all $r \in (0, \alpha_{k+1}/15)$. The constraint (4.4) and the continuity impose $\varphi(r) \leqslant \delta_0$ for all $r \in (0, \alpha_{k+1}/15)$. Taking $r \to 0$, a straightforward variation of the Lebesgue differentiation theorem then implies $z \notin A_{k+1}$ which contradicts the assumption. Hence A_{k+1} is covered by the family $\mathscr{F}$.

It implies in particular that A_{k+1} is covered by the family $\mathscr{F}'$ of cylinders $\mathfrak{C}_{3r}[z]$ with $z \in \mathscr{Q}^{k+1}$, $r \in (0, \alpha_{k+1}/15)$ and such that $|A_{k+1} \cap \mathfrak{C}_{15r}[z]| \leqslant \delta_0 |\mathfrak{C}_{15r}[z]|$ and $|A_{k+1} \cap \mathfrak{C}_r[z]| > \delta_0 |\mathfrak{C}_r[z]|$. The Vitali covering lemma then gives the existence of a countable sub-family, denoted $(\mathfrak{C}_{r_\ell}[z_\ell])_{\ell \geqslant 1}$, such that the $(\mathfrak{C}_{15 r_\ell}[z_\ell])_{\ell \geqslant 1}$ cover A_{k+1} and the $(\mathfrak{C}_{3r_\ell}[z_\ell])_{\ell \geqslant 1}$ are disjoint. The Vitali lemma applies thanks to the following property:

$$\Big[\mathfrak{C}_{r_1}[z_1]\cap\mathfrak{C}_{r_2}[z_2]\neq\varnothing \ \text{ and } \ r_1\leqslant 2r_2\Big] \implies \mathfrak{C}_{r_1}[z_1]\subset\mathfrak{C}_{5r_2}[z_2].$$

Take $z_0=(t_0,x_0,v_0)$ in the intersection and $z=(t,x,v)\in\mathfrak{C}_{r_1}[z_1]$. Inequalities $|t-t_2|\leqslant 18r_2^2$ and $|v-v_2|\leqslant 10r_2$ come naturally and $|x-[x_2+2r_2^2v_2+(t-t_2)v_2]|\leqslant 200r_2^3$ follows from

$$\begin{aligned}
&\big|x-\big[x_2+2r_2^2v_2+(t-t_2)v_2\big]\big|\\
&\quad\leqslant \big|x-\big[x_1+2r_1^2v_1+(t-t_1)v_1\big]\big|\\
&\qquad\qquad\qquad + \big|\big[x_2+2r_2^2v_2+(t-t_2)v_2\big]-\big[x_1+2r_1^2v_1+(t-t_1)v_1\big]\big|\\
&\quad\leqslant r_1^3+\big|\big[x_2+2r_2^2v_2+(t_0-t_2)v_2\big]-\big[x_1+2r_1^2v_1+(t_0-t_1)v_1\big]\big|\\
&\qquad\qquad\qquad +\big|(t-t_0)(v_2-v_1)\big|\\
&\quad\leqslant 128r_2^3+\big|x_0-\big[x_1+2r_1^2v_1+(t_0-t_1)v_1\big]\big|+\big|x_0-\big[x_2+2r_2^2v_2+(t_0-t_2)v_2\big]\big|\\
&\quad\leqslant 200r_2^3.
\end{aligned}$$

This finishes constructing the covering with the properties (1)-(2)-(3)-(4)-(5) above. Then Lemma 16 applied to each $\mathfrak{C}_{r_\ell}[z_\ell]$ implies $\mathfrak{C}_{r_\ell}[z_\ell]^+\subset A_k$, and the $\mathfrak{C}_{r_\ell}[z_\ell]^+\subset\mathfrak{C}_{3r_\ell}[z_\ell]$ are disjoint. We deduce

$$\begin{aligned}
|A_{k+1}|&\leqslant\sum_{\ell\geqslant1}|A_{k+1}\cap\mathfrak{C}_{15r_\ell}[z_\ell]|\leqslant\delta_0\sum_{\ell\geqslant1}|\mathfrak{C}_{15r_\ell}[z_\ell]|\\
&\leqslant 15^{4d+2}\delta_0\sum_{\ell\geqslant1}|\mathfrak{C}_{r_\ell}[z_\ell]|\leqslant 30^{4d+2}\delta_0\sum_{\ell\geqslant1}|\mathfrak{C}_{r_\ell}[z_\ell]^+|\\
&\leqslant 30^{4d+2}\delta_0|A_k|\leqslant\frac{30^{4d+2}\delta_0^2}{210^{(4d+2)k}}\leqslant\frac{\delta_0}{210^{(4d+2)(k+1)}}\left|\mathscr{Q}^{k+1}\right|
\end{aligned}$$

for δ_0 small enough which proves the induction claim (4.3) and concludes the proof.

4.2. The Hölder continuity. — De Giorgi's argument to Hölder continuity uses the measure-to-pointwise Lemma 16. We briefly sketch it in order to track the constant. Hölder regularity could also be deduced from the Harnack inequality in Theorem 5. Given f solution to (1.1)–(1.2) on Q_2 and $r_0=1/40$

$$\operatorname{osc}_{Q_{r_0}} f\leqslant\Big(1-\frac{\mu}{2}\Big)\max\big(\operatorname{osc}_{Q_1} f, e^{2(1+2^{10d+16})}\|S\|_{L^\infty(Q_2)}\big)\tag{4.5}$$

follows from Lemma 16 rescaled to Q_2 with $\delta=1/2$ and applied to whichever of F or $-F$ satisfies (3.17), where

$$F:=2\big[\max\big(\operatorname{osc}_{Q_1} f, e^{2(1+2^{10d+16})}\|S\|_{L^\infty(Q_2)}\big)\big]^{-1}\big[f-\tfrac12(\sup_{Q_1} f+\inf_{Q_1} f)\big].$$

By iteration we deduce

(4.6) $\quad\forall z_0\in Q_1,\forall\, r\in(0,r_0),$

$$\operatorname{osc}_{Q_r(z_0)} f\leqslant r^\alpha e^{2(1+2^{10d+16})}\left(1+\|S\|_{L^\infty(Q_2)}\right)\max\big(e^{2(1+2^{10d+16})}\|S\|_{L^\infty(Q_2)},\operatorname{osc}_{Q_1} f\big).$$

Indeed, the following sequence of solution of (1.1)–(1.2) in Q_1

$$f_n(\tau,y,w) = 2\frac{(1-\mu/2)^{1-n}\, f\left(t_0+r_0^{2n}\tau, x_0-r_0^{2n}\tau v_0+r_0^{3n}y, v_0+r_0^n w\right)}{\max\left(\operatorname{osc}_{Q_1} f, e^{2(1+2^{10d+16})}\|S\|_{L^\infty(Q_2)}\right)}$$

satisfies $\operatorname{osc}_{Q_1} f_n \leqslant 2e^{2(1+2^{10d+16})}(1+\|S\|_{L^\infty(Q_2)})$ by induction on $n\geqslant 1$ (the case $n=1$ is true by definition of f_n and it propagates thanks to (4.5)). If one defines $\alpha \in (0,1)$ such that $1-\mu/2 = r_0^\alpha$ (assuming that μ is small enough), the previous induction implies (4.6) by standard arguments. To deduce the Hölder estimate between $z, z' \in Q_1$, use intermediate points in $[z, z']$ at distance less than r_0 and the estimate (4.6).

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