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UNCONDITIONAL CHEBYSHEV BIASES IN NUMBER FIELDS

by Daniel Fiorilli & Florent Jouve

Abstract. — Chebyshev's bias is the phenomenon according to which for most x, the interval $[2, x]$ contains more primes congruent to 3 modulo 4 than primes congruent to 1 modulo 4. We present new families of examples of analogous phenomena when counting prime ideals in number fields of higher degree where the bias takes place for all large enough x. Our proofs are unconditional.

Résumé (Biais de Tchebychev inconditionnels dans les corps de nombres)

On appelle biais de Tchebychev le phénomène de prépondérance du nombre de premiers congrus à 3 modulo 4 par rapport aux premiers congrus à 1 modulo 4 dans l'intervalle $[2, x]$, pour la plupart des valeurs de x. Nous présentons de nouvelles familles d'exemples de phénomènes analogues où l'on compte des idéaux premiers dans des corps de nombres de degré supérieur et où l'on observe un biais pour tout x assez grand. Nos preuves sont inconditionnelles.

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1. Introduction and statement of results

In 1853, Chebyshev noticed in a letter to Fuss [Che53] that there seems to exist a bias in the distribution of primes modulo 4, that is in most intervals of the form $[2, x]$, there appears to be more primes of the form $4n + 3$ than of the form $4n + 1$. It turns out that the specific statements made in Chebyshev's letter are quite deep:

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the second is equivalent to the Riemann hypothesis for $L(s,\chi_{-4})$, and the first can be made explicit under an additional linear independence hypothesis on the zeros of $L(s,\chi_{-4})$. Chebyshev's observation has been widely generalized over the years; notably, Rubinstein and Sarnak [RS94] have shown that for two invertible residue classes a and b modulo q, there exists a bias towards a (that is $\pi(x;q,a)>\pi(x;q,b)$ is true more often than $\pi(x;q,a)<\pi(x;q,b)$) if and only if b is a quadratic residue and a is a non-quadratic residue. These theoretical results, as well as the numerical determinations of the bias in the paper, are conditional on the generalized Riemann hypothesis and a linear independence hypothesis on the non-trivial zeros of Dirichlet L-functions. In the same paper [RS94, §5], the authors mention several possible generalizations including biases in the distribution of prime ideals in Galois extensions of number fields. This context was explored by Ng in his Ph.D. thesis [Ng00]. Consider a Galois extension L/K of number fields, a conjugacy class $C\subset G=\mathrm{Gal}(L/K)$, and define the Frobenius counting function

$$\pi(x;L/K,C):=\sum_{\substack{\mathfrak{p}\lhd\mathscr{O}_K\text{ unram.}\\ \mathscr{N}\mathfrak{p}\leqslant x\\ \mathrm{Frob}_{\mathfrak{p}}=C}}1,$$

where $\mathrm{Frob}_{\mathfrak{p}}$ denotes the Frobenius conjugacy class associated to the unramified prime ideal $\mathfrak{p}$, and $\mathscr{N}\mathfrak{p}=|\mathscr{O}_K/\mathfrak{p}|$ denotes its norm. The Chebotarev density theorem asserts that

$$\pi(x;L/K,C)\sim\frac{|C|}{|G|}\int_2^x\frac{\mathrm{d}t}{\log t}.$$

More precisely, one is interested in understanding the size of the sets

$$P_{L/K;C_1,C_2}:=\{x\in\mathbb{R}_{\geqslant 1}\colon |C_2|\pi(x;L/K,C_1)>|C_1|\pi(x;L/K,C_2)\}.$$

Ng [Ng00] has shown under Artin's holomorphy conjecture, GRH, as well as a linear independence hypothesis on the set of zeros of Artin L-functions, that the set $P_{L/K;C_1,C_2}$ admits a logarithmic density, that is the limit

$$\delta(P_{L/K;C_1,C_2}):=\lim_{X\to\infty}\frac{1}{\log X}\int_{\substack{1\leqslant x\leqslant X\\ x\in P_{L/K;C_1,C_2}}}\frac{\mathrm{d}x}{x}$$

exists. Moreover, he computed this density in several explicit extensions, under the same hypotheses.

The goal of this paper is to show *unconditionally* the existence of the density $\delta(P_{L/K;C_1,C_2})$ in some families of extensions and for specific conjugacy classes. More precisely, we will exhibit a sufficient group-theoretic criterion on $G=\mathrm{Gal}(L/K)$ which implies in particular that $\delta(P_{L/K;C_1,C_2})=1$. This will involve the class function $r_G:G\to\mathbb{C}$ defined by

$$r_G(g):=\#\{h\in G\colon h^2=g\}.$$

We will require $L/\mathbb{Q}$ to be Galois, and for a conjugacy class $C\subset G$ we will denote by C^+ the unique conjugacy class of $G^+:=\mathrm{Gal}(L/\mathbb{Q})$ which contains C. Explicitly,

$$C^+:=\bigcup_{a\in G^+}aCa^{-1}. \tag{1}$$

Theorem 1.1. — *Let L/K be an extension of number fields for which $L/\mathbb{Q}$ is Galois. Assume that the conjugacy classes $C_1, C_2 \subset G = \mathrm{Gal}(L/K)$ are such that $C_1^+ = C_2^+$, but $r_G(g_{C_1}) < r_G(g_{C_2})$, where g_{C_i} is a representative of C_i. Then, for all large enough x we have the inequality $|C_2|\pi(x, L/K, C_1) > |C_1|\pi(x, L/K, C_2)$. In particular, the set $P_{L/K;C_1,C_2}$ has natural (and logarithmic) density equal to 1.*

Remark. — The fact that the natural density of $P_{L/K;C_1,C_2}$ exists in Theorem 1.1 is remarkable since it is widely believed that in the classical case of primes in arithmetic progressions as well as in the more general case of Galois extensions of number fields, the logarithmic density is the appropriate notion to work with. In general one cannot expect natural densities to exist (see [Kac95], as well as [RS94, p. 174] and the references therein).

Note also that in Theorem 1.1, one can further impose C_1 and C_2 to have the same size. Indeed, we will see in the proof of Proposition 1.2 (see Section 2) that there exists families of examples in which the group G is abelian.

Next we state a group theoretic result showing that the hypotheses of Theorem 1.1 are satisfied by infinitely many couples (G, G^+) and associated conjugacy classes $C_1, C_2 \subset G$.

Proposition 1.2. — *For $n \geqslant 8$ the symmetric group $G^+ = S_n$ admits a subgroup G which contains conjugacy classes C_1, C_2 satisfying $C_1^+ = C_2^+$, but $r_G(g_{C_1}) < r_G(g_{C_2})$, where $g_{C_i} \in C_i$ $(i = 1, 2)$.*

The combination of Theorem 1.1, Proposition 1.2 and the fact going back to Hilbert that the inverse Galois problem over $\mathbb{Q}$ is solved for the symmetric group S_n immediately yields the following consequence.

Corollary 1.3. — *There exists infinitely many Galois extensions L/K and conjugacy classes $C_1, C_2 \subset \mathrm{Gal}(L/K)$ for which $\delta(P_{L/K;C_1,C_2}) = 1$.*

The paper is organized as follows. Section 2 is devoted to the group theoretic aspects of our main result. In particular we prove Proposition 1.2 and discuss generalizations and related questions. In Section 3, we prove Theorem 1.1. We conclude the paper with Section 4 which is devoted to numerical computations and illustrations of Theorem 1.1.

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2. Group theoretical results

The goal of this section is to construct families of abelian extensions L/K satisfying the hypotheses of Theorem 1.1.

Proof of Proposition 1.2. — For $n \geqslant 8$, consider the permutations $g_1 := (12)(34)$ and $g_2 := (57)(68)$ as elements of S_n. Let $G := \langle (12)(34), (5678) \rangle < S_n$. We claim that the choices $C_1 = \{g_1\}$ and $C_2 = \{g_2\}$ satisfy the required properties. Indeed, $C_1^+ = C_2^+ = C_{(2,2)}$, where $C_{(2,2)}$ is the set of elements of S_n of cycle type $(2,2)$. Moreover, an enumeration of the elements of G shows that $r_G(g_1) = 0$ and $r_G(g_2) = 4$. □

The next lemma gives a group theoretical criterion which generalizes the construction in the proof of Proposition 1.2 and which implies the conditions of Theorem 1.1. (Here and later in the paper we make a slight abuse of notation by denoting $r_G(C)$ the common value $r_G(g)$ as g runs over the G-conjugacy class C.)

Lemma 2.1. — *Let G^+ be a group and let H and K be subgroups having trivial intersection and such that H centralizes K. Let $h \in H$ be a non-square (in H), and let $k \in K$ be a square (in K) which is a conjugate of h in G^+. Then, the conjugacy classes $C_1 = C_h$ and $C_2 = C_k$ in the group $G = HK$ are such that $r_G(C_2) > r_G(C_1)$; in other words, the conditions of Theorem* 1.1 *hold.*

Proof. — The fact that H centralizes K guarantees that $G = HK = KH$ is a subgroup of G^+. Moreover, any $x \in G$ such that $x^2 = k$ can be written $x = st$ with $s \in H$ and $t \in K$ (and in this decomposition there is a unique (s,t) corresponding to each x since $H \cap K = \{1\}$). Thus $k = s^2 t^2$, which implies that $s^2 \in H \cap K$. Therefore $s^2 = 1$, and as a result

$$\#\{x \in G \colon x^2 = k\} = \#\{x \in K \colon x^2 = k\} \cdot \#\{x \in H \colon x^2 = 1\} > 0.$$

By symmetry, we also have that

$$\#\{x \in G \colon x^2 = h\} = \#\{x \in H \colon x^2 = h\} \cdot \#\{x \in K \colon x^2 = 1\} = 0.$$ □

In order to apply Lemma 2.1, take for instance $G^+ = S_n$, and let $\sigma, \tau \in S_n$ be permutations of order divisible by 4 which have the same cycle type, but have disjoint supports. Consider the subgroups $H = \langle \sigma^2 \rangle$ and $K = \langle \tau \rangle$, and the elements $h = \sigma^2$ and $k = \tau^2$. We clearly have that $r_K(k) \geqslant 1$ and $r_H(h) = 0$, and Lemma 2.1 applies.

Remark. — From a group theoretical point of view, it would be interesting to classify the tuples (G, H, C_1, C_2) such that G is a finite group, $H < G$ and $C_1, C_2 \subset H$ are conjugacy classes such that $r_H(C_1) \neq r_H(C_2)$ and $C_1^+ = C_2^+$ (recall (1)). For example, one notices that no such tuple exists where H is a normal subgroup of G (see [FJ20, Proof of Lem. 3.13]). Beyond this case, one may ask the following questions: how rare is the property enjoyed by these tuples? What are the "minimal" examples? Such questions are the subject of Mounir Hayani's forthcoming Ph.D. thesis.

3. Proof of Theorem 1.1

To introduce the natural framework of Theorem 1.1, we will work in the setting of [Bel16], that is we will consider general class functions $t\colon \operatorname{Gal}(L/K) \to \mathbb{C}$, and define[(1)]

$$\psi(x;L/K,t) := \sum_{\substack{\mathfrak{p}\lhd\mathscr{O}_K\\ \mathscr{N}\mathfrak{p}\leqslant x\\ k\geqslant 1}} t(\mathrm{Frob}_{\mathfrak{p}}^k)\log(\mathscr{N}\mathfrak{p}); \qquad \theta(x;L/K,t) := \sum_{\substack{\mathfrak{p}\lhd\mathscr{O}_K\\ \mathscr{N}\mathfrak{p}\leqslant x}} t(\mathrm{Frob}_{\mathfrak{p}})\log(\mathscr{N}\mathfrak{p});$$

$$\pi(x;L/K,t) := \sum_{\substack{\mathfrak{p}\lhd\mathscr{O}_K\\ \mathscr{N}\mathfrak{p}\leqslant x\\ \mathfrak{p}\text{ unram.}}} t(\mathrm{Frob}_{\mathfrak{p}}).$$

When $L/\mathbb{Q}$ is Galois, we will use the shorthands $G := \operatorname{Gal}(L/K)$, $G^+ := \operatorname{Gal}(L/\mathbb{Q})$, as well as

$$t^+ = \operatorname{Ind}_G^{G^+} t\colon G^+ \longrightarrow \mathbb{C}, \qquad g \longmapsto \sum_{\substack{aG\in G^+/G:\\ a^{-1}ga\in G}} t(a^{-1}ga).$$

Finally, we recall that the inner product of class functions $t_1,t_2\colon G\to\mathbb{C}$ is defined by

$$\langle t_1,t_2\rangle_G := \frac{1}{|G|}\sum_{g\in G} t_1(g)\overline{t_2(g)}.$$

(We will simply write $\langle t_1,t_2\rangle$, dropping the subscript G, where the underlying group is clear from context.)

Lemma 3.1. — *Let L/K be an extension of number fields for which $L/\mathbb{Q}$ is Galois, and let $t\colon \operatorname{Gal}(L/K)\to\mathbb{C}$ be a class function. We have the estimate*

$$\pi(x;L/K,t) = \int_{2^-}^{x} \frac{\mathrm{d}\psi(u;L/\mathbb{Q},t^+)}{\log u} - \langle t, r_G\rangle\frac{x^{1/2}}{\log x} + o\Big(\frac{x^{1/2}}{\log x}\Big).$$

Proof. — For any integer $\ell\geqslant 2$, denote by $f_\ell\colon G\to G$ the class function defined by $f_\ell(g)=g^\ell$. Let μ denote the Möbius function; inclusion-exclusion implies that

$$\begin{aligned}\theta(x;L/K,t) &= \psi(x;L/K,t) + \sum_{\ell\geqslant 2}\mu(\ell)\psi(x^{1/\ell};L/K,t\circ f_\ell)\\ &= \psi(x;L/K,t) - \langle t,r_G\rangle x^{1/2}(1+o(1)) + O(x^{1/3}),\end{aligned}$$

by the Chebotarev density theorem and the identity $\frac{1}{|G|}\sum_{g\in G} t(g^2) = \langle t, r_G\rangle$. The claimed estimate follows from a summation by parts and an application of the identity

$$\psi(u;L/K,t) = \psi(u;L/\mathbb{Q},t^+),$$

which is a consequence of the invariance of Artin L-functions under induction ([Art31, §2], under the form used in [FJ20, Prop. 3.11]). □

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(1) See for instance [Mar77, Chap. 1 §4] for a definition of $\mathrm{Frob}_{\mathfrak{p}}$ in the case where $\mathfrak{p}$ is ramified.

Proof of Theorem 1.1. — We first compute, for any conjugacy class C of G, any fixed $g_C \in C$ and any irreducible character χ of G^+,

$$\langle 1_C^+, \chi\rangle_{G^+} = \langle 1_C, \chi_{|G}\rangle_G = \frac{|C|}{|G|}\,\overline{\chi(g_C)} = \frac{|C||G^+|}{|G||C^+|}\,\langle 1_{C^+}, \chi\rangle_{G^+},$$

where the first step uses Frobenius reciprocity. Therefore, denoting

$$t_{C_1,C_2}\colon\ \mathrm{Gal}(L/K) \longrightarrow \mathbb{C}$$

the class function $t_{C_1,C_2} = \frac{|G|}{|C_1|}1_{C_1} - \frac{|G|}{|C_2|}1_{C_2}$, one has $t^+_{C_1,C_2} = \frac{|G^+|}{|C_1^+|}1_{C_1^+} - \frac{|G^+|}{|C_2^+|}1_{C_2^+} \equiv 0$. Hence, Lemma 3.1 implies that

$$\pi(x; L/K, t_{C_1,C_2}) = -\langle t_{C_1,C_2}, r_G\rangle\,\frac{x^{1/2}}{\log x} + o\Big(\frac{x^{1/2}}{\log x}\Big).$$

However, $-\langle t_{C_1,C_2}, r_G\rangle = r_G(g_{C_2}) - r_G(g_{C_1}) > 0$, and thus $\pi(x; L/K, t_{C_1,C_2}) > 0$ for all large enough values of x. $\square$

We now discuss more precisely the oscillations of $\pi(x; L/K, C_1) - \pi(x; L/K, C_2)$ for triples $(L/K, C_1, C_2)$ chosen as in the proof of Proposition 1.2 and Corollary 1.3 (an explicit example of such a Galois extension produces Figure 1, and the purpose here is to discuss the rate of convergence of the function plotted to its asymptotic value). We recall that in the proof of Proposition 1.2, we have chosen $G = \langle (12)(34), (5678)\rangle$ and $t = 1_{C_1} - 1_{C_2}$, where $C_1 = \{(12)(34)\}$ and $C_2 = \{(57)(68)\}$. Since G is abelian of order 8, the class function $r_m(g) := \#\{h \in G\colon h^m = g\}$ is identically equal to 1 for all odd $m \geqslant 1$, and in particular, $\langle t\circ f_3, 1\rangle = 0$ (where we recall that f_ℓ is the function on G raising elements to their ℓ-th power). The identity $\psi(x; L/K, t) = \psi(x; L/\mathbb{Q}, t^+) \equiv 0$ and the Riemann Hypothesis for Artin L-functions then imply that

$$\begin{aligned}\theta(x; L/K, t) &= \psi(x; L/K, t) - \psi(x^{1/2}; L/K, t\circ f_2) - \psi(x^{1/3}; L/K, t\circ f_3) + O(x^{1/5})\\ &= -\langle t, r_G\rangle x^{1/2} + \sum_{\chi\in\mathrm{Irr}(G)} \overline{\langle \chi, t\circ f_2\rangle} \sum_{\rho_\chi} \frac{x^{1/4+(1/2)\Im(\rho_\chi)i}}{\rho_\chi} + O(x^{1/5}),\end{aligned}$$

by the explicit formula (see for instance [Ng00, Th. 3.4.9]). Here, $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G, and ρ_χ runs through the non-trivial zeros of the Artin L-function $L(s, L/K, \chi)$. Now, in this particular example

$$8t\circ f_2 = 1_{\{(5678)\}} + 1_{\{(5678)(12)(34)\}} + 1_{\{(5876)\}} + 1_{\{(5876)(12)(34)\}},$$

and thus

$$\langle \chi, t\circ f_2\rangle = \chi((5678)) + \chi((5678)(12)(34)) + \chi((5876)) + \chi((5876)(12)(34))$$

(which is not identically zero). This explains why we expect the difference between the solid line and the data in Figure 1 to be roughly of order $x^{-1/4}$. (More precisely, we expect order $x^{-1/4}$ almost everywhere, and maximal order $x^{-1/4}(\log\log\log x)^2$.)

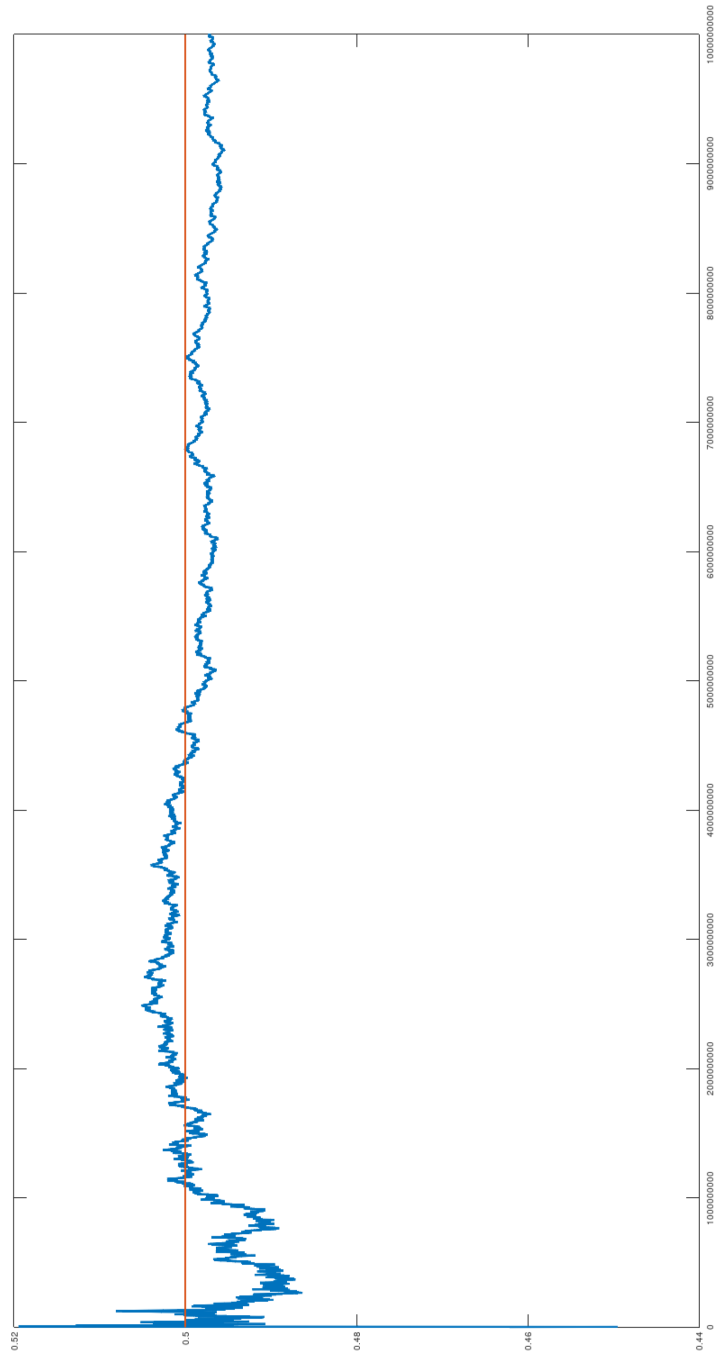

Figure 1. The normalized difference $(\pi(x; L/K, C_1) - \pi(x; L/K, C_2))/R(x)$ with $1 \leqslant x \leqslant 10^{10}$ (data due to B. Allombert)

4. Numerical examples

In this section we discuss our numerical verification of Theorem 1.1 and Proposition 1.2. It would be computationally very expensive to work with the full group S_8. However, it turns out that one can replace S_8 with a relatively small subgroup which has the required properties. Consider $G^+ := \langle (12)(34), (5678), (15)(27)(36)(48)\rangle$; let us show that G^+ is isomorphic to the wreath product of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, which is of order 32. Denote the permutations appearing in the generating set of G^+ by τ, σ, and γ, respectively, and note that $G^+ = \langle \sigma, \gamma\sigma\gamma, \gamma\rangle$ (since $\gamma\sigma\gamma = (1324)$, and thus $(\gamma\sigma\gamma)^2 = \tau$). The subgroup $\langle \sigma, \gamma\sigma\gamma\rangle$ is clearly isomorphic to $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$. Moreover, conjugating by γ on $\langle \sigma, \gamma\sigma\gamma\rangle$ amounts to exchanging the two factors $\mathbb{Z}/4\mathbb{Z}$, which is the definition of the wreath product.

Consider also the abelian subgroup $G := \langle (12)(34), (5678)\rangle < G^+$ as well as the conjugacy classes $C_1 := \{(12)(34)\}$ and $C_2 := \{(57)(68)\}$. In the group G^+, one has that $\gamma^{-1}C_1\gamma = C_2$, that is $C_1^+ = C_2^+$. It follows from Theorem 1.1 that for any Galois number field $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G^+$, the sub-extension $K = L^G$ has the property that for all large enough x,

$$\pi(x; L/K, C_1) > \pi(x; L/K, C_2)$$

(recall that $|C_1| = |C_2| = 1$). Bill Allombert has kindly provided us with the `pari/gp` code allowing for a numerical check of this inequality up to $x = 10^{10}$, for a particular number field $L/\mathbb{Q}$ of Galois group G^+. Explicitly, $L = \mathbb{Q}[x]/(f(x))$, where

$$\begin{aligned} f(x) = x^{32} &- 128x^{30} + 5680x^{28} - 120576x^{26} + 1386352x^{24} \\ &- 9267712x^{22} + 38233408x^{20} - 101305344x^{18} + 176213088x^{16} \\ &- 202610688x^{14} + 152933632x^{12} - 74141696x^{10} + 22181632x^{8} \\ &\qquad - 3858432x^{6} + 363520x^{4} - 16384x^{2} + 256. \end{aligned}$$

For the full code, visit
`https://www.math.u-bordeaux.fr/~fjouve001/UnconditionalBiasCode.gp`.
In Figure 1 we have plotted the difference $\pi(x; L/K, C_1) - \pi(x; L/K, C_2)$, normalized by the function

$$R(x) := \frac{x^{1/2}}{\log x} + \int_2^x \frac{\mathrm{d}u}{u^{1/2}(\log u)^2} \sim \frac{x^{1/2}}{\log x},$$

which can be shown following the proof of Lemma 3.1 to be the "natural approximation" for the order of magnitude of this difference. As expected, we see that the plotted function converges to 1/2, and to illustrate this we have added the solid line $y = 1/2$ on the plot. Finally, we see that as predicted in Section 3, the difference between the graph and the solid line is of order $x^{1/4}$.

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