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MOTION OF SEVERAL SLENDER RIGID FILAMENTS IN A STOKES FLOW

by Richard M. Höfer, Christophe Prange & Franck Sueur

Abstract. — We investigate the dynamics of several slender rigid bodies moving in a flow driven by the three-dimensional steady Stokes system in presence of a smooth background flow. More precisely, we consider the limit where the thickness of these slender rigid bodies tends to zero with a common rate ε, while their volumetric mass density is held fixed, so that the bodies shrink into separated massless curves. While for each positive ε, the bodies' dynamics are given by the Newton equations and correspond to some coupled second-order ODEs for the positions of the bodies, we prove that the limit equations are decoupled first-order ODEs whose coefficients only depend on the limit curves and on the background flow. We also determine the limit effect due to the limit curves on the fluid, in the spirit of the immersed boundary method.

Résumé (Mouvement de filaments rigides minces dans un fluide de Stokes)

Nous étudions la dynamique de filaments rigides minces se déplaçant dans un fluide qui est décrit par un état de base régulier perturbé par la présence des solides selon les équations du système de Stokes stationnaire tridimensionnel. Plus précisément, nous considérons la limite dans laquelle l'épaisseur des corps solides tend vers zéro avec un taux commun ε, tandis que leur densité de masse volumétrique est maintenue constante, de sorte que les solides limites occupent des courbes que l'on suppose d'intersections deux à deux vides, et ont une masse nulle. Pour $\varepsilon > 0$, la dynamique des solides est donnée par les équations de Newton et correspondent à des équations différentielles ordinaires du second ordre, couplées. Nous prouvons que les équations limites sont des équations différentielles ordinaires du premier ordre, découplées, dont les coefficients ne dépendent que des courbes limites et du flot de base. Nous déterminons également l'effet limite des courbes limites sur le fluide, dans l'esprit de la méthode des frontières immergées.

Mathematical subject classification (2020). — 74F10.
Keywords. — Slender rigid body, steady Stokes flow, fluid-solid interaction, singular perturbation.

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Contents

1. Introduction

In view of various applications in particular in biology (considering DNA) and in oceanography (considering sediment, plankton), the motion of rigid bodies with anisotropic shapes immersed in incompressible flows requests some mathematical analysis. In particular in view of modeling and numerics one may wish to consider asymptotic models where one gets rid of the dimensions corresponding to small extent of the rigid bodies. In this paper we consider the case of a finite number N of slender rigid bodies. We will use the terminology "filaments". Their radii have the same smallness parameter ε in $(0,1)$ and shrink into curves in $\mathbb{R}^3$ as $\varepsilon \to 0$. We will call these curves "centerline curves" or "filament centerlines". We will assume: (i) that these filaments are immersed into an incompressible fluid driven by the steady Stokes system in presence of a background flow, (ii) that their dynamics is driven by the Newton equations with forces acting on the filaments only due to the viscous stress tensor on their boundaries and (iii) that their volumetric mass density is independent of ε, so that their limit centerline curves are massless. This leads to a system of second-order ODEs all coupled to each other through the Stokes equations. We refer to Section 2 for a precise description of this so-called Newton-Stokes system, and for a straightforward local-in-time well-posedness result, see Lemma 2.2. In particular this result establishes the existence of smooth solutions as long as there is no collision between filaments.

The main result of this paper, Theorem 3.4 stated in Section 3.4, is the convergence of this Newton-Stokes system to a limit system describing the dynamics of the centerline curves. We will also determine the limit effect due to these centerline curves on the fluid, in the spirit of the immersed boundary method.

To give a flavor of our result to the reader we describe below how this limit system looks like. The precise description of the limit dynamics is given in Section 3.2. To emphasize what concerns limit objects when $\varepsilon \to 0$ we use the notation $\widehat{\bullet}$ for a quantity $\bullet$. On the other hand, to avoid heavy notations, we will only make the dependence on ε explicit when it is necessary in order to avoid confusion or to make precise in a quantitative way this dependence. Otherwise it is understood that the quantities at stake can depend on ε even if there is no corresponding index in the notation. The limit model drives the dynamics of the position at time t of a collection of smooth curves $\widehat{\mathcal{C}}_i(t)$ without any self-intersection, for $1 \leqslant i \leqslant N$, by the following rigid motion

$$\widehat{\mathcal{C}}_i(t) = \widehat{h}_i(t) + \widehat{Q}_i(t)\overline{\mathcal{C}}_i.$$

Here $\overline{\mathcal{C}}_i$ denotes a reference curve. The vector $\widehat{h}_i(t)$ in $\mathbb{R}^3$ and the matrix $\widehat{Q}_i(t)$ in $SO(3)$ satisfy some first-order ODEs:

$$\widehat{h}_i'(t) = \widehat{\mathrm{v}}_i(t), \quad \widehat{Q}_i'(t) = (\widehat{\omega}_i(t) \wedge \cdot)\widehat{Q}_i(t), \tag{1}$$

$$\text{and } (\widehat{\mathrm{v}}_i(t), \widehat{\omega}_i(t)) = F[\widehat{\mathcal{C}}_i(t), u^\flat(t,\cdot)|_{\widehat{\mathcal{C}}_i(t)}], \tag{2}$$

where the notation $(\widehat{\omega}_i(t) \wedge \cdot)$ is used for the skew-symmetric matrix canonically associated with the wedge product by the vector $\widehat{\omega}_i(t)$ in $\mathbb{R}^3$, $u^\flat(t,\cdot)|_{\widehat{\mathcal{C}}_i(t)}$ denotes the trace on $\widehat{\mathcal{C}}_i(t)$ of a smooth background flow velocity $u^\flat(t,\cdot)$. Moreover, $F[\cdot,\cdot]$ is a universal operator acting on smooth simple curves and smooth incompressible vector fields. This operator is given explicitly in (39). Let us highlight that, to ease the reading, in the left hand side of (2), and several more times below, we identify (v, w) with the corresponding column vector. We emphasize that the limit dynamics (1)–(2) is a system of uncoupled first-order ODEs which reflects that both the mass of the filaments and their perturbation on the fluid tends to zero as ε converges to 0.

The main novelty of our work is the mathematically study of the coupled dynamics of a collection of slender filaments. Hydrodynamic interactions between filaments have been extensively studied in the physics and engineering literature, in particular regarding the behavior of polymers and microorganisms, see for example [8, 9, 27, 28, 29]. As far as we know, previous mathematical works on filaments in Stokes flows are focused on static problems with only a single filament.

Related to these works, the first part of our analysis establishes approximations for the forces and torques acting on the filaments as well as on the fluid perturbation caused by the filaments. We show that explicit force distributions on the particle centerlines are sufficient to capture these quantities to leading order. This part of our analysis, carried out in Section 4, is related to so-called slender body theory, and we believe that our results there, Theorem 4.1 and Corollary 4.2 are of independent interest. The second part of our analysis consists in the study of a system of singularly perturbed ODEs relying on a modulated energy argument. We refer to Sections 3.5 and 3.6 for a more extensive outline of the key elements and the structure of the proof of the main result, to Section 3.7 for a discussion of some related results and to Section 3.8 for some open problems.

2. Setting of the problem

This section is devoted to the description of the setting of the problem.

2.1. Geometry of the filaments. — For each index i such that $1 \leqslant i \leqslant N$ we consider a filament $\mathcal{S}_i$ which can be closed or non-closed. For $\varepsilon \in (0, 1)$, the filament is given in terms of a reference filament $\overline{\mathcal{S}}_i$ which is described by a centerline and a shape function for the cross section as follows. For a non-closed filament, the centerline $\overline{\mathcal{C}_i}$ is assumed to be a curve of length $L_i > 0$, parametrized by arc length without self-intersections by a smooth function $\gamma_i\colon [0, L_i] \to \mathbb{R}^3$. We assume, for each index i such that $1 \leqslant i \leqslant N$, that the curve $\overline{\mathcal{C}_i}$ is not a straight line, i.e., $\gamma_i'' \neq 0$. The shape function is a smooth map $\Psi_i\colon [0, L_i] \times B_1(0) \to \mathbb{R}^2$, such that $\Psi_i(s, 0) = 0$ and $\Psi_i(s, \cdot)$ is a diffeomorphism to its image for all $s \in [0, L_i]$. Here, $B_1(0)$ denotes the open unit ball in $\mathbb{R}^2$. Moreover, let $R_i\colon [0, L_i] \to SO(3)$ be a smooth function such that $R_i e_3 = \gamma_i'$, where $e_3 = (0, 0, 1)$. Then, we define

$$\overline{\mathcal{S}}_i^\varepsilon = \overline{\mathcal{S}}_i := \{\gamma_i(s) + \varepsilon R_i(s)(\Psi_i(s, B_1(0)) \times \{0\}) : 0 \leqslant s \leqslant L_i\}. \tag{3}$$

In the case of a closed filament, the definition is analogous but we replace the interval $[0, L_i]$ by $\mathbb{R}/L_i\mathbb{Z}$ for γ_i, R_i and Ψ_i.

Remark 2.1. — Note that the non-closed filaments are not smooth but only Lipschitz due to corners at their ends. With minor modifications of some arguments, our analysis also applies to smooth non-closed filaments, which could be defined as

$$\overline{\mathcal{S}}_i^\varepsilon = \overline{\mathcal{S}}_i := \{\gamma_i(s) + \varepsilon a_\varepsilon(s) R_i(s)(\Psi_i(s, B_1(0)) \times \{0\}) : 0 \leqslant s \leqslant L_i\}, \tag{4}$$

where the additional function $a_\varepsilon(s) : [0, L_i] \to \mathbb{R}$ is given by

$$a_\varepsilon(s) = \begin{cases} \sqrt{s/\varepsilon} & \text{for } s \in [0, \varepsilon], \\ 1 & \text{for } s \in [\varepsilon, L_i - \varepsilon], \\ \sqrt{(L_i - s)/\varepsilon} & \text{for } s \in [L_i - \varepsilon, L_i]. \end{cases} \tag{5}$$

We assume that the reference centerlines are centered at the origin in the sense that

$$\int_{\overline{\mathcal{C}}_i} x \, d\mathcal{H}^1 = 0, \tag{6}$$

where $d\mathcal{H}^1$ is the one-dimensional Hausdorff measure. We emphasize that the center of mass of the reference filaments $\mathcal{S}_i$, for $1 \leqslant i \leqslant N$, depends on ε. For simplicity, we assume that the mass density is constant in each of the filaments. Then, their centers of mass are given as the following barycenters

$$\overline{h}_{i,\varepsilon} := \fint_{\overline{\mathcal{S}}_i} x \, dx. \tag{7}$$

By (6), we have

$$|\overline{h}_{i,\varepsilon}| \leqslant C\varepsilon, \tag{8}$$

where the constant C depends only on the functions specifying the reference filament, i.e., L_i, γ_i, R_i, and Ψ_i.

We are interested in the limit of the dynamics (specified below) as $\varepsilon \to 0$ for given, ε-independent initial data for the centerlines of the filaments. More precisely, we fix $\widehat{h}_i(0) \in \mathbb{R}^3$, $\widehat{Q}_i(0) \in SO(3)$ such that

$$\widehat{\mathcal{C}}_i(0) = \mathcal{C}_i(0) := \widehat{h}_i(0) + \widehat{Q}_i(0)\overline{\mathcal{C}}_i, \tag{9}$$

are the positions of the centerlines at time 0 for all $\varepsilon > 0$.

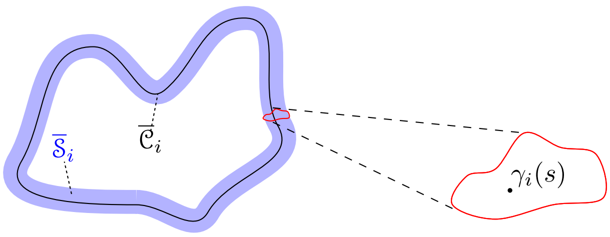

Figure 1. A closed reference filament

Then, the center of mass $h_{i,\varepsilon}(t)$ and the orientation $Q_{i,\varepsilon}(t)$ of the filament at time t have initial data

$$h_{i,\varepsilon}(0) = \widehat{h}_i(0) + \widehat{Q}_i(0)\overline{h}_{i,\varepsilon} \quad \text{and} \quad Q_{i,\varepsilon}(0) = \widehat{Q}_i(0), \tag{10}$$

and we denote the filament at time t with parameter ε by

$$\mathcal{S}_i(t) := \mathcal{S}_{i,\varepsilon}(t) = h_{i,\varepsilon}(t) + Q_{i,\varepsilon}(t)(\overline{\mathcal{S}}_i - \overline{h}_{i,\varepsilon}), \tag{11}$$

and similarly for the centerline $\mathcal{C}_i(t)$:

$$\mathcal{C}_i(t) := \mathcal{C}_{i,\varepsilon}(t) = h_{i,\varepsilon}(t) + Q_{i,\varepsilon}(t)(\overline{\mathcal{C}}_i - \overline{h}_{i,\varepsilon}).$$

2.2. Kinematics of the filaments. — For any $t \geqslant 0$, we denote by $\omega_i(t)$ in $\mathbb{R}^3$ the unique angular velocity of the i-th filament such that

$$Q'_{i,\varepsilon}(t)Q^T_{i,\varepsilon}(t) = (\omega_i(t) \wedge \cdot), \tag{12}$$

where $Q^T_{i,\varepsilon}$ denotes the transpose matrix of $Q_{i,\varepsilon}$ and $(\omega_i(t) \wedge \cdot)$ denotes the skew-symmetric matrix canonically associated with the wedge product by vector $\omega_i(t)$. We also set

$$\mathrm{v}_i(t) := h'_{i,\varepsilon}(t). \tag{13}$$

Accordingly, the solid velocities are given by

$$v^{\mathcal{S}_i}(t,x) := \mathrm{v}_i(t) + \omega_i(t) \wedge (x - h_{i,\varepsilon}(t)), \tag{14}$$

for all $x \in \mathcal{S}_i(t)$.

We highlight that all these quantities depend implicitly on ε which we usually omit in the notation except for the quantities $h_{i,\varepsilon}, Q_{i,\varepsilon}$. For these, we will always write the ε to avoid confusion with functions depending on variables h_i, Q_i that will appear later.

2.3. Inertia of the filaments. — We assume that the filaments' volumetric density is fixed, and we denote by $\varepsilon^2 m_i > 0$ the mass of $\mathcal{S}_i$ and by $\varepsilon^2 \mathcal{J}_i(t)$ the inertial matrix at time $t \geqslant 0$, so that m_i and $\mathcal{J}_i$ are of order one with respect to ε. Moreover the matrix $\mathcal{J}_i$ is positive definite, uniformly in ε (this only fails if γ_i was a straight line, which has been explicitly excluded) and evolves in time according to Sylvester's law:

$$\mathcal{J}_i(t) = Q_{i,\varepsilon}(t)\mathcal{J}_{0,i}Q^T_{i,\varepsilon}(t), \tag{15}$$

where $\mathcal{J}_{0,i}$ denotes the initial value $\mathcal{J}_{0,i} := \mathcal{J}_i(0)$.

2.4. Ambient fluid. — We assume that, for any $t \geqslant 0$, the open set

$$\mathcal{F}(t) := \mathbb{R}^3 \smallsetminus \bigcup_i \mathcal{S}_i(t),$$

is occupied by a fluid whose velocity u and pressure p are given as the sums

$$u := u^\flat + u^\mathfrak{p} \quad \text{and} \quad p := p^\flat + p^\mathfrak{p}, \tag{16}$$

where

$$\begin{aligned}(u^\flat, p^\flat) &\text{ in } C\big([0,+\infty); \dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\big)\\ &\text{and in } W^{2,\infty}((0,+\infty)\times\mathbb{R}^3)\times W^{1,\infty}((0,+\infty)\times\mathbb{R}^3), \text{ and satisfying } \operatorname{div} u^\flat = 0,\end{aligned} \tag{17}$$

is the background flow, and $(u^\mathfrak{p}, p^\mathfrak{p})$ is the perturbation flow due to the filaments, whose evolution is assumed to be driven by the steady Stokes equations:

$$-\Delta u^\mathfrak{p} + \nabla p^\mathfrak{p} = 0 \quad \text{and} \quad \operatorname{div} u^\mathfrak{p} = 0 \quad \text{in } \mathcal{F}(t), \tag{18a}$$

$$u^\mathfrak{p} = v^{\mathcal{S}_i} - u^\flat \quad \text{in } \mathcal{S}_i(t), \quad 1 \leqslant i \leqslant N, \tag{18b}$$

where we recall the notation (14). Above the notation $\dot H^1$ stands for the homogeneous Sobolev space of order 1 built on the Lebesgue space L^2, while $W^{1,\infty}$ and $W^{2,\infty}$ stand for the inhomogeneous Sobolev space respectively of order 1 and 2 built on the Lebesgue space L^∞.

2.5. Dynamics of the filaments. — The filaments are assumed to be only accelerated, for any $t \geqslant 0$, by the force exerted by the fluid on their boundaries $\partial\mathcal{S}_i(t)$ according to the Newton equations:

$$\varepsilon^2 m_i \mathrm{v}_i'(t) = -\int_{\partial\mathcal{S}_i(t)} \Sigma(u,p) n \, d\mathcal{H}^2, \tag{19a}$$

$$\varepsilon^2 (\mathcal{J}_i \omega_i)'(t) = -\int_{\partial\mathcal{S}_i(t)} (x - h_{i,\varepsilon}(t)) \wedge \Sigma(u,p) n \, d\mathcal{H}^2, \tag{19b}$$

where $d\mathcal{H}^2$ is the two-dimensional Hausdorff measure and n denotes the unit normal vector on $\partial\mathcal{S}_i(t)$ pointing outside the fluid domain $\mathcal{F}(t)$ and

$$\Sigma(u,p) := 2D(u) - p \operatorname{Id}, \tag{20}$$

where $D(u)$ is the deformation tensor defined by

$$D(u) := \frac{1}{2}(\partial_j u_i + \partial_i u_j)_{1\leqslant i,j\leqslant 3}. \tag{21}$$

2.6. The whole Newton-Stokes system at a glance. — Gathering (11), (12), (13), (14), (15), (16), (18) and (19) we arrive at the following Newton-Stokes system.

For $1 \leqslant i \leqslant N$,

$$h'_{i,\varepsilon}(t) = \mathrm{v}_i(t), \tag{22a}$$

$$Q'_{i,\varepsilon}(t) = (\omega_i(t) \wedge \cdot) Q_{i,\varepsilon}(t), \tag{22b}$$

$$\varepsilon^2\, m_i\, \mathrm{v}'_i(t) = -\int_{\partial \mathcal{S}_i(t)} \Sigma(u^\flat + u^\natural, p^\flat + p^\natural) n \, d\mathcal{H}^2, \tag{22c}$$

$$\varepsilon^2 (\mathcal{J}_i \omega_i)'(t) = -\int_{\partial \mathcal{S}_i(t)} (x - h_{i,\varepsilon}(t)) \wedge \Sigma(u^\flat + u^\natural, p^\flat + p^\natural) n \, d\mathcal{H}^2, \tag{22d}$$

$$\text{where } \mathcal{J}_i(t) = Q_{i,\varepsilon}(t) \mathcal{J}_{0,i} Q^T_{i,\varepsilon}(t) \ \text{ and } \ \mathcal{S}_i(t) = h_{i,\varepsilon}(t) + Q_{i,\varepsilon}(t)(\overline{\mathcal{S}}_i - \overline{h}_{i,\varepsilon}), \tag{22e}$$

and

$$-\Delta u^\natural + \nabla p^\natural = 0 \quad \text{and} \quad \operatorname{div} u^\natural = 0 \quad \text{in } \mathcal{F}(t), \tag{22f}$$

$$u^\natural = v^{\mathcal{S}_i} - u^\flat \quad \text{for } \ x \in \mathcal{S}_i(t), \quad \text{for } \ 1 \leqslant i \leqslant N, \tag{22g}$$

$$\text{where } v^{\mathcal{S}_i}(t,x) := \mathrm{v}_i(t) + \omega_i(t) \wedge (x - h_{i,\varepsilon}(t)) \ \text{ for } \ x \in \mathcal{S}_i(t). \tag{22h}$$

A reformulation of the Newton equations (22c)–(22d) into a compact form, involving in particular the so-called Stokes resistance matrices, will be given in Section 6.1.

2.7. A local-in-time well-posedness result. — Despite its apparent complexity, the system (22) can be considered as a system of second-order quasilinear ODEs on the $6N$ degrees of freedom of the rigid bodies, the fluid state being given by an auxiliary steady Stokes system for which time only appears as a parameter. Moreover the coefficients of this ODE, although their coefficients are given in a rather non explicit way, are smooth as long as the filaments $\mathcal{S}_i(t)$ remain separated; this follows from standard results on the regularity with respect to shape changes for which we refer for example to [41, 5, 6]. Therefore it follows from the Cauchy-Lipschitz theorem that, starting from separated positions with arbitrary velocities, we have the following local-in-time well-posedness result.

Proposition 2.2. — *For each ε in $(0,1)$, given some initial disjoint positions and some initial velocities of the filaments, there is $T^{\max}_\varepsilon \in (0,+\infty]$ and a unique smooth solution to* (22) *on $[0, T^{\max}_\varepsilon)$. Moreover, if $T^{\max}_\varepsilon < +\infty$, then*

$$\lim_{t \to T^{\max}_\varepsilon} \min_{i \neq j} \operatorname{dist}(\mathcal{S}_i(t), \mathcal{S}_j(t)) = 0. \tag{23}$$

Proof. — As mentioned above, the existence, locally in time, of a smooth solution to (22) is a straightforward consequence of the classical regularity properties of the Stokes system and of the Cauchy-Lipschitz theorem. It remains to prove the last statement regarding the lifetime of these solutions. To this end, we multiply, for $1 \leqslant i \leqslant N$, the equation (22c) by v_i and the equation (22d) by ω_i. By summing the resulting identities, and recalling (22h), we get

$$
\begin{aligned}
\varepsilon^2 \sum_{1\leqslant i\leqslant N}\Big(\frac{1}{2}m_i \mathrm{v}_i^2 + \mathcal{J}_i\omega_i\cdot\omega_i\Big)' &= -\sum_{1\leqslant i\leqslant N}\int_{\partial \mathcal{S}_i(t)} v^{\mathcal{S}_i}\cdot \Sigma(u^{\mathfrak{p}}+u^{\flat}, p^{\mathfrak{p}}+p^{\flat})n\, d\mathcal{H}^2 \\
&= -\sum_{1\leqslant i\leqslant N}\int_{\partial \mathcal{S}_i(t)} u^{\mathfrak{p}}\cdot \Sigma(u^{\mathfrak{p}}, p^{\mathfrak{p}})n\, d\mathcal{H}^2 \\
&\quad -\sum_{1\leqslant i\leqslant N}\int_{\partial \mathcal{S}_i(t)} u^{\flat}\cdot \Sigma(u^{\mathfrak{p}}, p^{\mathfrak{p}})n\, d\mathcal{H}^2 \\
&\quad -\sum_{1\leqslant i\leqslant N}\int_{\partial \mathcal{S}_i(t)} v^{\mathcal{S}_i}\cdot \Sigma(u^{\flat}, p^{\flat})n\, d\mathcal{H}^2,
\end{aligned}
\tag{24}
$$

thanks to (22g). Hence, integrating by parts in $\mathcal{F}$ for the two first terms in the right hand side of (24) taking into account (22f), and integrating by parts in $\bigcup_i \mathcal{S}_i$ for the last term in the right hand side of (24), we arrive at

$$
\begin{aligned}
\varepsilon^2 \sum_{1\leqslant i\leqslant N}\Big(\frac{1}{2}m_i \mathrm{v}_i^2 + \mathcal{J}_i\omega_i\cdot\omega_i\Big)' + \int_{\mathcal{F}} Du^{\mathfrak{p}} : Du^{\mathfrak{p}} \\
= -\int_{\mathcal{F}} Du^{\flat} : Du^{\mathfrak{p}} - \sum_{1\leqslant i\leqslant N}\int_{\mathcal{S}_i}(-\Delta u^{\flat}+\nabla p^{\flat})\cdot v^{\mathcal{S}_i}.
\end{aligned}
\tag{25}
$$

Above, the notation $A : B$, where $A := (a_{i,j})_{i,j}$ and $B := (b)_{i,j}$ are two 3×3 matrices stands for the scalar quantity $\sum_{1\leqslant i,j\leqslant 3} a_{i,j}b_{i,j}$. The above identity holds true as long as there is no collision. Then, by the Cauchy-Schwarz inequality, Young's inequality for products and a Gronwall argument, we deduce that the function

$$
t \longmapsto \varepsilon^2 \sum_{1\leqslant i\leqslant N}\Big(\frac{1}{2}m_i \mathrm{v}_i^2 + \mathcal{J}_i\omega_i\cdot\omega_i\Big)(t) + \int_0^t\int_{\mathcal{F}} Du^{\mathfrak{p}} : Du^{\mathfrak{p}},
$$

remains bounded as long as there is no collision. Then it follows from classical blowup criteria for ODEs that the solution can be continued as long as there is no collision. In particular, if the maximal lifetime $T_\varepsilon^{\max}$ of the smooth solution to (22) satisfies $T_\varepsilon^{\max} < +\infty$, then (23) holds true. □

Remark 2.3. — It is worth to observe that the energy identity (25) used in the proof above, alone, is not sufficient to obtain bounds on the filament velocities which are uniform with respect to ε as the $O(1)$ energy transfer with the background flow, see the right hand side of (25), that may a priori lead to high velocities due to the factor ε^2 associated with the filaments' inertia.

3. Main results

This section is devoted to the statements of the main results of the paper. More precisely, the main result of this paper, that is the convergence of the Newton-Stokes system to a limit system as the thickness parameter ε goes to 0, is given in Section 3.4, in particular in Theorem 3.4. To state this result, a few notations have to be introduced, which is the subject to the Sections 3.1, 3.2 and 3.3. In Section 3.5

we expose the strategy of the proof of Theorem 3.4 by considering a toy model. The organization of the proof of Theorem 3.4 is detailed in Section 3.6. We will also draw some comparisons with the existing literature on close issues, see Section 3.7, and we will finally mention a few open problems, see Section 3.8.

3.1. A few general notations. — First we introduce, for $1 \leqslant i \leqslant N$, the vector fields:

$$v_{i,\alpha}[h_i](x) := \begin{cases} e_\alpha & \text{if } \alpha = 1,2,3, \\ e_{\alpha-3} \wedge (x - h_i) & \text{if } \alpha = 4,5,6, \end{cases} \tag{26}$$

where e_α, for $\alpha = 1,2,3$, denotes the α-th unit vector of the canonical basis of $\mathbb{R}^3$. These vector fields are elementary rigid velocities with respect to the i-th filament.

We define, for p in $\mathbb{R}^3$, the 3×3 matrix

$$k(p) := 8\pi\Big(\mathrm{Id} - \frac{1}{2}\, p \otimes p\Big). \tag{27}$$

The matrix k is related to the Stokes kernel S, defined for x in $\mathbb{R}^3 \smallsetminus \{0\}$, by

$$S(x) = \frac{1}{8\pi|x|}\Big(\mathrm{Id} + \frac{x}{|x|} \otimes \frac{x}{|x|}\Big) = \frac{1}{|x|}\, S_0\Big(\frac{x}{|x|}\Big), \tag{28}$$

where for p in the euclidean unit sphere S^2,

$$S_0(p) := \frac{1}{8\pi}(\mathrm{Id} + p \otimes p). \tag{29}$$

Indeed, by the Sherman-Morrison formula, see for example [40, Prop. 3.21], we observe that for any p in S^2, the matrix $S_0(p)$ is invertible and its inverse is precisely $k(p)$ defined above. As a matter of fact, the use of the identity

$$\forall p \in S^2, \quad S_0(p)k(p) = \mathrm{Id}, \tag{30}$$

is crucial in our analysis below, see (99).

Next we associate with a smooth oriented curve $\mathcal{C}$ without self-intersections and with two vector fields v and $\widetilde{v}$ defined on $\mathcal{C}$ with values in $\mathbb{R}^3$, the following real-valued functional:

$$I_{\mathcal{C}}[v, \widetilde{v}] := \frac{1}{2} \int_{\mathcal{C}} k(\tau) v \cdot \widetilde{v}\, d\mathcal{H}^1, \tag{31}$$

where we recall that $d\mathcal{H}^1$ is the one-dimensional Hausdorff measure and τ denotes the unit tangent vector field along $\mathcal{C}$. Since the matrix k is symmetric, the operator $I_{\mathcal{C}}[\cdot,\cdot]$ is bilinear symmetric.

3.2. Limit dynamics. — Now we define the objects which occur in the limit dynamics of the filaments when the thickness parameter ε converges to 0. To do so, we will define several functions depending on the filament positions denoted by (h_i, Q_i), for $1 \leqslant i \leqslant N$.

For $1 \leqslant \alpha \leqslant 6$ and for $1 \leqslant i,j \leqslant N$, we set

$$\widehat{\mathcal{K}}_{i,\alpha,j,\beta}(h_i, Q_i) := \delta_{ij}\, I_{\mathcal{C}_i(h_i,Q_i)}\big[v_{i,\alpha}[h_i], v_{i,\beta}[h_i]\big], \tag{32}$$

where δ_{ij} is the Kronecker symbol, and, for $\alpha = 1,2,3$, for $1 \leqslant i \leqslant N$, for $t \geqslant 0$,

$$\begin{aligned} \widehat{F}^\flat_{i,\alpha}(t,h_i,Q_i) &:= I_{\mathcal{C}_i(h_i,Q_i)}[v_{i,\alpha}[h_i],u^\flat(t)] \\ \widehat{T}^\flat_{i,\alpha}(t,h_i,Q_i) &:= I_{\mathcal{C}_i(h_i,Q_i)}[v_{i,\alpha+3}[h_i],u^\flat(t)]. \end{aligned} \tag{33}$$

Let us emphasize that the matrices $\widehat{\mathcal{K}}_{i,\alpha,i,\beta}$ and the vectors $\widehat{F}^\flat_{i,\alpha}$ and $\widehat{T}^\flat_{i,\alpha}$ do not depend on ε but on the positions, considered here as variables, of the filament centerlines denoted by $\mathcal{C}_i(h_i,Q_i)$ and defined by

$$\mathcal{C}_i(h_i,Q_i) := h_i + Q_i\overline{\mathcal{C}}_i.$$

In addition, the vectors $\widehat{F}^\flat_{i,\alpha}$ and $\widehat{T}^\flat_{i,\alpha}$ depend explicitly on time through the time dependence of $u^\flat$. We refer to them respectively as the Stokes resistance matrix associated with the filament centerline $\mathcal{C}_i$, and the Faxén force and torque associated with the filament centerline $\mathcal{C}_i$ and with the background flow $u^\flat$.

For $1 \leqslant i \leqslant N$, let us consider the 6×6 matrices diagonal blocks

$$\widehat{\mathcal{K}}_{i,i} = \widehat{\mathcal{K}}_{i,i}(h_i,Q_i) := (\widehat{\mathcal{K}}_{i,\alpha,i,\beta})_{1\leqslant\alpha,\beta\leqslant 6}, \tag{34}$$

and, for $1 \leqslant i \leqslant N$, the vectors of $\mathbb{R}^6$:

$$\widehat{\mathfrak{f}}^\flat_i = \widehat{\mathfrak{f}}^\flat_i(t,h_i,Q_i) := ((\widehat{F}^\flat_{i,\alpha})_{1\leqslant\alpha\leqslant 3},(\widehat{T}^\flat_{i,\alpha})_{1\leqslant\alpha\leqslant 3}). \tag{35}$$

By a change of coordinates, it is easy to see that $\widehat{\mathcal{K}}_{i,i}$ satisfies

$$\widehat{\mathcal{K}}_{i,i}(h_i,Q_i) = \begin{pmatrix} Q_i & 0 \\ 0 & Q_i \end{pmatrix} \widehat{\mathcal{K}}_{i,i}(0,\mathrm{Id}) \begin{pmatrix} Q_i^T & 0 \\ 0 & Q_i^T \end{pmatrix}. \tag{36}$$

Lemma 3.1. — *For any (h_i,Q_i) in $\mathbb{R}^3\times SO(3)$, for $1\leqslant i\leqslant N$, the matrix $\widehat{\mathcal{K}}_{i,i}(h_i,Q_i)$ is symmetric positive definite.*

Proof. — By (36), it suffices to consider the case where $(h_i,Q_i) = (0,\mathrm{Id})$, and we will omit to write this variable. Since for any $p\in S^2$, the matrix $k(p)$ is positive symmetric and satisfies $k(p)\geqslant 4\pi\,\mathrm{Id}$, we deduce that for all $(v,\omega)\in\mathbb{R}^3\times\mathbb{R}^3$

$$\widehat{\mathcal{K}}_{i,i}\begin{pmatrix} v \\ \omega \end{pmatrix}\cdot(v,\omega) \geqslant 2\pi\int_{\mathcal{C}_i}|v+\omega\wedge x|^2\,d\mathcal{H}^1(x). \tag{37}$$

Indeed, for $\omega\neq 0$, the integrand is non-constant since $\mathcal{C}_i$ is not a straight line. Thus, the integral is positive for $(v,\omega)\neq 0$. □

With these tools in hands we can now explicitly present the system which will prove to be the zero-thickness limit of the Newton-Stokes system (22). This system drives the dynamics of the positions at time t of the centerline curves $\widehat{\mathcal{C}}_i(t)$, for $1\leqslant i\leqslant N$, by the rigid motions

$$\widehat{\mathcal{C}}_i(t) = \widehat{h}_i(t) + \widehat{Q}_i(t)\overline{\mathcal{C}}_i. \tag{38}$$

Here the vector $\widehat{h}_i(t) \in \mathbb{R}^3$ and the matrix $\widehat{Q}_i(t) \in SO(3)$ satisfy the following first-order ODEs:

$$\widehat{h}_i'(t) = \widehat{\mathrm{v}}_i(t), \quad \widehat{Q}_i'(t) = (\widehat{\omega}_i(t) \wedge \cdot)\widehat{Q}_i^T(t), \tag{39}$$

$$(\widehat{\mathrm{v}}_i(t), \widehat{\omega}_i(t)) = \widehat{\mathcal{K}}_{i,i}^{-1}(\widehat{h}_i(t), \widehat{Q}_i(t))\, \widehat{\mathfrak{f}}_i^{\flat}(t, \widehat{h}_i(t), \widehat{Q}_i(t)). \tag{40}$$

For $1 \leqslant i \leqslant N$, the right hand side of (40) only depends on $\widehat{h}_i(t)$ and $\widehat{Q}_i(t)$, not on the positions of the other centerline curves corresponding to $j \neq i$.

On the other hand neither the matrices $\widehat{\mathcal{K}}_{i,i}$, nor their inverses, usually referred to as mobility matrices, are diagonal, not even by 3×3 blocks. This coupling between translation/rotation velocities and force/torque is typical of the case of rigid bodies with shape anisotropies. It is usually called the Jeffery effect, see [20, 23, 43].

Finally, since the coefficients of (40) are smooth and globally Lipschitz (this follows immediately from (36), (35), (31) and the smoothness assumption for $u^\flat$), the Cauchy-Lipschitz theorem applies again and guarantees the following global-in-time well-posedness result.

Proposition 3.2. — *Given some initial disjoint positions of the centerline curves, given a smooth background flow $u^\flat$ satisfying* (17)*, there is a unique smooth global-in-time solution to* (38)–(39)–(40) *on* $[0, +\infty)$.

Although each of these decoupled ODEs admits a unique smooth global-in-time solution, it could be that some of the positions of the centerline curves which they define collide in finite time.

Definition 3.3. — Let us denote by $\widehat{T}$ in $(0, +\infty]$ the time of the first collision in the limit dynamics, that is the first time for which at least two of the centerline curves $\widehat{\mathcal{C}}_i(t)$ defined by (38)–(39)–(40) have a non-empty intersection, with the convention that $\widehat{T} = +\infty$ if there is no such collision. More precisely, we define

$$\widehat{d}_{\min}(t) := \min_{i \neq j} \operatorname{dist}(\widehat{\mathcal{C}}_i(t), \widehat{\mathcal{C}}_j(t)), \tag{41}$$

$$\widehat{T} := \inf\{t \geqslant 0 : \widehat{d}_{\min}(t) = 0\}. \tag{42}$$

3.3. Asymptotic fluid behaviour. — Regarding the fluid behaviour when the thickness parameter ε converges to 0, it is only a matter to understand the behaviour of the perturbation flow $(u^\mathfrak{p}, p^\mathfrak{p})$ due to the filaments, since on the other hand the background flow $(u^\flat, p^\flat)$ is fixed. Precisely, the steady Stokes system in presence of several thin filaments has been the object of several studies usually referred to as the slender body theory or as the immersed boundary method. It can also be viewed as a Stokesian counterpart of the issue of Newtonian capacity, see [10].

To capture the leading term of the perturbation flow $(u^\mathfrak{p}, p^\mathfrak{p})$ as ε converges to 0, the key idea is to consider the Stokes system in the full space $\mathbb{R}^3$ with an appropriate source term given as Dirac masses along the centerline curves $\mathcal{C}_i$. The intensity of these Dirac masses is related to the bilinear operator $I_\mathcal{C}[\cdot,\cdot]$ defined in (31) in the following way. Let v a divergence-free vector field in $W^{1,\infty}(\bigcup_{1\leqslant i\leqslant N}\mathcal{S}_i)$. We define $\mu_{\mathcal{C}_i}[v]$ as the vector measure, supported on $\mathcal{C}_i$, defined by

$$\langle \mu_{\mathcal{C}_i}[v],\phi\rangle := I_{\mathcal{C}_i}[v,\phi], \qquad \text{for any } \phi\in C_c(\mathbb{R}^3;\mathbb{R}^3). \tag{43}$$

Moreover we define the vector field

$$U_{\mathcal{C}_i}[v] := S * \mu_{\mathcal{C}_i}[v], \tag{44}$$

where the symbol $*$ stands for the convolution in $\mathbb{R}^3$, S is the Stokes kernel defined by (28). This reads

$$U_{\mathcal{C}_i}[v](x) = \frac{1}{2}\int_{\mathcal{C}_i} S(x-y)k(\tau(y))v(y)\,d\mathcal{H}^1(y), \tag{45}$$

for any $x\in\mathbb{R}^3\smallsetminus\mathcal{C}_i$, where we recall that τ is the unit tangent vector defined below (31).

Let us recall that the counterpart of the Stokes kernel S for the pressure is the vector $P(x)$, defined for x in $\mathbb{R}^3\smallsetminus\{0\}$, by

$$P(x) = \frac{x}{4\pi|x|^3}. \tag{46}$$

Notice that

$$-\Delta S + \nabla P = \delta_0\,\mathrm{Id} \qquad \text{and} \qquad \operatorname{div} S = 0, \tag{47}$$

in the sense of distributions, where the differential operators are applied column-wise. Then we associate with the operator $U_{\mathcal{C}_i}$ the following counterpart for the pressure

$$P_{\mathcal{C}_i}[v](x) := P * \mu_{\mathcal{C}_i}[v], \tag{48}$$

which reads for any $x\in\mathbb{R}^3\smallsetminus\mathcal{C}_i$,

$$P_{\mathcal{C}_i}[v](x) = \frac{1}{2}\int_{\mathcal{C}_i} P(x-y)\cdot k(\tau(y))v(y)\,d\mathcal{H}^1(y).$$

Thus it follows from (47) that, in the sense of distributions in the variable x,

$$-\Delta U_{\mathcal{C}_i}[v] + \nabla P_{\mathcal{C}_i}[v] = \mu_{\mathcal{C}_i}[v] \qquad \text{and} \qquad \operatorname{div} U_{\mathcal{C}_i}[v] = 0. \tag{49}$$

Our main result below, see Theorem 3.4, establishes that the vector field

$$\widehat{u}^{\mathfrak{p}}(t,\cdot) := \sum_{1\leqslant i\leqslant N} U_{\widehat{\mathcal{C}}_i}[\widehat{v}^{\mathcal{S}_i}(t,\cdot) - u^{\flat}(t,\cdot)], \tag{50}$$

where

$$\widehat{v}^{\mathcal{S}_i}(t,x) := \widehat{h}_i'(t) + \widehat{Q}_i'(t)\widehat{Q}_i(t)^T(x-\widehat{h}_i(t)), \tag{51}$$

is the leading part of the perturbation flow $u^{\mathfrak{p}}$ up to a renormalization factor $|\log\varepsilon|^{-1}$.

3.4. CONVERGENCE RESULT. — The main result of this paper is the following theorem, which contains two points: (i) an estimate of the time of the first collision and (ii) the convergence of the dynamics of the filaments and of a renormalized fluid perturbation velocity as the thickness parameter ε of the filaments converges to zero. The following statement aims at providing a simple description of our results while some complementary more technical elements will be discussed below.

THEOREM 3.4. — *We consider some initial disjoint positions of the centerline curves, a smooth background flow $u^\flat$ satisfying* (17)*, and the solutions $(\widehat{h}_i, \widehat{Q}_i)_{1\leqslant i\leqslant N}$ given by Proposition* 3.2*. Let $\widehat{T}$ in $(0,+\infty]$ be the time of the first collision associated with this solution as defined in Definition* 3.3*. For each ε in $(0,1)$, the initial positions of the filaments of thickness parameter ε are deduced from the ones for the centerline curves by* (3) *and* (11)*. Let κ in $(0,1)$ be such that for any ε in $(0,\kappa)$ the initial positions of the filaments are disjoint. Let us consider some initial rigid velocities, all independent of ε in $(0,\kappa)$, for the N filaments. For ε in $(0,\kappa)$, we denote by $(h_{i,\varepsilon}, Q_{i,\varepsilon})_{1\leqslant i\leqslant N}$ the corresponding solutions to the Newton-Stokes system* (22) *up to the time $T_\varepsilon^{\max}$ as given by Proposition* 2.2*.*

Then on the one hand,

$$\liminf_{\varepsilon\to 0} T_\varepsilon^{\max} \geqslant \widehat{T}, \tag{52}$$

and, on the other hand, for any $1\leqslant i\leqslant N$, for all $T<\widehat{T}$ there exists C depending only on $u^\flat$, on the filaments $\overline{\mathcal{S}}_i^\kappa$ with thickness κ, on $\inf_{t\in[0,T]}\widehat{d}_{\min}(t)$ (see Definition 3.3*) and on the initial velocities, and there exists $\varepsilon_0>0$ depending in addition on T such that, for all ε in $(0,\varepsilon_0)$,*

$$\|(h_{\varepsilon,i}, Q_{\varepsilon,i}) - (\widehat{h}_i, \widehat{Q}_i)\|_{L^\infty(0,T)} \leqslant C\left(\varepsilon + |\log\varepsilon|^{-1/2}\, T\right) e^{CT}. \tag{53}$$

The perturbation flow $u^{\mathfrak{p}}$ due to the filaments, extended by the filament velocity inside each filament, satisfies the following estimates: for any compact subset K of $\mathbb{R}^3$ and for any p in $[1,2)$, for all $t<T$, for all ε in $(0,\varepsilon_0)$,

$$\|u^{\mathfrak{p}}(t,\cdot) - |\log\varepsilon|^{-1}\,\widehat{u}^{\mathfrak{p}}(t,\cdot)\|_{L^p(K)} \leqslant C\,|\log\varepsilon|^{-1}\left(e^{-Ct/\varepsilon^2|\log\varepsilon|} + |\log\varepsilon|^{-1/2}\, e^{Ct}\right). \tag{54}$$

where $\widehat{u}^{\mathfrak{p}}$ is given by (50) *and* (51)*.*

A few comments on Theorem 3.4 are in order. Let us start with saying that Theorem 3.4 establishes the convergence of the original system (22) for the filaments to the reduced model (40) for their centerline curves, as the thickness parameter ε converges to zero. Let us highlight that the system (22) is a coupled system made of the Newton equations associated with all the filaments whereas the limit equation (40) for each filament is decoupled from the others. Such a phenomenon enters the scope of the theme of *hydrodynamic decoupling*. Here it states that the main effect on each limit centerline curve is due to the background flow and not from the other filaments. The limit equations (40) have the advantage in view of applications to only involve the geometry of the centerline curves rather than the one of the whole filaments.

The system (22) is a second-order system whereas the limit equations (40) are first-order equations. Therefore one initial data has to be dropped for the limit system (40). Unless the initial data for the system (22) satisfies the compatibility conditions $(\mathrm{v}_i(0), \omega_i(0)) = (\widehat{\mathrm{v}}_i(0), \widehat{\omega}_i(0))$, for all $1 \leqslant i \leqslant N$, with $(\widehat{\mathrm{v}}_i(t), \widehat{\omega}_i(t))$ given by (40), the velocities dynamics exhibit an initial layer, which prevents uniform convergence of the filament velocities down to the initial time. Indeed a byproduct of our analysis is that we are able to describe the nature of the initial stage: it is an exponential relaxation within a time interval of order $O(\varepsilon^2 \log \varepsilon)$. During that time, a transition of the amplitudes of the filament velocities occurs which is of order $O(1)$. After this initial stage, the dynamics of the filaments is adapted to the first-order dynamics of the limit system and the convergence occurs at least with a rate $O(|\log \varepsilon|^{-1/2})$. More precisely, for any $T < \widehat{T}$ there is $\varepsilon_0 > 0$ and $C > 0$ as in Theorem 3.4 such that for any ε in $(0, \varepsilon_0)$, for any $1 \leqslant i \leqslant N$, and for any t in $[0, T]$,

$$
\begin{aligned}
(55) \quad & |(\mathrm{v}_i, \omega_i)(t) - (\widehat{\mathrm{v}}_i, \widehat{\omega}_i)(t)| \\
& \qquad \leqslant |(\mathrm{v}_i, \omega_i)(0) - (\widehat{\mathrm{v}}_i, \widehat{\omega}_i)(0)|\; e^{-Ct/\varepsilon^2 |\log \varepsilon|} + C\, |\log \varepsilon|^{-1/2}\, e^{Ct}.
\end{aligned}
$$

Our analysis allows us to give an even better approximation of the solutions to (22) when ε goes to 0, by a family of velocities, indexed by ε, given by a quasi-static balance similar to (40), but with the Stokes resistance matrices and the Faxén force and torque associated with the whole set of filaments rather than their sole centerlines, see Theorem 6.5.

Regarding the fluid part of the system, Theorem 3.4 establishes that after an initial relaxation stage the perturbation of the fluid velocity $u^{\mathfrak{p}}$ is well-approximated in L^p_{loc} by $|\log \varepsilon|^{-1}\, \widehat{u}^{\mathfrak{p}}$ which is explicitly given (see (50)) in terms of the limit dynamics of the filament centerlines only. The estimate (54) should be interpreted in the sense that, firstly, the fluid perturbation is of order $|\log \varepsilon|^{-1}$ in L^p_{loc}, $p < 2$ which corresponds to the Stokes resistance of the filaments. Secondly, the perturbation, rescaled to order 1, is well approximated by $\widehat{u}^{\mathfrak{p}}$ up to an error which corresponds to the sum of the errors of the positions (53) and of the velocities (55) of the filament centerlines. As we will see, it is possible to improve the estimate to L^p_{loc}, $p < 6$ on the expense of the rate of convergence. More precisely, for $2 \leqslant p < 6$ and for all $\delta > 0$,

$$
\begin{aligned}
(56) \quad & \|u^{\mathfrak{p}}(t,\cdot) - |\log \varepsilon|^{-1}\, \widehat{u}^{\mathfrak{p}}(t,\cdot)\|_{L^p(K)} \\
& \leqslant C\, |\log \varepsilon|^{-1} \Bigg(\sum_{1 \leqslant i \leqslant N} |(\mathrm{v}_i, \omega_i)(0,\cdot) - (\widehat{\mathrm{v}}_i, \widehat{\omega}_i)(0,\cdot)|\; e^{-Ct/\varepsilon^2 |\log \varepsilon|} \\
& \qquad\qquad + |\log \varepsilon|^{-1/2(3/p - 1/2 - \delta)}\, e^{Ct} \Bigg).
\end{aligned}
$$

We observe that, at any time, the leading part $\widehat{u}^{\mathfrak{p}}$ of the perturbation flow given by (50) satisfies the modified Stokes equation in the sense of distributions in the variable x, in $\mathbb{R}^3$,

$$
-\Delta \widehat{u}^{\mathfrak{p}} + \nabla \widehat{p}^{\mathfrak{p}} = \sum_{1 \leqslant i \leqslant N} \widehat{\mu}_i^{\mathfrak{p}} \quad \text{and} \quad \operatorname{div} \widehat{u}^{\mathfrak{p}} = 0,
$$

where for $1 \leqslant i \leqslant N$, the term $\mu_i^{\mathfrak{p}}$ is the vector measure given by

$$\widehat{\mu}_i^{\mathfrak{p}} := \mu_{\widehat{\mathcal{C}}_i}[\widehat{v}^{\mathcal{S}_i} - u^{\flat}],$$

where $\widehat{v}^{\mathcal{S}_i}$ is given by (51) and $\widehat{p}^{\mathfrak{p}} := P * \widehat{\mu}_i^{\mathfrak{p}}$, with P given by (46). Moreover, for any $1 \leqslant i \leqslant N$, it follows from the definition of the vector fields $v_{i,\alpha}$, for $\alpha = 1, 2, 3$, in (26), that the total mass of the measure $\mu_i^{\mathfrak{p}}$ is

$$\int_{\widehat{\mathcal{C}}_i} d\widehat{\mu}_i^{\mathfrak{p}} = (\widehat{\mathcal{K}}_{i,\alpha,i,\beta}(\widehat{h}_i, \widehat{Q}_i))_{1\leqslant\alpha\leqslant3;1\leqslant\beta\leqslant6}\,(\widehat{\mathrm{v}}_i, \widehat{\omega}_i) - (\widehat{F}^{\flat}_{i,\alpha})_{1\leqslant\alpha\leqslant3},$$

where we also recall the definitions (32) and (33). The right hand side above is precisely the leading part of the force due to the fluid on the i-th filament, up to the renormalization factor $|\log\varepsilon|^{-1}$ and to the sign, so that its vanishing is precisely the part of (40) which concerns the force. This is reminiscent of Newton's third law of motion (a.k.a. the action-reaction principle).

We emphasize that the perturbation flow $u^{\mathfrak{p}}$ is not well approximated by $|\log\varepsilon|^{-1}\,\widehat{u}^{\mathfrak{p}}$ in the homogeneous Sobolev space $\dot{H}^1$. On the one hand, the perturbation in $\dot{H}^1$ is actually of order $|\log\varepsilon|^{-1/2}$ instead of $|\log\varepsilon|^{-1}$ (since the Stokes resistance $|\log\varepsilon|^{-1}$ corresponds to the square of the $\dot{H}^1$-norm). On the other hand, the $\dot{H}^1$-norm turns out to be concentrated in a region of order ε around the filaments. Since the errors of the positions compared to the limit system is much larger (of order $|\log\varepsilon|^{-1/2}$), $|\log\varepsilon|^{-1}\,\widehat{u}^{\mathfrak{p}}$ is not a good approximation in $\dot{H}^1$. However, we will show, see Proposition 7.1, that $u^{\mathfrak{p}}$ is well approximated in $\dot{H}^1$ by

$$|\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i}[v^{\mathcal{S}_i}(t,\cdot) - u^{\flat}(t,\cdot)]. \tag{57}$$

This estimate is actually an important ingredient in the proof of our main result.

3.5. Strategy of the proof of Theorem 3.4. — Let us give here a glimpse of some elements of the proof of Theorem 3.4, whose detailed proof is the purpose of the rest of the paper. Let us focus first on the way we deal with the ε-dependence in the filaments dynamics, letting aside for a while the role played by the Stokes system.

– A first ingredient is a reformulation of the Newton equations into a second-order ODE for the $6N$ degrees of freedom of the filaments, see (190). This singularly perturbed ODE looks like the following toy-model:

$$\varepsilon^2\, q'' = -\,|\log\varepsilon|^{-1}\,(k^{\varepsilon}(q)q' - f^{\varepsilon}(q)) + r^{\varepsilon}, \tag{58}$$

where the scalar unknown q stands for the variables encoding the positions of the filaments (with a mute dependence on ε), k^{ε} are positive Lipschitz functions uniformly with respect to ε, f^{ε} are Lipschitz functions of uniformly with respect to ε, and r_{ε} are remainders with nice estimates.

– A second ingredient is a modulated energy argument which consists in estimating the dynamics of

$$\frac{1}{2}(q' - V^{\varepsilon}(q))^2 \quad \text{with} \quad V^{\varepsilon}(q) := (k^{\varepsilon}(q))^{-1}\, f^{\varepsilon}(q).$$

This leads to

$$q' = V^{\varepsilon}(q) + \widetilde{r}^{\varepsilon}, \tag{59}$$

with $\widetilde{r}^{\varepsilon}$ satisfying some relevant estimates.

– A third ingredient is to prove that, roughly speaking,

$$\forall \overline{q}, \quad V^{\varepsilon}(\overline{q}) \longrightarrow \widehat{V}(\overline{q}) \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{60}$$

– This finally allows to compare q and the solution $\widehat{q}$ of the limit ODE:

$$\widehat{q}' = \widehat{V}(\widehat{q}).$$

Of course this protocol relies on a detailed analysis of the asymptotic behaviour on the fluid part, to obtain the behaviour with respect to ε of the coefficients in (58), and to prove (60). This analysis uses properties of the Stokes system in the presence of several filaments in the zero-thickness limit, for which time only plays the role of a parameter through the positions of the filaments. We will therefore devote a separate section to this issue first, see Section 4. This analysis will also allow to obtain the part of Theorem 3.4 which concerns the asymptotic behaviour of the fluid.

Remark 3.5. — The idea of using a modulated energy to deal with singular ODEs is rather ubiquitous in nature; let us mention the paper [4] for a spectacular use in the context of the analysis of the motion of a charged particle in a slowly varying electromagnetic field when the particle mass converges to zero. An important difference with the case of the equation (190) is that in [4] the term without derivative is a gyroscopic term, rather than a damping term, so that the modulation provides a center-guide along which the exact solution oscillates.

3.6. Organization of the proof of Theorem 3.4. — In Section 4 we analyze the asymptotic behaviour of the solution of the steady Stokes system in presence of several thin filaments with Dirichlet data at the interface between the fluid and the filaments. A well-known approximation consists in replacing the presence of the slender filaments by appropriate source terms which are measures supported on the filament centerlines in the steady Stokes system set in the whole space $\mathbb{R}^3$. The precise definition of this approximation is given in Section 4 together with an error estimate of the difference between this approximation and the exact solution in the natural energy space, see Theorem 4.1. This analysis holds for any given configuration of the filaments as long as there is no intersection of two or more filaments.

In Section 5, we bound the shape derivatives of the Dirichlet energy of solutions of the steady Stokes system in presence of several thin filaments.

Section 6 is devoted to the proof of the part of Theorem 3.4 which concerns the asymptotic behaviour of the filament centerlines.

On the other hand the part of Theorem 3.4 which concerns the fluid asymptotic behaviour is proved in Section 7.

3.7. Comparison with the literature. — It is well known, see for example the classical textbooks [19, 38], that the solution to the steady Stokes system in the exterior of bodies can be written in terms of boundary integral operators over the surfaces of the bodies. The purpose of the slender body theory is to approximate this solution in the case where the bodies are thin filaments by replacing the integral operators over the surfaces by integral operators over the filament centerlines. This idea dates back to Hancock [16], Cox [7], Batchelor [3], Keller and Rubinow [24], Johnson [22] and had a regain of interest with the numerical work by Peskin [37]; see also the more recent papers [36, 42]. More precisely in the slender body theory, in the case where one considers the steady Stokes equations in the exterior of a single ε-thick filament $\overline{\mathcal{S}}_i^\varepsilon$, as defined in (3), with some boundary data v on $\partial\overline{\mathcal{S}}_i^\varepsilon$, one substitutes to the exact solution u of this exterior problem, the solution u_f to the steady Stokes equations in the full space $\mathbb{R}^3$ with as source term Dirac masses along the centerline curve $\overline{\mathcal{C}}_i$ of $\overline{\mathcal{S}}_i^\varepsilon$, that is a measure μ_f defined by

$$\langle \mu_f, \phi\rangle := \int_{\overline{\mathcal{C}}_i} f\cdot\phi\, d\mathcal{H}^1, \qquad \text{for any } \phi\in C_c(\mathbb{R}^3;\mathbb{R}^3), \tag{61}$$

where the (vector) density f has to be chosen in a relevant way. Indeed u_f is given by $u_f := S * \mu_f$, where the symbol $*$ stands for the convolution in $\mathbb{R}^3$ and S is the Stokes kernel defined by (28), which satisfies the steady Stokes equations, with zero source, in the exterior of the centerline curve $\overline{\mathcal{C}}_i$ and a fortiori in the exterior of the ε-thick filament $\overline{\mathcal{S}}_i^\varepsilon$. Therefore, when comparing u and u_f, the key point is that the trace of u_f on $\partial\overline{\mathcal{S}}_i^\varepsilon$, which is a linear integral operator acting on the density f, matches with v. However it has been shown in [15] that this operator is actually not invertible. On the other hand, as already observed in [7], the leading order part of the integral operator is completely local, and gives rise to the correspondence $f(y) = \frac{1}{2}k(\tau(y))v(y)$ as in (45). To our knowledge, we provide here for the first time rigorous quantitative error estimates for the zero order slender body approximation given by this correspondence. Neglecting higher-order terms has the advantage of an explicit approximation but restricts to errors of order $|\log\varepsilon|$. However, it seems that in the case of non-circular cross-section, errors of this order are unavoidable anyway if one only relies on approximations through force densities on the centerline. On the other hand, in the case of a filament with circular cross sections, one may consider refined approximations by adding to Dirac masses along the centerline curve some other higher-order singularities, in particular the so-called doublets which correspond to ΔS. In this case, invertible regularizations of the integral operator mentioned above have been studied in [32, 33]. Let us also mention the recent papers [34, 35] which provide rigorous justifications of the slender body theory in the case where the density of force on the centerline curve of a single filament with circular cross sections is prescribed.

In [14], Gonzalez has tackled the zero-radius limit of the quasi-static motion of a single massless filament. His result establishes a limit balance similar to our result, however only under an extra assumption on the asymptotic behaviour of the density of forces acting on the filament. His conditional result relies on a different approach than ours, that is on the boundary integral formulation of the Stokes equations.

The aforementioned papers are mostly concerned with the quasi-static Stokes problem in the exterior of a given filament and not with the time evolution of the filament. On the other hand, in [23] a rigid body of arbitrary shape is considered, moving in a viscous incompressible flow driven by the unsteady incompressible Navier-Stokes equations. The authors provide a formal derivation of the motion in the limit where the size of the body converges to 0 and the mass density is fixed. This asymptotic analysis relies on the assumption that the fluid is undisturbed by the particle at the main order and that the rotation of the rigid body is $O(1)$, while it results from the analysis that at the leading order the particle behaves as a passive tracer in the fluid. In [43] the authors have extended the analysis to other inertia regimes.

Readers familiar with the vortex filament conjecture for Euler Flows may be tempted to draw a comparison with the present work. This conjecture concerns the 3D incompressible Euler equations in the case where the initial vorticity is concentrated along a smooth curve. It is believed, see for instance [2, 31], that the curve evolves in time by binormal curvature flow, to leading order. Therefore two huge differences in this problematic, compared to the present setting, are that: (i) it concerns a single phase problem, rather than a diphasic system where fluid and rigid bodies are considered, and (ii) the dynamics of the curve is way more intricate since it can deform in time, which corresponds to an infinite number of degrees of freedom. An important step toward this conjecture has recently been achieved by Jerrard and Seis in [21] where it is shown that under the assumption that the vorticity remains concentrated along a smooth curve when time proceeds, then this curve approximately evolves by binormal curvature flow. Despite these important differences, the mathematical analysis shares some common features, for example in the way to deal with singular line integral. In this respect, it is interesting to compare Lemma 4.4 with [21, §4.5].

Let us also mention another possible comparison to a setting where the fluid is also assumed to be driven by the incompressible Euler equations: the work [13] where the zero radius limit of the dynamics of several solids in a 2D perfect incompressible fluid is studied. In particular it shares with the present setting the feature to deal with the case where the inertia of some rigid bodies converges to zero in the limit so that their limit dynamics is a first-order equation rather than a second-order equation. Accordingly the proofs both use some modulated energy arguments, compare [13, §7] and Section 6.2 below. However the forces which drive the limit dynamics are rather different in both settings, on the one hand they are gyroscopic type forces in the case of [13], similarly to the setting evoked in Remark 3.5, and on the other hand they are viscous drag type forces in the present paper. Another difference is that in [13] the limit dynamics of the particles are still coupled in the limit and they influence the fluid, as point vortices. On the other hand we deal here with some 3D rigid bodies shrinking to 1D limit rigid bodies instead of 2D rigid bodies shrinking to point particles.

3.8. A few possible extensions as open problems. — In this subsection, we state a few open problems regarding some extensions of the analysis performed in this paper.

Open Problem 3.6. — We let aside the particular case of rod-like filaments whose centerlines are line segments, which seems to require additional work due to the degeneracy of the limit Stokes resistance matrix $\widehat{\mathcal{K}}$, for which Lemma 3.1 does not hold true. Indeed, the resistance to rotations around the orientation of the rod like filaments scales like ε^2 rather than $|\log\varepsilon|^{-1}$. In the case where the cross sections of the filaments are circular, some decoupling of the dynamics occurs and one can substitute an orientation vector ξ in S^2 to the orientation matrix Q_i in $SO(3)$ in order to describe the filaments' rotations. In such a case, it seems possible to adjust our arguments in order to obtain a result similar to Theorem 3.4. However, in the case where the cross sections are not circular, the analysis seems more delicate.

Open Problem 3.7. — In view of the quantitative convergence result obtained in Theorem 3.4, a natural issue is to obtain, in the general case as in the case of line segments, higher-order asymptotic expansions of the dynamics with respect to ε. In particular, it would be interesting to analyze the influence of the cross sections on the dynamics. As it can be seen from the toy-model (58), and from the compressed form of the Newton equations given in (190) where we highlight that the coefficients are related to the fluid state and depend on ε, establishing such asymptotic expansions in time requires to prove some precise asymptotic description of the fluid state. In this direction it would be interesting to investigate if the analysis performed in [30, Chap. 12.2], which overcomes the difficulties related to the boundary layers associated with non circular cross-sections in the case of the Laplace equations with a circular centerline could be adapted to the present setting. Let us also mention that the influence of small scales in the cross sections can also be encoded by a different choice of the boundary conditions at the interface between the fluid phase and the solid phase. In this paper we concentrate on the case of the no-slip condition at the interface, but some other conditions could be considered as well, such as the Navier slip conditions, see [17] and the references therein. Hence, it would be interesting to investigate whether or not the results of Theorem 3.4 can be adapted to other boundary conditions. Moreover, one may wonder how a change of shape of the centerline curve influences the dynamics, and the convergence of the dynamics, as the thickness parameter ε goes to zero. Another natural issue to consider is whether the asymptotic description can be extended up to a collision. For a similar issue in a close setting let us mention the papers [5, 6].

Open Problem 3.8. — It would be interesting to investigate the case where the number N of filaments goes to $+\infty$, while the thickness parameter ε and the length ℓ of the filaments go to 0 with $\varepsilon \ll \ell \ll 1$, so that at the limit the phase corresponding to the rigid filaments is then a cloud of point particles. A first question is to identify the limit dynamics of these particles. Moreover one may identify a case where the density of these particles is sufficient to create a collective effect at the main order on the fluid. This would extend the investigations on the Brinkman force for arbitrary shapes done in [10, 18] from a case where anisotropy corresponds to a finite ratio to the case of an infinite ratio.

3.9. A few more notations. — For $E \subset \mathbb{R}^d$ and $r > 0$ we denote

$$B_r(E) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, E) < r\}. \tag{62}$$

We use the convention that in our estimates the constant C might change from line to line and might depend on the background velocity, on the number N of filaments and on the functions specifying the reference filaments, i.e., Ψ_i, γ_i, R_i, $1 \leqslant i \leqslant N$. We will always specify other dependencies and will make any dependence of C on ε explicit.

We also point out that the following convention is used throughout the paper: the letter u always stands for the fluid velocity, while the letter v stands for the solid velocities.

There are several smallness requirements on ε throughout the paper, typically denoted by $\varepsilon < \varepsilon_0$. Similarly as for the constant C we will for simplicity allow ε_0 to change its value throughout the proofs of our results. Notice that we will usually take ε_0 smaller than κ, where κ is defined in Theorem 3.4.

4. Immersed boundary method for the steady Stokes system in presence of several thin filaments

This section is devoted to the asymptotic behaviour, in the limit where the thickness ε of the filaments $(\mathcal{S}_j)_j$ converges to zero, of the solution u in $\dot{H}^1(\mathbb{R}^3)$ to the problem

$$\begin{aligned} -\Delta u + \nabla p = 0 \quad &\text{and} \quad \operatorname{div} u = 0 \quad \text{in } \mathcal{F}, \\ u(x) = v(x) \quad &\text{in } \mathcal{S}_i, \\ u(x) = 0 \quad &\text{in } \mathcal{S}_j, \quad \text{for } j \neq i, \end{aligned} \tag{63}$$

where a given index i such that $1 \leqslant i \leqslant N$ has been specified, as well as the inhomogeneous data v on the associated filament $\mathcal{S}_i$ which is assumed to satisfy

$$v \in W^{1,\infty}(\mathcal{S}_i) \quad \text{with} \int_{\mathcal{S}_i} \operatorname{div} v = 0. \tag{64}$$

Here these filaments are supposed to be given and fixed in terms of the reference filaments $\overline{\mathcal{S}}_j$ and some translations and rotations h_j, Q_j as in (11) but without any time dependence. The quantities h_j, Q_j are supposed to be given in such a way that the filaments $(\mathcal{S}_j)_j$ do not overlap or touch. In fact all the results in this section that concern several filaments will be stated under the assumption that the minimal distance between the filament centerlines

$$d_{\min} := \min_{i \neq j} \operatorname{dist}(\mathcal{C}_i, \mathcal{C}_j), \tag{65}$$

is bounded from below and under a smallness condition on ε. Together, this implies a lower bound on the distance between the filaments $\mathcal{S}_i$.

To approximate the solution u to (63) we rely on the auxiliary velocity field $U_{\mathcal{C}_i}[v]$ given in (44). This velocity field solves the Stokes system in the full space $\mathbb{R}^3$ with an appropriate source term given as Dirac masses along the limit curve $\mathcal{C}_i$. It follows from (45), from the decay of the kernel S defined by (28) (and its derivative), and from the boundedness of k from (27) that for all $x \in \mathbb{R}^3 \smallsetminus \mathcal{C}_i$

$$|U_{\mathcal{C}_i}[v](x)| \leqslant C\|v\|_{L^\infty} \min\Big\{\log\Big(1+\frac{1}{\operatorname{dist}(x,\mathcal{C}_i)}\Big), \frac{1}{\operatorname{dist}(x,\mathcal{C}_i)}\Big\}, \tag{66}$$

$$|\nabla U_{\mathcal{C}_i}[v](x)| \leqslant C\|v\|_{L^\infty} \min\Big\{\frac{1}{\operatorname{dist}(x,\mathcal{C}_i)}, \frac{1}{(\operatorname{dist}(x,\mathcal{C}_i))^2}\Big\}. \tag{67}$$

The following result establishes that $U_{\mathcal{C}_i}[v]$ is the leading part of the solution u to (63), up to a renormalization factor $|\log\varepsilon|^{-1}$ as long as the filaments are sufficiently separated in terms of $d_{\min}$ given by (65).

Theorem 4.1. — *For all $d>0$ there exists $\varepsilon_0(d)>0$ and $C(d)>0$, for all filament configuration with $d_{\min}\geqslant d$ and for all ε in $(0,\varepsilon_0)$, for any v satisfying (64) we have the following result. The solution u to (63) satisfies*

$$\|u\|_{\dot H^1(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-1/2}\,\|v\|_{W^{1,\infty}(\mathcal{S}_i)}. \tag{68}$$

Moreover,

$$\|u-|\log\varepsilon|^{-1}\,U_{\mathcal{C}_i}[v]\|_{\dot H^1(\mathbb{R}^3\smallsetminus\mathcal{S}_i)} \leqslant C\,|\log\varepsilon|^{-1}\,\|v\|_{W^{1,\infty}(\mathcal{S}_i)}, \tag{69}$$

$$\|u-|\log\varepsilon|^{-1}\,U_{\mathcal{C}_i}[v]\|_{W^{1,q}(K)} \leqslant C\,|\log\varepsilon|^{-3/2}\,\|v\|_{W^{1,\infty}(\mathcal{S}_i)}, \tag{70}$$

for any q in $[1,3/2)$ and any compact $K\subset\mathbb{R}^3$, where C in (70) depends in addition on q and K.

To prove Theorem 4.1, we will proceed in several steps. First, in Section 4.2, we will establish pointwise estimates of $U_{\mathcal{C}_i}[v]$. Then in Section 4.4, we will deduce uniform estimates in $\dot H^1(\mathbb{R}^3)$ based on Helmholtz' minimum dissipation theorem, see Theorem 4.9. This enables to tackle the very proof of Theorem 4.1 in Section 4.4.

Theorem 4.1 will be used in Section 7 to prove the part of Theorem 3.4 devoted to the asymptotic behavior of the fluid, once the asymptotic behavior of the dynamics of the filaments is obtained.

Theorem 4.1 is also useful to establish approximation results of the force exerted by the fluid on the filaments. To cover the different uses which we will need, we first show a rather general result, where we make use of the elementary rigid velocities $v_{i,\alpha}$ defined in (26). We associate with these fields, for $1\leqslant\alpha\leqslant 6$ and $1\leqslant i\leqslant N$, the unique solutions $V_{i,\alpha}$ in $\dot H^1(\mathbb{R}^3)$ to

$$-\Delta V_{i,\alpha}+\nabla P_{i,\alpha}=0 \quad\text{and}\quad \operatorname{div} V_{i,\alpha}=0, \quad \text{in } \mathcal{F}, \tag{71a}$$

$$V_{i,\alpha}=\delta_{i,j}v_{i,\alpha}, \quad \text{in } \mathcal{S}_j. \tag{71b}$$

The vector fields $V_{i,\alpha}$ are smooth, decay as $1/|x|$ at infinity, their first-order derivatives and the associated pressures $P_{i,\alpha}$ decay as $1/|x|^2$.

The next result concerns the approximation of force and torque.

Corollary 4.2. — *For all $d>0$ there exists a constant $C=C(d)>0$ such that for all ε in $(0,\varepsilon_0(d))$, for all filament configuration with $d_{\min}\geqslant d$, for all divergence-free functions $v\in W^{1,\infty}(\bigcup_{j=1}^N \mathcal{S}_j)$ and all $1\leqslant i\leqslant N$, for $1\leqslant\alpha\leqslant 6$,*

$$\left|\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} (\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v\, d\mathcal{H}^2 - |\log\varepsilon|^{-1} I_{\mathcal{C}_i}[v_{i,\alpha},v]\right| \leqslant C\,|\log\varepsilon|^{-3/2}\,\|v\|_{W^{1,\infty}(\bigcup_{j=1}^N \mathcal{S}_j)}. \tag{72}$$

The proof of Corollary 4.2 will be given in Section 4.5. A first particular useful application of Corollary 4.2 corresponds to the case where $v=\delta_{i,j}v_{j,\beta}$ for $1\leqslant\beta\leqslant 6$ and $1\leqslant i,j\leqslant N$. It entails that for $1\leqslant\alpha,\beta\leqslant 6$ and $1\leqslant i,j\leqslant N$, the quantity

$$\mathcal{K}_{i,\alpha,j,\beta} := \int_{\partial\mathcal{S}_j} (\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v_{j,\beta}\, d\mathcal{H}^2, \tag{73}$$

satisfies

$$\left|\mathcal{K}_{i,\alpha,j,\beta} - |\log\varepsilon|^{-1}\,\widehat{\mathcal{K}}_{i,\alpha,j,\beta}\right| \leqslant C\,|\log\varepsilon|^{-3/2}. \tag{74}$$

Recall that the limit Stokes resistance matrices $\widehat{\mathcal{K}}_{i,\alpha,j,\beta}$ are defined in (32) and are considered here as being associated with a fixed position of the centerline curves.

We will denote by $\mathcal{K}$ the $6N\times 6N$ matrix whose coefficients are these quantities $\mathcal{K}_{i,\alpha,j,\beta}$, for $1\leqslant\alpha,\beta\leqslant 6$ and $1\leqslant i,j\leqslant N$. Recall that the matrix $\mathcal{K}$ is referred to as the steady Stokes resistance tensor, that it depends on all the positions h_i and orientations Q_i and is symmetric positive definite, as a consequence of integrations by parts, energy and uniqueness properties of the exterior steady Stokes system. Let us refer for example to [25, Chap. 2], [12, Chap. 5], [26, Chap. 2 & 3].

Moreover it follows immediately from (74) and the coercivity of $\widehat{\mathcal{K}}$ that we observed in (37) (recall that $\widehat{\mathcal{K}}$ is block-diagonal) that

$$\mathcal{K} \geqslant \frac{1}{C}\,|\log\varepsilon|^{-1}\,\mathrm{Id} \quad\text{and}\quad |\mathcal{K}^{-1}|\leqslant C|\log\varepsilon|. \tag{75}$$

Another particular use of Corollary 4.2 is the case where $v=u^\flat$. It will provide some estimates on the so-called Faxén forces and torques defined by

$$\mathfrak{f}^\flat := ((F_i^\flat, T_i^\flat))_{1\leqslant i\leqslant N}. \tag{76}$$

where

$$F_i^\flat := (F_{i,\alpha}^\flat)_{\alpha=1,2,3} \quad\text{and}\quad T_i^\flat := (T_{i,\alpha}^\flat)_{\alpha=1,2,3}, \tag{77}$$

with, for $\alpha=1,2,3$, $1\leqslant i\leqslant N$,

$$F_{i,\alpha}^\flat := \int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} (\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot u^\flat\, d\mathcal{H}^2, \tag{78}$$

$$T_{i,\alpha}^\flat := \int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} (\Sigma(V_{i,\alpha+3},P_{i,\alpha+3})n)\cdot u^\flat\, d\mathcal{H}^2. \tag{79}$$

Indeed applying (72) to $v=u^\flat$, we arrive at

$$|\mathfrak{f}^\flat - |\log\varepsilon|^{-1}\,\widehat{\mathfrak{f}}^\flat| \leqslant C\,|\log\varepsilon|^{-3/2}\,\|u^\flat\|_{W^{1,\infty}}. \tag{80}$$

Above

$$\widehat{\mathfrak{f}}^{\flat} := (\widehat{\mathfrak{f}}_i^{\flat})_{1\leqslant i\leqslant N},$$

where we recall that, for $1 \leqslant i \leqslant N$, the vector $\widehat{\mathfrak{f}}_i^{\flat}$ gathering the limit Faxén forces and torques is defined in (35) and is here considered as being associated with a fixed position of the centerline curves.

4.1. Modified centerlines for the non-closed filaments. — To simplify the proof of Theorem 4.1, we introduce slightly modified centerline curves in the case of non-closed filaments, i.e., the case when γ_i is not periodic. In this case, we cut an ε layer at both endpoints. More precisely, we define

$$\overline{\mathcal{C}}_i^{\varepsilon} := \gamma_i([\varepsilon, L_i - \varepsilon]), \tag{81}$$

and correspondingly, we write $\mathcal{C}_i^{\varepsilon}$ for the curve which is obtained from $\overline{\mathcal{C}}_i^{\varepsilon}$ through translation and rotation. This cut-off version satisfies

$$\operatorname{dist}(\mathcal{C}_i^{\varepsilon}, \partial\mathcal{S}_i) \geqslant c\varepsilon \tag{82}$$

for all $\varepsilon < \varepsilon_0$ and some $c > 0$ independent of ε We remark that $\mathcal{C}_i^{\varepsilon}$ resembles the so-called effective centerline in [35].

In the case of a closed filament, i.e., when γ_i is periodic, (82) is automatically satisfied for $\mathcal{C}_i^{\varepsilon} := \mathcal{C}_i$.

We show the following lemma, which allows us to prove Theorem 4.1 by replacing $U_{\mathcal{C}_i}$ in (69) by $U_{\mathcal{C}_i^{\varepsilon}}$.

Lemma 4.3. — *Let v as in (64). Then there exists $C > 0$ such that for all ε in $(0, \varepsilon_0)$,*

$$\|U_{\mathcal{C}_i^{\varepsilon}}[v] - U_{\mathcal{C}_i}[v]\|_{\dot H^1(\mathbb{R}^3\smallsetminus\mathcal{S}_i)} \leqslant C\sqrt{\varepsilon}\,\|v\|_{L^\infty(\mathcal{S}_i)}. \tag{83}$$

Proof. — By linearity, $U_{\mathcal{C}_i}[v] - U_{\mathcal{C}_i^{\varepsilon}}[v] = U_{\mathcal{C}_i\smallsetminus\mathcal{C}_i^{\varepsilon}}[v]$. Similarly to the pointwise estimate (67), we observe that for all $x \in \mathbb{R}^3 \smallsetminus \mathcal{C}_i$,

$$|\nabla U_{\mathcal{C}_i\smallsetminus\mathcal{C}_i^{\varepsilon}}[v](x)| \leqslant \|v\|_{L^\infty(\mathcal{S}_i)} \min\Big\{\frac{1}{\operatorname{dist}(x, \mathcal{C}_i\smallsetminus\mathcal{C}_i^{\varepsilon})}, \frac{\varepsilon}{(\operatorname{dist}(x, \mathcal{C}_i\smallsetminus\mathcal{C}_i^{\varepsilon}))^2}\Big\}. \tag{84}$$

Denote by $\partial\mathcal{C}_i$ the two endpoints of the curve $\mathcal{C}_i$. Then, using that for all $x \in \mathbb{R}^3 \smallsetminus \mathcal{S}_i$

$$\operatorname{dist}(x, \mathcal{C}_i \smallsetminus \mathcal{C}_i^{\varepsilon}) \geqslant c\operatorname{dist}(x, \partial\mathcal{C}_i), \tag{85}$$

the estimate (84) yields (83). □

4.2. Pointwise estimates. — This section is devoted to the analysis of the behavior of $U_{\mathcal{C}_i^{\varepsilon}}[v]$ on the boundary of the filament $\mathcal{S}_i$.

We introduce ξ_i as the orthogonal projection from $\partial\mathcal{S}_i$ to $\mathcal{C}_i$ which is well-defined for ε sufficiently small since, by assumption, γ_i has no self-intersections. Moreover, we denote by $\partial\mathcal{C}_i^{\varepsilon}$ the boundary of the 1-dimensional manifold $\mathcal{C}_i^{\varepsilon} \subset \mathbb{R}^3$, which is empty if γ_i is a closed curve (as we defined $\mathcal{C}_i^{\varepsilon} = \mathcal{C}_i$ in this case) and contains precisely 2 points otherwise.

Lemma 4.4. — *Let v as in (64). Then there exists $C>0$ such that for all ε in $(0,\varepsilon_0)$,*

$$|U_{\mathcal{C}_i}[v] - |\log\varepsilon|\; v\circ\xi_i|(x) \leqslant C\|v\|_{W^{1,\infty}(\mathcal{S}_i)}(1+|\log(\operatorname{dist}(x,\partial\mathcal{C}_i^\varepsilon))|), \tag{86}$$

on $\partial\mathcal{S}_i$. Moreover,

$$\|U_{\mathcal{C}_i}[v]\|_{\dot H^1(\mathbb{R}^3\smallsetminus\mathcal{S}_i)} \leqslant C\,|\log\varepsilon|^{1/2}\,\|v\|_{L^\infty(\mathcal{S}_i)}, \tag{87}$$

and

$$\|U_{\mathcal{C}_i}[v]\|_{W^{1,p}(K)} \leqslant C_{K,p}\|v\|_{L^\infty(\mathcal{S}_i)}, \tag{88}$$

for all p in $[1,2)$ and all compact $K\subset\mathbb{R}^3$. Furthermore, for all $d>0$, there exists C depending on d such that

$$\|U_{\mathcal{C}_i}[v]\|_{W^{1,\infty}(\mathbb{R}^3\smallsetminus B_d(\mathcal{C}_i))} \leqslant C\|v\|_{L^\infty(\mathcal{S}_i)} \quad \textit{for } j\neq i. \tag{89}$$

Proof. — The estimates in (87), (88) and (89) are direct consequences of the pointwise estimates (66) and (67).

It remains to prove (86). We will assume that the filament is non-closed. The case of a closed filament is slightly easier. Since the statement concerns only a single filament, we might assume that $\mathcal{C}_i=\overline{\mathcal{C}_i}$, and in particular $\mathcal{C}_i^\varepsilon=\gamma_i([\varepsilon,L_i-\varepsilon])$. Then, we introduce the function $\overline{s}$ from $\partial\mathcal{S}_i$ to $[0,L_i]$ by the formula $\gamma_i(\overline{s}(x))=\xi_i(x)$ for any x in $\partial\mathcal{S}_i$. In the rest of the proof, we will drop the index i.

Fix $x\in\partial\mathcal{S}$. Note that $\overline{s}(x)\in[0,\varepsilon]\cup[L-\varepsilon,L]$ implies $\operatorname{dist}(x,\partial\mathcal{C}^\varepsilon)\leqslant C\varepsilon$. In this case, (86) follows immediately (66) applied to $\mathcal{C}^\varepsilon$ and (82). Therefore, we might assume in the following that $\overline{s}(x)\in(\varepsilon,L-\varepsilon)$.

Let us denote the straight line approximation at $\overline{s}=\overline{s}(x)$ by

$$\gamma_0(s):=\gamma(\overline{s})+(s-\overline{s})\gamma'(\overline{s}).$$

We observe that for all $s\in[\varepsilon,L-\varepsilon]$ we have

$$|\gamma(s)-\gamma_0(s)|\leqslant C|s-\overline{s}|^2,\quad |\gamma'(s)-\gamma_0'(s)|\leqslant C|s-\overline{s}|, \tag{90}$$

$$|x-\gamma(s)|\geqslant c|s-\overline{s}| \quad\text{and}\quad |x-\gamma_0(s)|\geqslant|s-\overline{s}|. \tag{91}$$

Then, we split the integral

$$\begin{aligned} U_{\mathcal{C}_i}[v](x) &= \frac12\int_\varepsilon^{L-\varepsilon} S(x-\gamma(s))k(\gamma'(s))v(\gamma(s))\,ds \\ &= \frac12\big(U_1(x)+U_2(x)\big), \end{aligned} \tag{92}$$

where

$$U_1(x):=\int_\varepsilon^{L-\varepsilon} S(x-\gamma_0(s))k(\gamma'(\overline{s}))v(\gamma(\overline{s}))\,ds, \tag{93}$$

$$U_2(x):=\int_\varepsilon^{L-\varepsilon}\big(S(x-\gamma(s))k(\gamma'(s))v(\gamma(s))-S(x-\gamma_0(s))k(\gamma'(\overline{s}))v(\gamma(\overline{s}))\big)\,ds. \tag{94}$$

We decompose $U_1(x)$ further by observing the following. By definition of S,

$$S(x-\gamma_0(s))=\frac{1}{8\pi}\frac{1}{|x-\gamma_0(s)|}\big(\operatorname{Id}+A(x-\gamma_0(s))\big), \tag{95}$$

where, for p in $\mathbb{R}^3 \smallsetminus \{0\}$,
$$A(p) := \frac{p}{|p|} \otimes \frac{p}{|p|}.$$
Moreover,
$$\mathrm{Id} = 8\pi S_0(\gamma'(\overline{s})) - \gamma'(\overline{s}) \otimes \gamma'(\overline{s}) = 8\pi S_0(\gamma'(\overline{s})) - A((\overline{s}-s)\gamma'(\overline{s})), \tag{96}$$
where S_0 is defined in (29), so that
$$S(x-\gamma_0(s)) = \frac{1}{8\pi}\frac{1}{|x-\gamma_0(s)|}\big(8\pi S_0(\gamma'(\overline{s})) + A(x-\gamma_0(s)) - A((\overline{s}-s)\gamma'(\overline{s}))\big). \tag{97}$$
This leads to
$$U_1(x) = U_{1,a}(x) + U_{1,b}(x), \tag{98}$$
with
$$\begin{aligned} U_{1,a}(x) &:= \Big(\int_\varepsilon^{L-\varepsilon} \frac{1}{|x-\gamma_0(s)|}\,ds\Big) S_0(\gamma'(\overline{s}))k(\gamma'(\overline{s}))v(\gamma(\overline{s})),\\ U_{1,b}(x) &:= \frac{1}{8\pi}\Big(\int_\varepsilon^{L-\varepsilon} \frac{A(x-\gamma_0(s)) - A((\overline{s}-s)\gamma'(\overline{s})}{|x-\gamma_0(s)|}\,ds\Big) k(\gamma'(\overline{s}))v(\gamma(\overline{s})).\end{aligned}$$
Then, thanks to the identity (30), to the fact that $\gamma(\overline{s})$ is the orthogonal projection of x on $\mathcal{C}_i$, to the change of variables $z = (s-\overline{s})\gamma'(\overline{s})$ and to the fact that γ is a parametrization by arc length, we obtain
$$\begin{aligned} U_{1,a}(x) &= \Big(\int_\varepsilon^{L-\varepsilon} \frac{1}{|x-\gamma_0(s)|}\,ds\Big) v(\gamma(\overline{s}))\\ &= \Big(\int_{-\overline{s}+\varepsilon}^{L-\overline{s}-\varepsilon} \frac{1}{(|x-\gamma(\overline{s})|^2+z^2)^{1/2}}\,dz\Big) v(\gamma(\overline{s}))\\ &= (\sinh^{-1}(|x-\gamma(\overline{s})|^{-1}(L-\overline{s}-\varepsilon)) - \sinh^{-1}(-|x-\gamma(\overline{s}-\varepsilon)|^{-1}\overline{s}))v(\gamma(\overline{s})).\end{aligned} \tag{99}$$
Using that $\sinh^{-1}(z) = \log(z+\sqrt{1+z^2})$, for any real z, and that $c\varepsilon \leqslant |x-\gamma(\overline{s})| \leqslant C\varepsilon$, elementary but tedious estimates show that
$$\big|U_{1,a}(x) - 2\,|\log\varepsilon|\,v(\gamma(\overline{s}))\big| \leqslant C|v(\gamma(\overline{s}))|\big(1+|\log(\mathrm{dist}(x,\partial\mathcal{C}^\varepsilon))|\big), \tag{100}$$
where we used that $\partial\mathcal{C}^\varepsilon = \{\gamma(\varepsilon), \gamma(L-\varepsilon)\}$.

To estimate $U_{1,b}(x)$, we use
$$|A(p)-A(q)| \leqslant C\min\Big\{1, |p-q|\max\big\{1/|p|, 1/|q|\big\}\Big\}. \tag{101}$$
Thus, $|U_{1,b}(x)|$ is bounded by
$$C|v(\gamma(\overline{s}))|\int_\varepsilon^{L-\varepsilon} \frac{1}{|x-\gamma_0(s)|}\min\Big\{1, |x-\gamma(\overline{s})|\max\Big\{\frac{1}{|x-\gamma_0(s)|}, \frac{1}{|s-\overline{s}|}\Big\}\Big\}\,ds,$$
and therefore by
$$C|v(\gamma(\overline{s}))|\Big(\int_{\overline{s}-\varepsilon}^{\overline{s}+\varepsilon} \frac{1}{(\varepsilon^2+(s-\overline{s})^2)^{1/2}}\,ds + \int_{[\varepsilon,L-\varepsilon]\smallsetminus[\overline{s}-\varepsilon,\overline{s}+\varepsilon]} \frac{\varepsilon}{(s-\overline{s})^2}\,ds\Big),$$

so that finally

$$|U_{1,b}(x)| \leqslant C|v(\gamma(\overline{s}))|. \tag{102}$$

We now estimate $U_2(x)$. Using that

$$|S(x_1) - S(x_2)| \leqslant C|x_1 - x_2| \max\Big\{\frac{1}{|x_1|^2}, \frac{1}{|x_2|^2}\Big\}, \tag{103}$$

we deduce that

$$|S(x - \gamma(s))k(\gamma'(s))v(\gamma(s)) - S(x - \gamma_0(s))k(\gamma'(\overline{s}))v(\gamma(\overline{s}))| \leqslant C\|v\|_{W^{1,\infty}(\mathcal{S}_i)}.$$

Therefore,

$$|U_2(x)| \leqslant C\|v\|_{W^{1,\infty}(\mathcal{S}_i)}. \tag{104}$$

Combining (92), (98), (100), (102) and (104), we arrive at (86). □

4.3. Bogovskiĭ and extension operators with some uniformity with respect to the domain. — A second ingredient of the proof of Corollary 4.2 is the use of some Bogovskiĭ operators with some natural norms that are bounded uniformly with respect to the domain.

We will make use of a statement regarding Bogovskiĭ operators associated with John domains. Roughly speaking, an open bounded domain Ω is a John domain with respect to a point x_0 if each point y in Ω can be reached by a Lipschitz curve beginning at x_0 and contained in Ω in such a way that, for every point x in the curve, the distance from x to y is proportional to the distance from x to the boundary of Ω. This class strictly contains the Lipschitz domains. Notice nevertheless that external cusps are not allowed. Let us now give the precise definition of John domains following the definition of [1].

Definition 4.5. — Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Then, Ω is called a John domain with constant $Z > 0$ if there exists $x_0 \in \Omega$ such that for all $x \in \Omega$ there is an Z-Lipschitz map $\rho\colon [0, |x - x_0|] \to \Omega$ such that $\rho(0) = x$, $\rho(|x - x_0|) = x_0$ and for all $t \in [0, |x - x_0|]$,

$$\operatorname{dist}(\rho(t), \partial\Omega) \geqslant t/Z. \tag{105}$$

The filaments $\mathcal{S}_i$ are Lipschitz domains and therefore any smooth neighborhood of $\mathcal{S}_i$ is a John domain. In the next Lemma, we prove that suitable neighborhoods are John domains uniformly in ε.

Lemma 4.6. — *Let $d > 0$. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ is a John domain with a constant Z independent of ε.*

Proof. — We first consider the case of a closed filaments. The necessary adaptations for non-closed filaments will be discussed at the end of the proof.

Fix $c < d$ such that the projection $\xi_i\colon B_c(\mathcal{C}_i) \to \mathcal{C}_i$ as in Lemma 4.4 is well defined. We will consider ε_0 in $(0, c)$.

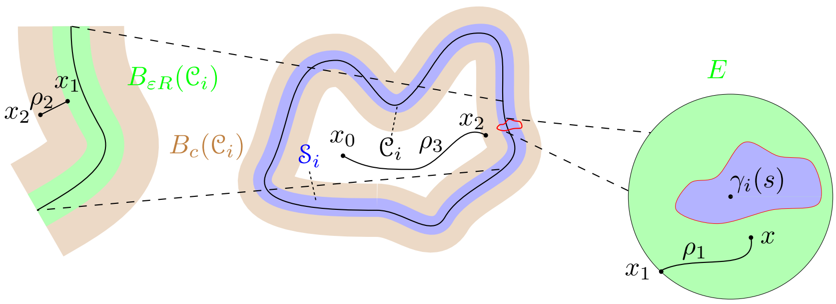

Figure 2. Illustration of the construction of the curve ρ from x to x_0 via x_1 and x_2

Choose any reference point $x_0 \in B_d(\mathcal{C}_i) \smallsetminus B_c(\mathcal{C}_i)$. To construct a curve from $x \in B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ to x_0, we proceed in three steps: we construct three curves ρ_1, ρ_2 and ρ_3 that once pasted together connect x to x_0. We parametrize the curves by arc length and reparametrize at the end to obtain a curve $\rho\colon [0, |y - x_0|] \to B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ connecting x to x_0. In the following construction, the constants $\eta, C > 0$ will be independent of ε. The construction is visualized in Figure 2.

Let $R := 2\|\Psi_i\|_\infty$, where we recall from (3) that Ψ_i specifies the cross section of the reference filament. In particular, we have $\mathcal{S}_i \subset \overline{B_{\varepsilon R/2}(\mathcal{C}_i)}$.

First, if $x \in B_{\varepsilon R}(\mathcal{C}_i)$, we construct a Lipschitz curve $\rho_1\colon [0, T_1]$ from x to $x_1 \in \partial B_{\varepsilon R}(\mathcal{C}_i)$ which satisfies $T_1 \leqslant C\varepsilon R$ and $\operatorname{dist}(\rho_1(t), \partial\mathcal{S}_i) \geqslant \eta t$. If $x \notin B_{\varepsilon R}(\mathcal{C}_i)$, we set $x_1 = x$. To construct ρ_1, we consider $\xi_i(x)$ the projection of x on $\mathcal{C}_i$ and $\bar{s}$ such that $\gamma_i(\bar{s}) = \xi_i(x)$. Let A be the plane through $\xi_i(x)$ perpendicular to $\gamma_i'(\bar{s})$. Note that $x \in A$. Then, for ε sufficiently small, $E = A \cap (B_{\varepsilon R}(\mathcal{C}_i) \smallsetminus \mathcal{S}_i)$ is a smooth two dimensional domain, and $\widehat{E} := (1/\varepsilon)E$ is independent of ε and depends smoothly on $\xi_i(x)$. Therefore, we may construct $\widehat{\rho}_1$ in $\widehat{E}$ in order to obtain an appropriate curve ρ_1 by rescaling.

Second, we construct a Lipschitz curve $\rho_2\colon [0, T_2] \to B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ from x_1 to some $x_2 \in B_d(\mathcal{C}_i) \smallsetminus B_c(\mathcal{C}_i)$ such that $T_2 = (c - (\varepsilon R + \operatorname{dist}(x_1, B_{\varepsilon R}(\mathcal{C}_i)))_+$ and

$$\operatorname{dist}(\rho_2(t), \mathcal{S}_i) \geqslant \operatorname{dist}(\rho_2(t), B_{\varepsilon R}(\mathcal{C}_i)) + \frac{1}{2}\varepsilon R = t + \operatorname{dist}(x_1, B_{\varepsilon R}(\mathcal{C}_i)) + \frac{1}{2}\varepsilon R. \tag{106}$$

To construct ρ_2, we just move along the gradient of $\operatorname{dist}(\cdot, \partial B_{\varepsilon R}(\mathcal{C}_i))$. The gradient of $\operatorname{dist}(\cdot, \partial B_{\varepsilon R}(\mathcal{C}_i))$ coincides with the gradient of $\operatorname{dist}(\cdot, \mathcal{C}_i)$ outside of $B_{\varepsilon R}(\mathcal{C}_i)$. The gradient is well defined through our choice of c, and (106) holds.

Third, we construct a Lipschitz curve $\rho_3\colon [0, T_3] \to B_d(\mathcal{C}_i) \smallsetminus B_c(\mathcal{C}_i)$ from x_2 to x_0 such that $|T_3| \leqslant C|x_2 - x_0|$. The existence of ρ_3 is straightforward since $B_d(\mathcal{C}_i) \smallsetminus B_c(\mathcal{C}_i)$ is a Lipschitz domain independent of ε which contains x_0. In particular observe that the part of the condition (105) which concerns the distance to the external boundary $\partial B_d(\mathcal{C}_i)$ of the domain $B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ is clear.

Now, we glue the three curves together and rescale to obtain $\rho\colon [0,|x-x_0|] \to B_d(\mathcal{C}_i) \smallsetminus \mathcal{S}_i$ which has a Lipschitz constant

$$Z_0 = \frac{T}{|x-x_0|} =: \frac{T_1+T_2+T_3}{|x-x_0|}. \tag{107}$$

Moreover, ρ satisfies

$$\operatorname{dist}(\rho(t),\partial\mathcal{S}_i) \geqslant \begin{cases} \dfrac{\eta t|x-x_0|}{T} & 0 \leqslant t \leqslant \dfrac{T_1|x-x_0|}{T}, \\ c\Big(t-\dfrac{T_1|x-x_0|}{T}\Big)+\dfrac{1}{2}\varepsilon R & \dfrac{T_1|x-x_0|}{T} \leqslant t \leqslant \dfrac{(T_1+T_2)|x-x_0|}{T}, \\ c & \dfrac{(T_1+T_2)|x-x_0|}{T} \leqslant t \leqslant |x-x_0|. \end{cases} \tag{108}$$

Note that the additional constant c in the second line above arises because the distance to $\partial\mathcal{S}_i$ is considered instead of the distance to $\mathcal{C}_i$. We claim that

$$T \leqslant C|x-x_0|. \tag{109}$$

which implies that Z_0 is bounded independently of ε and x.

To prove the claim note that

$$T \leqslant C\varepsilon R\mathbf{1}_{x\in B_{\varepsilon R}(\mathcal{C}_i)} + (c-(\varepsilon R+\operatorname{dist}(x_1,B_{\varepsilon R}(\mathcal{C}_i)))_+ + C|x_2-x_0|, \tag{110}$$

where the notation $\mathbf{1}$ is used for the indicator function of the set written as an index. Consider first the case $x \in B_{\varepsilon R}(\mathcal{C}_i)$. Then, for ε sufficiently small, $\varepsilon R \leqslant c \leqslant 2|x-x_0|$. Moreover, $x_1 \in \partial B_{\varepsilon R}(\mathcal{C}_i)$. Thus

$$(c-(\varepsilon R+\operatorname{dist}(x_1,B_{\varepsilon R}(\mathcal{C}_i)))_+ = |x_1-x_2| \leqslant c \leqslant 2|x-x_0|, \tag{111}$$

and finally

$$|x_2-x_0| \leqslant |x-x_0| + |x-x_1| - |x_1-x_2| \leqslant |x-x_0| + 2c \leqslant C|x-x_0| \tag{112}$$

such that we conclude (109).

If $x \notin B_{\varepsilon R}(\mathcal{C}_i)$ but $x \in B_c(\mathcal{C}_i)$, then we use that x_2 is the orthogonal projection of $x = x_1$ to $\partial B_c(\mathcal{C}_i)$. Thus

$$T_2 = |x-x_2| = \operatorname{dist}(x,\partial B_c(\mathcal{C}_i)) \leqslant |x-x_0| \tag{113}$$

and we deduce again (109) by the triangle inequality. In the case $x \notin B_c(\mathcal{C}_i)$, the claim is also trivially satisfied since $T_1 = T_2 = 0$.

It remains to verify that $\operatorname{dist}(\rho(t),\partial\mathcal{S}_i) \geqslant t/Z$, for some Z independent of x and ε. By (108), we see that this is satisfied on the first and on the third part of the curve. Recalling $|T_1| \leqslant C\varepsilon R$, the same holds for the second part of the curve. This finishes the proof.

In the case of a non-closed filament, the proof works almost the same. The only necessary change is due to the fact that the plane A as defined in the construction of ρ_1 above does not always contain x. Indeed $x \notin A$ if the segment $[x,\xi(x)]$ is not perpendicular to $\gamma_i(\overline{s})$ which can only happen if $\xi(x) \in \partial\mathcal{C}_i$. Fix such an $x \in B_c(\mathcal{C}_i)$

and let $P(x)$ be the projection of x to $\partial \mathcal{S}_i$. Then, since the faces of ∂S_i, i.e., the surfaces $\{y \in \partial S_i : \xi(y) \in \partial \mathcal{C}_i\}$, are flat, the straight curve

$$\rho_{1,2}(t) := x + t\frac{x - P(x)}{|x - P(x)|} \quad \text{satisfies } \operatorname{dist}(\rho_{1,2}(t), \partial \mathcal{S}_i) \geqslant t, \tag{114}$$

for all $t \leqslant T_2$ which we again take to be the time where $\rho_{1,2}(t) \in \partial B_c(\mathcal{C}_i)$. From there, we can continue with the curve ρ_3 as above. □

Now the statement that we will use is the following particular case of [1, Th. 4.1], where L^2_0 denotes the space of L^2-functions with vanishing mean.

Theorem 4.7 ([1, Th. 4.1]). — *Let $\Omega \subset \mathbb{R}^3$ be a John domain with constant Z. Then there exists a bounded linear operator* $\operatorname{Bog}\colon L^2_0(\Omega) \to H^1_0(\Omega)$ *such that for all $f \in L^2_0(\Omega)$*

$$\operatorname{div} \operatorname{Bog} f = f, \tag{115}$$

and the operator norm $\| \operatorname{Bog} \|_{L^2_0(\Omega)\to H^1_0(\Omega)}$ depends only on Z and $\operatorname{diam}(\Omega)$.

Relying on this result, we prove the following lemmas about extending functions defined on $\partial \mathcal{S}_j$.

Lemma 4.8. — *Let $d > 0$. Then, there exists $\varepsilon_0(d) > 0$ and $C(d) > 0$ such that for all ε in $(0, \varepsilon_0)$ the following holds. Let χ in $H^1(\partial \mathcal{S}_j)$ satisfying*

$$\int_{\partial S_j} \chi \cdot n \, d\mathcal{H}^2 = 0.$$

Then, there exists a divergence-free function $\psi \in \dot H^1(\mathbb{R}^3 \smallsetminus \mathcal{S}_j)$ such that $\psi = \chi$ on $\partial \mathcal{S}_j$, $\operatorname{supp} \psi \subset B_d(\mathcal{C}_j)$ *and*

$$\|\psi\|^2_{\dot H^1(\mathbb{R}^3 \smallsetminus \mathcal{S}_j)} \leqslant \frac{C}{\varepsilon} \|\chi\|^2_{L^2(\partial \mathcal{S}_j)} + C\varepsilon \|\nabla \chi\|^2_{L^2(\partial \mathcal{S}_j)}. \tag{116}$$

Proof. — We consider first the case of a closed filament $\mathcal{S}_j$. The necessary adaptations for a non-closed filament will be discussed at the end of the proof. Let $c > 0$ and $T_\varepsilon = B_{\varepsilon c}(\mathcal{S}_j) \smallsetminus \mathcal{S}_j$. Denote by P_ε the projection from T_ε to $\partial \mathcal{S}_j$. Then, there exists (c, ε_0) (depending only on $\mathcal{C}_j$) such that for all $\varepsilon < \varepsilon_0$, P_ε is well-defined, smooth and $|\nabla P_\varepsilon| \leqslant C$. By further reducing ε_0 (depending on d), we ensure that $T_\varepsilon \subset B_d(\mathcal{C}_j)$. Let θ_ε be a smooth cutoff function supported in T_ε such that $\theta_\varepsilon = 1$ on $\partial \mathcal{S}_j$ and $|\nabla \theta_\varepsilon| \leqslant C/\varepsilon$.

Consider the function

$$\psi(x) = \theta_\varepsilon(x)\chi(P_\varepsilon(x)) - \operatorname{Bog}(\operatorname{div}(\theta_\varepsilon(\cdot)\chi(P_\varepsilon(\cdot)))(x), \tag{117}$$

where Bog denotes suitable Bogovskiĭ operators on $B_d(\mathcal{C}_j) \smallsetminus \mathcal{S}_j$, provided by Theorem 4.7. We readily check the condition

$$\int_{B_d(\mathcal{C}_j)\smallsetminus \mathcal{S}_j} \operatorname{div}(\theta_\varepsilon(x)\chi(P_\varepsilon(x)) = \int_{\partial \mathcal{S}_j} \chi \cdot n \, d\mathcal{H}^2 = 0. \tag{118}$$

Therefore, we have

$$\|\psi\|^2_{\dot H^1(\mathbb{R}^3\smallsetminus \mathcal{S}_j)} \leqslant C\|\nabla(\theta_\varepsilon \chi\circ P_\varepsilon)\|^2_{L^2(\mathbb{R}^3\smallsetminus\mathcal{S}_j)} \tag{119}$$

$$\leqslant C\varepsilon^{-2}\,\|\chi\circ P_\varepsilon\|^2_{L^2(T_\varepsilon)} + C\|\nabla\chi\circ P_\varepsilon\|^2_{L^2(T_\varepsilon)}. \tag{120}$$

Finally, we observe that, by a change of coordinates and Fubini's principle,

$$\|\chi\circ P_\varepsilon\|^2_{L^2(T_\varepsilon)} \leqslant \varepsilon\|\chi\|^2_{L^2(\partial\mathcal{S}_j)}, \tag{121}$$

and analogously for the gradient. This implies the result. For the change of coordinates we used that for each $r\in(0,\varepsilon c)$, P_ε is a diffeomorphism (uniformly in r and ε) from $\partial B_r(\mathcal{S}_j)$ to $\partial\mathcal{S}_j$ for c,ε_0 sufficiently small.

This is not the case for a non-closed filament $\mathcal{S}_j$ which is only Lipschitz. Thus, we need to slightly modify the definition of P_ε. To this end, we first define $\widetilde P_\varepsilon$ as the projection from T_ε to

$$\mathcal{Z}_j := \{x\in\mathcal{S}_j : \operatorname{dist}(x,\partial\mathcal{S}_j)\geqslant c\varepsilon\}. \tag{122}$$

After possibly reducing c and ε_0, $\mathcal{Z}_j$ satisfies an exterior sphere condition with $R\geqslant 4c\varepsilon$, which makes this projection well-defined and also $P_\varepsilon\colon T_\varepsilon\to\partial\mathcal{S}_j$,

$$P_\varepsilon(x) := \widetilde P_\varepsilon(x) + c\varepsilon\,\frac{x-\widetilde P_\varepsilon(x)}{|x-\widetilde P_\varepsilon(x)|}. \tag{123}$$

Then, P_ε is again a diffeomorphism (uniformly in r and ε) from $\partial B_r(\mathcal{S}_j)$ to $\partial\mathcal{S}_j$. Indeed, the exterior sphere condition yields for all $x,y\in T_\varepsilon$

$$|\widetilde P_\varepsilon(x)-\widetilde P_\varepsilon(y)|\leqslant 2|x-y|. \tag{124}$$

Thus, for all $x,y\in\partial B_r(\mathcal{S}_j)$

$$|P_\varepsilon(x)-P_\varepsilon(y)|\leqslant 2|x-y| + c\varepsilon\,\frac{|x-y|+|\widetilde P_\varepsilon(x)-\widetilde P_\varepsilon(y)|}{r+c\varepsilon}\leqslant 5|x-y|.$$

It remains to check that

$$|P_\varepsilon(x)-P_\varepsilon(y)|\geqslant c_0|x-y|. \tag{125}$$

To this end, we distinguish two cases: in the first case, $|\widetilde P_\varepsilon(x)-\widetilde P_\varepsilon(y)|\geqslant |x-y|/8$. In this case (125) follows from (124) applied to $P_\varepsilon(x),P_\varepsilon(y)$ instead of x,y and using that $\widetilde P_\varepsilon(P_\varepsilon(x))=\widetilde P_\varepsilon(x)$. In the opposite case, $|\widetilde P_\varepsilon(x)-\widetilde P_\varepsilon(y)|\leqslant|x-y|/8$. Then, using $r\leqslant c\varepsilon$, (123) implies

$$|P_\varepsilon(x)-P_\varepsilon(y)|\geqslant\frac{|x-y|}{2}-2|\widetilde P_\varepsilon(x)-\widetilde P_\varepsilon(y)|\geqslant\frac{|x-y|}{4}. \qquad\square$$

4.4. Proof of Theorem 4.1. — Let us first focus on proving (69), observing that the estimate (68) will follow from (69), (87) and (89).

By Lemma 4.3 it suffices to show (69) with $U_{\mathcal{C}_i}$ replaced by $U_{\mathcal{C}_i^\varepsilon}$, namely

$$\|u-|\log\varepsilon|^{-1}\,U_{\mathcal{C}_i^\varepsilon}[v]\|_{\dot H^1(\mathbb{R}^3\smallsetminus\mathcal{S}_i)}\leqslant C\,|\log\varepsilon|^{-1}\,\|v\|_{W^{1,\infty}(\mathcal{S}_i)}, \tag{126}$$

Let

$$u_r := u-|\log\varepsilon|^{-1}\,U_{\mathcal{C}_i^\varepsilon}[v].$$

Using (89) for inside $\mathcal{S}_j$, for $j \neq i$, and the definition of u, we observe that, to prove the estimate (126), it suffices to show that

$$\|u_r\|_{\dot{H}^1(\mathcal{F})} \leqslant C\,|\log\varepsilon|^{-1}\,\|v\|_{W^{1,\infty}(\mathcal{S}_i)}. \tag{127}$$

We apply Lemma 4.8 with $\chi = u_r$, which satisfies the condition

$$\int_{\partial\mathcal{S}_j} u_r \cdot n\, d\mathcal{H}^2 = \int_{\mathcal{S}_j} \operatorname{div} u_r = 0, \tag{128}$$

for $1 \leqslant j \leqslant N$.

By (86), since

$$\int_{\partial\mathcal{S}_i} |\log(\operatorname{dist}(x, \partial\mathcal{C}_i^\varepsilon))|^2\, d\mathcal{H}^2 \leqslant C\varepsilon,$$

we have

$$\|u_r\|^2_{L^2(\partial\mathcal{S}_i)} \leqslant C\varepsilon\,|\log\varepsilon|^{-2}\,\|v\|^2_{W^{1,\infty}(\mathcal{S}_i)}. \tag{129}$$

Moreover from the pointwise estimate (67) applied to $U_{\mathcal{C}_i^\varepsilon}$ and recalling (82), we find

$$\|\nabla_\tau u_r\|^2_{L^2(\partial\mathcal{S}_i)} \leqslant \frac{C}{\varepsilon}\,|\log\varepsilon|^{-2}\,\|v\|^2_{W^{1,\infty}(\mathcal{S}_i)}, \tag{130}$$

where ∇_τ denotes the tangential part of the gradient. Finally, for $j \neq i$, we observe that (89) implies

$$\|u_r\|^2_{L^2(\partial\mathcal{S}_j)} \leqslant C\varepsilon\,|\log\varepsilon|^{-2}\,\|v\|^2_{W^{1,\infty}(\mathcal{S}_i)}, \tag{131}$$

$$\|\nabla_\tau u_r\|^2_{L^2(\partial\mathcal{S}_i)} \leqslant C\varepsilon\,|\log\varepsilon|^{-2}\,\|v\|^2_{W^{1,\infty}(\mathcal{S}_j)}, \tag{132}$$

for some constant depending on d.

Thus, Lemma 4.8 yields a divergence-free function $\widetilde{u}_r = \sum \psi_i \in H^1(\mathcal{F})$ with $\widetilde{u}_r = u_r$ on $\partial\mathcal{F}$ and

$$\|\widetilde{u}_r\|_{\dot{H}^1(\mathcal{F})} \leqslant C\,|\log\varepsilon|^{-1}\,\|v\|_{W^{1,\infty}}. \tag{133}$$

We conclude by recalling that u_r solves the Stokes equations in $\mathcal{F}$. Thus, it minimizes the $\dot{H}^1$-norm among all divergence-free functions which satisfy the same boundary conditions according to the Helmholtz minimum dissipation theorem which we now recall.

THEOREM 4.9. — *If u in $\dot{H}^1(\mathcal{F})$ satisfies*

$$-\Delta u + \nabla p = 0, \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \mathcal{F},$$

and $\widetilde{u}$ in $\dot{H}^1(\mathcal{F})$ satisfies

$$\operatorname{div}\widetilde{u} = 0 \quad \text{in } \mathcal{F} \quad \text{and} \quad \widetilde{u} = u \quad \text{on } \partial\mathcal{F},$$

then

$$\|u\|_{\dot{H}^1(\mathcal{F})} \leqslant \|\widetilde{u}\|_{\dot{H}^1(\mathcal{F})}.$$

Proof. — Using $\widetilde{u} - u$ as a test function in the weak formulation of the PDE for u, we find $(\widetilde{u} - u, u)_{\dot{H}^1(\mathcal{F})} = 0$ and thus $\|\widetilde{u}\|^2_{\dot{H}^1(\mathcal{F})} = \|\widetilde{u} - u\|^2_{\dot{H}^1(\mathcal{F})} + \|u\|^2_{\dot{H}^1(\mathcal{F})}$. □

For the proof of Theorem 4.1, it remains to show (70). This estimate follows directly from (69), (88) and the following lemma.

Lemma 4.10. — *For all $d > 0$ and $p < 3/2$ there exists $\varepsilon_0(d) > 0$ and $C_p(d) > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $d_{\min} \geqslant d$ the following holds. Let $q \in [p, \infty]$ and let $v \in \dot H^1(\mathcal{F}) \cap W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3)$ be divergence-free and solve the homogeneous Stokes equations in $\mathcal{F}$, that is*

$$-\Delta v + \nabla p = 0 \quad \text{and} \quad \operatorname{div} v = 0 \quad \text{in } \mathcal{F}.$$

Then,

$$\|v\|_{W^{1,p}_{\mathrm{loc}}} \leqslant C_p \left|\log \varepsilon\right|^{-1/2} \|\nabla v\|_{L^2(\mathcal{F})} + \varepsilon^{2/q'-2/p'} \|\nabla v\|_{L^q(\bigcup_i \mathcal{S}_i)}. \tag{134}$$

Proof. — Let K be compact, $g \in L^{p'}$, $\operatorname{supp} g \subset K$ and let w solve

$$-\Delta w + \nabla \pi = \operatorname{div} g \quad \text{and} \quad \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^3.$$

Then, by standard regularity theory and Sobolev embedding with $1/p' = 1/r - 1/3$

$$\|\nabla w\|_{L^{p'}(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \leqslant C_p(\|\nabla w\|_{L^{p'}(\mathbb{R}^3)} + \|\nabla w\|_{L^r(\mathbb{R}^3)}) \leqslant C_{K,p} \|g\|_{L^{p'}(\mathbb{R}^3)}, \tag{135}$$

where we used in the last step that $g \in L^r(\mathbb{R}^3)$ due to its compact support. Thus the desired estimate for $\|\nabla v\|_{L^p(K)}$ will follow by duality once we have shown

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla v \cdot g &= 2 \int_{\mathbb{R}^3} Dw : Dv \\ &\leqslant C_p \Big(\left|\log \varepsilon\right|^{-1/2} \|\nabla v\|_{L^2(\mathcal{F})} + \varepsilon^{2/q'-2/p'} \|\nabla v\|_{L^q(\bigcup_i \mathcal{S}_i)} \Big) \Big(\|\nabla w\|_{L^{p'}(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \Big),\end{aligned} \tag{136}$$

where we recall that D is the deformation tensor, see (21). The estimate for $\|v\|_{L^p(K)}$ follows along the same lines by considering the problem $-\Delta w' + \nabla \pi' = g$ and the fact that

$$\int v \cdot g = 2 \int_{\mathbb{R}^3} Dw' : Dv.$$

Note that the regularity of w' is even better than the regularity of w.

To show (136), we split the left hand side into

$$2 \int_{\mathbb{R}^3} Dw : Dv = 2 \int_{\cup \mathcal{S}_i} Dw : Dv + 2 \int_{\mathcal{F}} Dw : Dv =: I_1 + I_2. \tag{137}$$

By Hölder's inequality, we estimate

$$I_1 \leqslant C \varepsilon^{2/q'-2/p'} \|\nabla v\|_{L^q(\bigcup_i \mathcal{S}_i)} \|\nabla w\|_{L^{p'}}, \tag{138}$$

where we used $|\bigcup \mathcal{S}_i| \leqslant C\varepsilon^2$. Moreover, by some integrations by parts, we have that

$$I_2 = \sum_{1 \leqslant i \leqslant N} \int_{\partial \mathcal{S}_i} \Sigma(v,p) n \cdot w = 2 \int_{\mathbb{R}^3 \smallsetminus \mathcal{F}} D\varphi : Dv \leqslant 2 \|v\|_{\dot H^1(\mathcal{F})} \|\varphi\|_{\dot H^1(\mathcal{F})} \tag{139}$$

for all divergence-free functions $\varphi \in \dot H^1(\mathbb{R}^3)$ with $\varphi = w$ in S_i.

Therefore, it remains to show that such a function φ exists which satisfies

$$\|\varphi\|_{\dot H^1} \leqslant C \left|\log \varepsilon\right|^{-1/2} \big(\|\nabla w\|_{L^{p'}(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \big). \tag{140}$$

The construction of such a function φ is similar to the construction in the proof of Lemma 4.8. However, since we have better control of the function which we want to extend, we will see that using a different cut-off function yields better estimates than in Lemma 4.8. More precisely, we consider $R \geqslant 0$ (independent of ε) and ε_0 sufficiently small, such that $\mathcal{S}_i \subset B_{\varepsilon R}(\mathcal{C}_i) \subset B_d(\mathcal{C}_i)$ for all $1 \leqslant i \leqslant N$. Then, we define

$$\theta_\varepsilon(x) := \begin{cases} \dfrac{\log(\operatorname{dist}(x, \bigcup_i \mathcal{C}_i)) - \log d}{\log(\varepsilon R) - \log d} & \text{in } \bigcup_i (B_d(\mathcal{C}_i) \smallsetminus B_{\varepsilon R}(\mathcal{C}_i)), \\ 1 & \text{in } \bigcup_i B_{\varepsilon R}(\mathcal{C}_i), \\ 0 & \text{in } \mathbb{R}^3 \smallsetminus (\bigcup_i B_d(\mathcal{C})). \end{cases} \tag{141}$$

Note that this cut-off function corresponds to the 2-dimensional capacitary function of a ball $B_\varepsilon(x)$ within $B_d(x)$.

Now we define

$$\varphi := \theta_\varepsilon w - \operatorname{Bog}(\nabla\theta_\varepsilon \cdot w). \tag{142}$$

Note that $\operatorname{supp} \nabla\theta_\varepsilon \subset \cup B_d(\mathcal{C}_i) \smallsetminus B_{\varepsilon R}(\mathcal{C}_i)$. We may apply Lemma 4.6 to this domain to estimate the Bogovskiĭ operator since $B_{\varepsilon R}(\mathcal{C}_i)$ just corresponds to a filament with centerline $\mathcal{C}_i$ and circular cross section.

Thus, we can estimate

$$\begin{aligned} \|\varphi\|_{\dot H^1} &\leqslant \|w\nabla\theta_\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\theta_\varepsilon \nabla w\|_{L^2(\mathbb{R}^3)} \\ &\leqslant \|w\|_{L^\infty(\mathbb{R}^3)} \|\nabla\theta_\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\theta_\varepsilon\|_{L^r(\mathbb{R}^3)} \|\nabla w\|_{L^{p'}(\mathbb{R}^3)}, \end{aligned} \tag{143}$$

where $1/r + 1/p' = 1/2$. To conclude, we assume that ε_0 is chosen small enough such that $\log d - \log(\varepsilon R) \geqslant \frac{1}{2}|\log\varepsilon|$. Then,

$$\|\nabla\theta_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-2} \sum_{1\leqslant i\leqslant N} \int_{B_d(\mathcal{C}_i)\smallsetminus B_{\varepsilon R}(\mathcal{C}_i)} \frac{1}{\operatorname{dist}^2(x,\mathcal{C}_i)}\,dx \leqslant C\,|\log\varepsilon|^{-1}, \tag{144}$$

and for all $r < \infty$,

$$\begin{aligned} &\|\theta_\varepsilon\|^r_{L^r(\mathbb{R}^3)} \\ &\qquad \leqslant C\,|\log\varepsilon|^{-r} \sum_{1\leqslant i\leqslant N} \int_{B_d(\mathcal{C}_i)} |\log(\operatorname{dist}(x,\mathcal{C}_i))|^r + |\log d|^r\,dx \leqslant C_r\,|\log\varepsilon|^{-r}. \end{aligned} \tag{145}$$

Inserting (144) and (145) in (143) yields (140). This concludes the proof. □

4.5. Proof of Corollary 4.2. — This subsection is devoted to the proof of Corollary 4.2. Let us therefore consider a vector field v in $W^{1,\infty}(\bigcup_{j=1}^N \mathcal{S}_j)$ and divergence-free. Our aim is to establish the inequality (72) regarding the approximation of

$$\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} (\Sigma(V_{i,\alpha}, P_{i,\alpha})n)\cdot v\,d\mathcal{H}^2 \quad \text{by} \quad |\log\varepsilon|^{-1} I_{\mathcal{C}_i}[v_{i,\alpha}, v], \tag{146}$$

where the functions $(V_{i,\alpha}, P_{i,\alpha})$ are defined in (71). Let

$$u_{i,\alpha} := |\log\varepsilon|^{-1} U_{\mathcal{C}_i}[v_{i,\alpha}] \quad \text{and} \quad p_{i,\alpha} := |\log\varepsilon|^{-1} P_{\mathcal{C}_i}[v_{i,\alpha}], \tag{147}$$

where $U_{\mathcal{C}_i}$ and $P_{\mathcal{C}_i}$ are the operators respectively defined in (44) and (48). By (49),

$$-\Delta u_{i,\alpha} + \nabla p_{i,\alpha} = |\log\varepsilon|^{-1}\,\mu_{\mathcal{C}_i}[v_{i,\alpha}], \quad \operatorname{div} u_{i,\alpha} = 0, \tag{148}$$

in the sense of distributions in $\mathbb{R}^3$. Let us decompose

$$\begin{aligned}\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} &(\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v\,d\mathcal{H}^2 - |\log\varepsilon|^{-1}\,I_{\mathcal{C}_i}[v_{i,\alpha},v]\\ &= \int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot v\,d\mathcal{H}^2 - |\log\varepsilon|^{-1}\,I_{\mathcal{C}_i}[v_{i,\alpha},v]\\ &\quad+\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j}(\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v\,d\mathcal{H}^2 - \int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot v\,d\mathcal{H}^2.\end{aligned}\tag{149}$$

By an integration by parts inside the filament $\mathcal{S}_i$ and recalling the definition (31), we deduce from (148) that

$$\begin{aligned}\int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot v\,d\mathcal{H}^2 &= \int_{\mathcal{S}_i}(-\Delta u_{i,\alpha}+\nabla p_{i,\alpha})\cdot v + \int_{\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha}):D(v)\\ &= |\log\varepsilon|^{-1}\,I_{\mathcal{C}_i}[v_{i,\alpha},v] + \int_{\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha}):D(v).\end{aligned}\tag{150}$$

By (67), and by observing that the pressure $P_{\mathcal{C}_i}$ satisfies the same pointwise decay estimates as the velocity gradient $\nabla U_{\mathcal{C}_i}$

$$\Big|\int_{\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha}):D(v)\Big| \leqslant C\|v\|_{W^{1,\infty}(\mathcal{S}_i)}\int_{\mathcal{S}_i}\frac{1}{\operatorname{dist}(x,\mathcal{C}_i)}\,dx \leqslant C\varepsilon\|v\|_{W^{1,\infty}(\mathcal{S}_i)}.\tag{151}$$

Let $w := \sum_{i=1}^N w_i$, where w_i is the solution to (63). Then $w = v$ in $\bigcup_{j=1}^N \mathcal{S}_j$ and thus,

$$\begin{aligned}\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j}(\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v\,d\mathcal{H}^2 &= \int_{\bigcup_{j=1}^N \partial\mathcal{S}_j}(\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot w\,d\mathcal{H}^2\\ &= \int_{\mathcal{F}} D(V_{i,\alpha}):D(w)\\ &= \int_{\mathbb{R}^3\smallsetminus\mathcal{S}_i} D(V_{i,\alpha}):D(w),\end{aligned}$$

since $D(V_{i,\alpha}) = 0$ in $\bigcup_{j\neq i}\mathcal{S}_j$. On the other hand, it follows from (148) that

$$\begin{aligned}\int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot v d\mathcal{H}^2 &= \int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot w\;d\mathcal{H}^2\\ &= \int_{\mathbb{R}^3\smallsetminus\mathcal{S}_i} D(u_{i,\alpha}):D(w)\;d\mathcal{H}^2,\end{aligned}$$

so that

$$\begin{aligned}\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j}(\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot v\,d\mathcal{H}^2 - \int_{\partial\mathcal{S}_i}\Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot w d\mathcal{H}^2&\\ = \int_{\mathbb{R}^3\smallsetminus\mathcal{S}_i} D(V_{i,\alpha}-u_{i,\alpha}):D(w).&\end{aligned}$$

Therefore, by Theorem 4.1, we deduce that

$$\Big|\int_{\bigcup_{j=1}^N \partial\mathcal{S}_j} (\Sigma(V_{i,\alpha},P_{i,\alpha})n)\cdot h\, d\mathcal{H}^2 - \int_{\partial\mathcal{S}_i} \Sigma(u_{i,\alpha},p_{i,\alpha})n\cdot v d\mathcal{H}^2\Big| \leqslant \|u_{i,\alpha}-V_{i,\alpha}\|_{\dot H^1(\mathbb{R}^3\smallsetminus\mathcal{S}_i)}\|w\|_{\dot H^1(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-3/2}\,\|v\|_{W^{1,\infty}(\bigcup_j \mathcal{S}_j)}. \tag{152}$$

Gathering (149), (150), (151) and (152) we arrive at (72) and this finishes the proof of Corollary 4.2.

5. Shape derivatives

Since the filaments evolve in time, it is necessary to tackle the behaviour of the solutions to the Stokes system with Dirichlet data in the filaments under rigid displacements of the filaments. To this end we establish the following bound on the shape derivatives, with respect to rigid motions of the filaments, of the interaction energy of two solutions to the Stokes system with fixed values in the filaments. This bound is uniform with respect to ε and to the positions for which a positive minimal distance between the centerlines is guaranteed.

Proposition 5.1. — *For all $d>0$, there are $C(d)>0$ and $\varepsilon_0(d)>0$ such that for all ε in $(0,\varepsilon_0)$, for all (h,Q) in $\mathbb{R}^{3N}\times SO(3)^N$ such that the corresponding minimal distance $d_{\min}$ between the centerlines defined in* (65)*, satisfies $d_{\min}\geqslant d$, and for all divergence-free vector fields φ_1 and φ_2 in $W^{2,\infty}(\mathbb{R}^3)$, the following holds true. The corresponding solutions ψ_1 and ψ_2 in $\dot H^1(\mathcal{F})$ to the Stokes problem*

$$\begin{aligned} -\Delta\psi_i+\nabla p_i=0 \quad \text{and} \quad \operatorname{div}\psi_i=0 \quad \text{in } \mathcal{F},\\ \psi_i=\varphi_i \quad \text{on } \bigcup_{j=1}^N \partial\mathcal{S}_j, \end{aligned} \tag{153}$$

where the position $\mathcal{S}_j$ of the filaments are deduced from their original positions by (h,Q) as in (11)*, satisfy*

$$\begin{aligned} &|\nabla_{h,Q}\big(D(\psi_1),D(\psi_2)\big)_{L^2(\mathcal{F})}| \\ &\qquad \leqslant C\,|\log\varepsilon|^{-1}\Big(\|\varphi_1\|_{W^{1,\infty}(\mathbb{R}^3)}\|\varphi_2\|_{W^{2,\infty}(\mathbb{R}^3)}+\|\varphi_1\|_{W^{2,\infty}(\mathbb{R}^3)}\|\varphi_2\|_{W^{1,\infty}(\mathbb{R}^3)}\Big). \end{aligned} \tag{154}$$

Above the notation $(\cdot,\cdot)_{L^2(\mathcal{F})}$ stands for the inner product in $L^2(\mathcal{F})$.

Proof of Proposition 5.1. — To establish the bound (154) of the shape derivative of the interaction energy

$$\big(D(\psi_1),D(\psi_2)\big)_{L^2(\mathcal{F})},$$

we estimate the difference of such interaction energies corresponding to two close configurations of the filaments. To this end, let $d>0$, let φ_1,φ_2 divergence-free vector fields in $W^{2,\infty}(\mathbb{R}^3)$. Let $(h,Q)\in\mathbb{R}^{3N}\times SO(3)^N$ with $d_{\min}\geqslant d$ and let $(\check h,\check Q)\in\mathbb{R}^{3N}\times SO(3)^N$ such that $|(h,Q)-(\check h,\check Q)|\leqslant\delta$ small enough, to be chosen later. For $i=1,2$, let ψ_i and $\check\psi_i$, the solutions to (153) corresponding to the same boundary data φ_i, and to the filaments' positions (h,Q) and $(\check h,\check Q)$, respectively. Corresponding pressures are denoted by p_i and $\breve{p}_i$. Thus, for $i=1,2$, on the one hand (ψ_i, p_i) satisfies (153) and on the other hand $(\breve{\psi}_i, \breve{p}_i)$ satisfies

$$
\begin{gathered}
-\Delta\breve{\psi}_i + \nabla\breve{p}_i = 0 \quad \text{and} \quad \operatorname{div}\breve{\psi}_i = 0 \quad \text{in } \breve{\mathcal{F}}, \\
\breve{\psi}_i = \varphi_i \quad \text{on } \bigcup_{j=1}^{N} \partial\breve{\mathcal{S}}_j,
\end{gathered}
\tag{155}
$$

where $\mathcal{F}$ and $\breve{\mathcal{F}}$ denote the fluid domain respectively corresponding to (h,Q) and $(\breve{h},\breve{Q})$, while the sets $\mathcal{S}_j$ and $\breve{\mathcal{S}}_j$ are the positions respectively occupied by the filaments in the two configurations.

To prove Proposition 5.1 we are going to prove that there are $C(d)>0$ and $\varepsilon_0(d)>0$ such that for all ε in $(0,\varepsilon_0)$,

$$
\begin{aligned}
&|(D(\psi_1), D(\psi_2))_{L^2(\mathcal{F})} - (D(\breve{\psi}_1), D(\breve{\psi}_2))_{L^2(\breve{\mathcal{F}})}| \\
&\qquad \leqslant C\delta\,|\log\varepsilon|^{-1}\Big(\|\varphi_1\|_{W^{1,\infty}(\mathbb{R}^3)}\|\varphi_2\|_{W^{2,\infty}(\mathbb{R}^3)} + \|\varphi_1\|_{W^{2,\infty}(\mathbb{R}^3)}\|\varphi_2\|_{W^{1,\infty}(\mathbb{R}^3)}\Big).
\end{aligned}
\tag{156}
$$

Without loss of generality, we may restrict the proof to the case where only one filament is displaced, say the first one, so that the positions of the other filaments is the same for the two configurations, that is $(\breve{h}_j,\breve{Q}_j)=(h_j,Q_j)$ for all $2\leqslant j\leqslant N$. As a consequence,

$$
\mathcal{S}_j = \breve{\mathcal{S}}_j, \quad \text{for all } j \text{ such that } 2\leqslant j\leqslant N. \tag{157}
$$

Moreover, up to a change of frame, we may also assume without loss of generality that the position of the first filament satisfies $(h_1,Q_1)=(0,\mathrm{Id})$, and we recall that the position $(\breve{h}_1,\breve{Q}_1)$ of the first filament in the second configuration is in general different from $(0,\mathrm{Id})$ but δ-close so that

$$
|\breve{h}_1| + |\breve{Q}_1 - \mathrm{Id}\,| \leqslant \delta. \tag{158}
$$

Step 1: Construction of a suitable deformation. — In this first step we introduce an auxiliary vector field associated with $\breve{\psi}_i$, see (167) for the definition, but which solves a Stokes system in $\mathcal{F}$, see (174) for the exact system.

We choose a neighborhood $\mathcal{T}_1$ defined by $\mathcal{T}_1 = \mathcal{S}_1^{2\varepsilon}$ (i.e., the filament corresponding to 2ε instead of ε). Lemma 4.6 ensures that the set

$$
\mathcal{J} := B_{d/4}(\mathcal{C}_1) \smallsetminus \mathcal{T}_1 \tag{159}
$$

is a John domain with a constant Z independent of ε for ε sufficiently small. Let $\eta\in C_c^\infty(B_{d/4}(\mathcal{C}_1))$ be a nonnegative cut-off function such that $\eta=1$ in $\mathcal{T}_1$.

Let ϕ be the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that for all x in $\mathbb{R}^3$,

$$
\phi(x) := x + \big((\breve{Q}_1 - \mathrm{Id})x + \breve{h}_1\big)\eta(x). \tag{160}
$$

By construction,

$$
\phi(\mathcal{F}) = \breve{\mathcal{F}}, \quad \text{in particular} \quad \phi(\mathcal{S}_1) = \breve{\mathcal{S}}_1, \tag{161}
$$

and

$$
\phi|_{\mathbb{R}^3\smallsetminus(B_{d/4}(\mathcal{C}_1))} = \mathrm{Id}_{\mathbb{R}^3\smallsetminus(B_{d/4}(\mathcal{C}_1))}\,. \tag{162}
$$

Moreover ϕ and ϕ^{-1} are diffeomorphisms from $\mathbb{R}^3$ to $\mathbb{R}^3$ with

$$|\nabla\phi| + |\nabla\phi^{-1}| \leqslant C, \tag{163}$$

for a constant C independent of d for δ sufficiently small. To see that ϕ is injective for δ sufficiently small, and to estimate $\nabla\phi^{-1}$, we estimate $|\phi(x_1)-\phi(x_2)|$ from below for all $x_1, x_2 \in \mathbb{R}^3$. Clearly, if $x_i \notin \operatorname{supp}\eta$ for $i=1,2$, the estimate is trivial. Let us assume $x_2 \in \operatorname{supp}\eta$ and note that since $h_1 = 0$ this implies $|x_2| \leqslant C + d$, where the constant C depends only on the reference filament $\overline{\mathcal{S}}_1$. Thus, using (160) and (158), we have that

$$\begin{aligned}
&|\phi(x_1) - \phi(x_2)| \\
&\qquad \geqslant |x_1 - x_2| - \big|\big((\check{Q}_1 - \mathrm{Id})x_1 + \check{h}_1\big)\eta(x_1) - \big((\check{Q}_1 - \mathrm{Id})x_2 + \check{h}_1\big)\eta(x_2)\big| \\
&\qquad \geqslant |x_1 - x_2| - |\check{Q}_1 - \mathrm{Id}\,||x_1 - x_2|\eta(x_1) - \big|(\check{Q}_1 - \mathrm{Id})x_2 + \check{h}_1\big||\eta(x_1) - \eta(x_2)| \\
&\qquad \geqslant \frac{1}{2}|x_1 - x_2|,
\end{aligned}$$

for δ sufficiently small.

Furthermore, for all x in $\mathbb{R}^3$, we set

$$\Phi(x) := \nabla\phi(x) = \mathrm{Id} + (\check{Q}_1^T - \mathrm{Id})\eta(x) + \big((\check{Q}_1 - \mathrm{Id})x + \check{h}_1\big) \otimes \nabla\eta, \tag{164}$$

with the convention that $\nabla\phi = (\partial_i\phi_j)_{i,j}$. From the definition of ϕ and Φ in (160) and (164) it follows that

$$|(\nabla\phi^T)\nabla\phi - \mathrm{Id}\,| + |\nabla\Phi| + |\nabla^2\Phi| \leqslant C\delta\, \mathbf{1}_{\mathcal{J}}, \tag{165}$$

where $\mathbf{1}_{\mathcal{J}}$ is the indicator function of the set $\mathcal{J}$ defined in (159).

Since $\eta = 1$ in $\mathcal{T}_1$,

$$\text{for all}\, x \in \mathcal{T}_1, \quad (\Phi\check{\psi}_i \circ \phi)(x) = \check{Q}_1^T \psi_i(\check{Q}_1 x + \check{h}_1). \tag{166}$$

In particular the vector field $\Phi\check{\psi}_i \circ \phi$ is divergence-free in $\mathcal{T}_1$. We define

$$\widetilde{\psi}_i := \Phi\check{\psi}_i \circ \phi - \operatorname{Bog}(\operatorname{div}(\Phi\check{\psi}_i \circ \phi)) \quad \text{and} \quad \widetilde{p}_i = \check{p}_i \circ \phi, \tag{167}$$

where Bog denotes a Bogovskiĭ operator provided by Theorem 4.7 in the domain $\mathcal{J}$, which satisfies

$$\operatorname{supp}(\operatorname{div}(\Phi\check{\psi}_i \circ \phi)) \subset \mathcal{J}. \tag{168}$$

Recall that, according to Theorem 4.7 and the fact that $\mathcal{J}$ is a John domain with a constant Z independent of ε for ε sufficiently small, the operator Bog mentioned above satisfies that there exists $C > 0$ such that for ε sufficiently small,

$$\text{for all }\; f \in L^2_0(\mathcal{J}), \quad \|\operatorname{Bog} f\|_{H^1(\mathcal{J})} \leqslant C\|f\|_{L^2(\mathcal{J})}. \tag{169}$$

Step 2: The divergence of $\Phi\breve{\psi}_i\circ\phi$. — In this step, we prove the following identity, which in combination with (165), is helpful below, see (178), to prove that in $\mathcal{J}$, $\operatorname{div}(\Phi\breve{\psi}_i\circ\phi)$ is a $O(\delta)$:

$$\operatorname{div}(\Phi\breve{\psi}_i\circ\phi)=(\operatorname{div}\Phi)\cdot(\psi_i\circ\phi)+\Big((\nabla\phi)^T\nabla\phi-\operatorname{Id}\Big):((\nabla\breve{\psi}_i)\circ\phi). \tag{170}$$

Let us first recall that for some regular enough fields of matrices A and of vectors v, the following identity holds true:

$$\operatorname{div}(Av)=(\operatorname{div}A)\cdot v+A:\nabla v, \tag{171}$$

where the operator div has to be applied row-wise to A.

In particular, by applying (171) to the case where $A=\Phi$ and $v=\breve{\psi}_i\circ\phi$, and recalling that $\Phi=\nabla\phi$, we obtain that

$$\begin{aligned}\operatorname{div}(\Phi\breve{\psi}_i\circ\phi)&=(\operatorname{div}\Phi)\cdot(\psi_i\circ\phi)+\nabla\phi:\nabla(\breve{\psi}_i\circ\phi)\\&=(\operatorname{div}\Phi)\cdot(\psi_i\circ\phi)+(\nabla\phi)^T\nabla\phi:((\nabla\breve{\psi}_i)\circ\phi),\end{aligned} \tag{172}$$

where we used in the last identity that $A:BC=B^TA:C$ for any $A,B,C\in R^{3\times 3}$.

Finally, by definition $\operatorname{div}\breve{\psi}_i=\operatorname{Id}:\nabla\breve{\psi}=0$. Using this in (172) yields (170).

Step 3: The Stokes system solved by $\widetilde{\psi}_i$, for $i=1,2$. — Observe the following fact:

$$\text{for all } x\in\mathcal{T}_1,\quad \widetilde{\psi}_i(x)=\breve{Q}_1^T\breve{\psi}_i(\breve{Q}_1x+\breve{h}_1)\quad\text{and}\quad \widetilde{p}_i(x)=\breve{p}_i(\breve{Q}_1x+\breve{h}_1). \tag{173}$$

Using $\Phi=(\nabla\phi)^T$ and some tensor calculus similar as in the previous step, we find in $\mathcal{F}$

$$\begin{aligned}-\Delta\widetilde{\psi}_i+\nabla\widetilde{p}_i&=-(\Delta\Phi)\breve{\psi}_i\circ\phi+\Delta\operatorname{Bog}(\operatorname{div}(\Phi\breve{\psi}_i\circ\phi))-2\nabla\Phi\nabla(\breve{\psi}_i\circ\phi)\\&\qquad-\Phi\Big((\nabla\phi)^T\nabla\phi:\nabla^2\breve{\psi}_i\circ\phi\Big)+\Phi\nabla\breve{p}_i\circ\phi\\&=-(\Delta\Phi)\breve{\psi}_i\circ\phi+\Delta\operatorname{Bog}(\operatorname{div}(\Phi\breve{\psi}_i\circ\phi))-2\nabla\Phi\nabla(\breve{\psi}_i\circ\phi)\\&\qquad+\Phi(\operatorname{Id}-(\nabla\phi)^T\nabla\phi):\nabla^2\breve{\psi}_i\circ\phi,\end{aligned}$$

where we used that $\nabla\breve{p}_i=\Delta\breve{\psi}_i$ in $\mathcal{F}$.

Concerning the last term, a further manipulation leads to

$$\begin{aligned}\Phi(\operatorname{Id}-(\nabla\phi)^T\nabla\phi):\nabla^2\breve{\psi}_i\circ\phi&=\Phi(\operatorname{Id}-(\nabla\phi)^T\nabla\phi):(\nabla\phi)^{-T}\nabla(\nabla\breve{\psi}_i\circ\phi)\\&=\operatorname{div}\Big(\Phi(\operatorname{Id}-(\nabla\phi)^T\nabla\phi)(\nabla\phi)^{-T}:\nabla\breve{\psi}_i\circ\phi\Big)\\&\qquad-\operatorname{div}\big(\Phi(\operatorname{Id}-(\nabla\phi)^T\nabla\phi):(\nabla\phi)^{-T}\big)\nabla\breve{\psi}_i\circ\phi.\end{aligned}$$

Therefore, and relying on (155), (160), (167), (173) and (161), we obtain that $\widetilde{\psi}_i$ solves the following Stokes system:

$$-\Delta\widetilde{\psi}_i+\nabla\widetilde{p}_i=\operatorname{div}g_i+f_i\quad\text{and}\quad\operatorname{div}\widetilde{\psi}_i=0\quad\text{in }\mathcal{F}, \tag{174a}$$

$$\widetilde{\psi}_i(x)=\breve{Q}_1^T\varphi_1(\breve{Q}_1x+\breve{h}_1)\quad\text{in }\mathcal{S}_1, \tag{174b}$$

$$\widetilde{\psi}_i=\varphi_i\quad\text{in }\mathcal{S}_j\quad\text{for all }j\neq 1. \tag{174c}$$

where

$$g_i := -\nabla \operatorname{Bog}(\operatorname{div}(\Phi\breve{\psi}_i \circ \phi)) + \Phi(\operatorname{Id} - (\nabla\phi)^T\nabla\phi)(\nabla\phi)^{-T}\nabla\breve{\psi}_i \circ \phi, \tag{175}$$

$$f_i := -(\Delta\Phi)\breve{\psi}_i \circ \phi - 2\nabla\Phi\nabla(\breve{\psi}_i \circ \phi) + \operatorname{div}\big(\Phi(\operatorname{Id} - (\nabla\phi)^T\nabla\phi)(\nabla\phi)^{-T}\big)\nabla\breve{\psi}_i \circ \phi. \tag{176}$$

We observe that g_i and f_i are compactly supported in $\mathcal{J}$, see (165).

Step 4: Estimate of g_i and f_i, for $i = 1, 2$. — We start with estimating the term of g_i involving the Bogovskiĭ operator. First, using that ϕ is a diffeomorphism satisfying (163), we obtain that

$$\|\nabla\breve{\psi}_i \circ \phi\|_{L^2(\mathbb{R}^3)} \leqslant C\|\nabla\breve{\psi}_i\|_{L^2(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-1/2}\,\|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^3)}, \tag{177}$$

by applying Theorem 4.1. Moreover, by applying (171), it follows

$$\big|\operatorname{div}(\Phi\breve{\psi}_i \circ \phi)\big| \leqslant C\delta\big(|\breve{\psi}_i \circ \phi| + |\nabla\breve{\psi}_i \circ \phi|\big). \tag{178}$$

Therefore, by using (169)

$$\|\nabla(\operatorname{Bog}(\operatorname{div}(\Phi\breve{\psi}_i \circ \phi)))\|_{L^2} \leqslant C\big\|\operatorname{div}(\Phi\breve{\psi}_i \circ \phi)\big\|_{L^2} \leqslant C\delta\,|\log\varepsilon|^{-1/2}\,\|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^3)},$$

where we used (177) to get the last inequality.

Similarly, we can estimate all the other terms on the right hand sides of (176) and (175) by

$$\|g_i\|_{L^2(\mathcal{F})} + \|f_i\|_{L^{6/5}(\mathcal{F})} \leqslant C\delta\,|\log\varepsilon|^{-1/2}\,\|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^3)}. \tag{179}$$

Step 5: The interaction energy $2(D(\widetilde{\psi}_1), D(\widetilde{\psi}_2))_{L^2(\mathcal{F})}$. — By (162) and (157), for $2 \leqslant j \leqslant N$,

$$\int_{\partial\breve{\mathcal{S}}_j} \Sigma(\breve{\psi}_1, \breve{p}_1)n \cdot \breve{\psi}_2 = \int_{\partial\mathcal{S}_j} \Sigma(\widetilde{\psi}_1, \widetilde{p}_1)n \cdot \widetilde{\psi}_2.$$

On the other hand, by (173), (160) and (161), the chain rule and a change of variable (observe that the normal is also rotated),

$$\int_{\partial\breve{\mathcal{S}}_1} \Sigma(\breve{\psi}_1, \breve{p}_1)n \cdot \breve{\psi}_2 = \int_{\partial\mathcal{S}_1} \Sigma(\widetilde{\psi}_1, \widetilde{p}_1)n \cdot \widetilde{\psi}_2.$$

Therefore, by some integrations by parts, from the two previous identities, (155) and (174), we arrive at

$$\begin{aligned}
2(D(\breve{\psi}_1), D(\breve{\psi}_2))_{L^2(\breve{\mathcal{F}})} &= \sum_{j=1}^N \int_{\partial\breve{\mathcal{S}}_j} \Sigma(\breve{\psi}_1, \breve{p}_1)n \cdot \breve{\psi}_2 \\
&= \sum_{j=1}^N \int_{\partial\mathcal{S}_j} \Sigma(\widetilde{\psi}_1, \widetilde{p}_1)n \cdot \widetilde{\psi}_2 \\
&= 2(D(\widetilde{\psi}_1), D(\widetilde{\psi}_2))_{L^2(\mathcal{F})} + (g_1, \nabla\widetilde{\psi}_2)_{L^2(\mathcal{F})} + (f_1, \widetilde{\psi}_2)_{L^2(\mathcal{F})}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
2&(D(\check{\psi}_1), D(\check{\psi}_2))_{L^2(\check{\mathcal{F}})} - 2(D(\psi_1), D(\psi_2))_{L^2(\mathcal{F})} \\
&= 2(D(\widetilde{\psi}_1), D(\widetilde{\psi}_2))_{L^2(\mathcal{F})} - 2(D(\psi_1), D(\psi_2))_{L^2(\mathcal{F})} + (g_1, \nabla\widetilde{\psi}_2)_{L^2(\mathcal{F})} + (f_1, \widetilde{\psi}_2)_{L^2(\mathcal{F})} \\
&= 2(D(\widetilde{\psi}_1 - \psi_1), D(\widetilde{\psi}_2))_{L^2(\mathcal{F})} - 2(D(\psi_1), D(\psi_2 - \widetilde{\psi}_2))_{L^2(\mathcal{F})} \\
&\qquad\qquad + (g_1, \nabla\widetilde{\psi}_2)_{L^2(\mathcal{F})} + (f_1, \widetilde{\psi}_2)_{L^2(\mathcal{F})}.
\end{aligned}
$$

Thus, by the Cauchy-Schwarz inequality, the Hölder inequality and the Sobolev embedding of $\dot{H}^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, we obtain that

$$
\begin{aligned}
(180)\quad &|2(D(\check{\psi}_1), D(\check{\psi}_2))_{L^2(\check{\mathcal{F}})} - 2(D(\psi_1), D(\psi_2))_{L^2(\mathcal{F})}| \\
&\leqslant \|\widetilde{\psi}_1 - \psi_1\|_{\dot{H}^1(\mathcal{F})}\|\widetilde{\psi}_2\|_{\dot{H}^1(\mathcal{F})} + \|\widetilde{\psi}_1\|_{\dot{H}^1(\mathcal{F})}\|\widetilde{\psi}_2 - \psi_2\|_{\dot{H}^1(\mathcal{F})} \\
&\qquad + \big(\|g_1\|_{L^2(\mathcal{F})} + C\|f_1\|_{L^{6/5}(\mathcal{F})}\big)\|\widetilde{\psi}_2\|_{\dot{H}^1(\mathbb{R}^3)},
\end{aligned}
$$

recalling that g_1 and f_1 are compactly supported in $\mathcal{J} \subset \mathcal{F}$.

Step 6: Estimates of $\widetilde{\psi}_i$ and of $\widetilde{\psi}_i - \psi_i$, for $i = 1, 2$. — First, we decompose $\widetilde{\psi}_i$, for $i = 1, 2$, into

$$\widetilde{\psi}_i = w_i^I + w_i^B,$$

where w_i^I and w_i^B are the solutions to the following Stokes systems respectively corresponding to the interior source term and to the boundary data in (174):

$$
\begin{aligned}
-\Delta w_i^I + \nabla p_i^I = -\operatorname{div} g_i + f_i \quad &\text{and} \quad \operatorname{div} w_i^I = 0 \quad \text{in } \mathcal{F}, \\
w_i^I = 0 \quad &\text{in } \bigcup_{j=1}^N \mathcal{S}_j,
\end{aligned}
$$

and

$$
\begin{aligned}
-\Delta w_i^B + \nabla p_i^B = 0 \quad &\text{and} \quad \operatorname{div} w_i^B = 0 \quad \text{in } \mathcal{F}, \\
w_i^B(x) &= \check{Q}_1^T\varphi_i(\check{Q}_1 x + \check{h}_j) \quad \text{in } \mathcal{S}_1, \\
w_i^B(x) &= \varphi_i(x) \quad \text{in } \bigcup_{2\leqslant j\leqslant N} \mathcal{S}_j.
\end{aligned}
$$

On the one hand, by a straightforward energy estimate, we have that

$$\|w_i^I\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant \|g_1\|_{L^2(\mathcal{F})} + C\|f_1\|_{L^{6/5}(\mathcal{F})}.$$

On the other hand, by Theorem 4.1 we have that

$$\|w_i^B\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-1/2}\,\|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^3)}.$$

Thus, we arrive at

$$
(181)\qquad \|\widetilde{\psi}_i\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant C\,|\log\varepsilon|^{-1/2}\,\|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^3)} + \|g_i\|_{L^2(\mathcal{F})} + C\|f_i\|_{L^{6/5}(\mathcal{F})}.
$$

Similarly, for $i = 1, 2$, using (153) and (174), we decompose $\widetilde{\psi}_i - \psi_i$ into

$$\widetilde{\psi}_i - \psi_i = w_i^I + w_i^{B,\mathrm{diff}},$$

with, this time, the boundary term $w_i^{B,\mathrm{diff}}$ satisfying the following Stokes system:

$$\begin{aligned} -\Delta w_i^{B,\mathrm{diff}} + \nabla p_i^{B,\mathrm{diff}} = 0 \quad \text{and} \quad \operatorname{div} w_i^{B,\mathrm{diff}} = 0 \quad \text{in } \mathcal{F}, \\ w_i^{B,\mathrm{diff}}(x) = \check{Q}_1^T \varphi_i(\check{Q}_1 x + \check{h}_1) - \varphi_i(x) \quad \text{in } \mathcal{S}_1, \\ w_i^{B,\mathrm{diff}}(x) = 0 \quad \text{in } \bigcup_{2\leqslant j\leqslant N} \mathcal{S}_j. \end{aligned}$$

By Theorem 4.1 and (158), we have that

$$\|w_i^{B,\mathrm{diff}}\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant C\delta\, |\log\varepsilon|^{-1/2}\, \|\varphi_i\|_{W^{2,\infty}(\mathbb{R}^3)}.$$

Thus we obtain that

$$\|\widetilde{\psi}_i - \psi_i\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant C\delta\, |\log\varepsilon|^{-1/2}\, \|\varphi_i\|_{W^{2,\infty}(\mathbb{R}^3)} + C\|g_i\|_{L^2(\mathcal{F})} + C\|f_i\|_{L^{6/5}(\mathcal{F})}. \tag{182}$$

Step 7: Conclusion. — Gathering (180), (181), (182) and (179), we arrive at (156). This finishes the proof of Proposition 5.1. □

6. Asymptotic behaviour of the filament centerlines

This section is devoted to the proof of the part of Theorem 3.4 devoted to the asymptotic behaviour of the filament centerlines, that is to the proof of (53), (55) and of Theorem 6.5, which, as mentioned in the comments after Theorem 3.4, provides a more precise approximation of the asymptotic behaviour of the filament velocities than the one in Theorem 3.4, at the expense of ε-dependent positions. This section is divided into three subsections.

First Section 6.1 is devoted to reformulation of the Newton equations (19) into a system of second-order quasilinear ODEs on the $6N$ degrees of freedom of the rigid bodies, which does not involve the fluid pressure anymore and reveals the role played by the Stokes resistance matrices.

In Section 6.2 we consider the time evolution of a modulated energy which measures, for each positive ε, the difference between the filaments velocities for positive ε and the so-called "Faxén" velocities, which are given by the quasi-static balance of the Stokes resistance force and torque with the force and torque due to the background flow. The latter are a family of velocities which depend on the positions of the filaments velocities of ε-thickness. Unlike the total energy of the system considered in (25), this modulated energy has the advantage to circumvent the part of the energy corresponding to the motion of the filaments, under the influence of the fluid.

Finally in Section 6.3 we take advantage of the previous subsections to prove the part of Theorem 3.4 which concerns the filaments.

6.1. Reformulation of the Newton equations.

— This subsection is devoted to reformulation of the Newton equations into a compressed form which does not involve the fluid pressure anymore, and reveals the role played by the Stokes resistance matrices.

We introduce first a few notations. Let us emphasize that all quantities here are defined with respect to the filaments of ε-thickness at time t. Indeed the result below concerns the solutions $(h_{i,\varepsilon}, Q_{i,\varepsilon})_{1\leqslant i\leqslant N}$ to the Newton-Stokes system (22) up to the time $T_\varepsilon^{\max}$ as given by Proposition 2.2.

– Let us first gather all the translation and rotation velocities corresponding to the motions of the N filaments into the following vector of $\mathbb{R}^{6N}$:

$$\mathrm{Y} := \begin{pmatrix} \mathrm{v}_i \\ \omega_i \end{pmatrix}_{1\leqslant i\leqslant N} = (\mathrm{Y}_{j,\beta})_{1\leqslant j\leqslant N,\, 1\leqslant \beta\leqslant 6}. \tag{183}$$

– Similarly let

$$\mathrm{Y}^\flat := \begin{pmatrix} \mathrm{v}_i^\flat \\ \omega_i^\flat \end{pmatrix}_{1\leqslant i\leqslant N} \quad \text{such that} \quad \mathrm{Y}^\flat = \mathcal{K}^{-1}\mathfrak{f}^\flat, \tag{184}$$

recalling that $\mathcal{K}$ is the $6N \times 6N$ matrix defined in (73) and that $\mathfrak{f}^\flat$ is defined by (76). These are the so-called “Faxén” velocities. Let us observe that it follows from (75) and (80) that

$$|\mathrm{Y}^\flat| \leqslant C \tag{185}$$

with a constant depending only on $d_{\min}$ (see (65)) for all ε sufficiently small. We will also use the following notations, where on the one hand translation velocities are gathered, and on the other hand rotation velocities are gathered:

$$\mathrm{v}^\flat := (\mathrm{v}_i^\flat)_{1\leqslant i\leqslant N} \quad \text{and} \quad \omega^\flat := (\omega_i^\flat)_{1\leqslant i\leqslant N}. \tag{186}$$

– Let

$$\mathfrak{f}^a := \begin{pmatrix} F^a \\ T^a \end{pmatrix}, \tag{187}$$

where

$$F^a := (F_i^a)_{1\leqslant i\leqslant N} \quad \text{and} \quad T^a := (T_i^a)_{1\leqslant i\leqslant N},$$

with for $1 \leqslant i \leqslant N$,

$$F_i^a := \int_{\partial \mathcal{S}_i} \Sigma(u^\flat, p^\flat) n \, d\mathcal{H}^2 \quad \text{and} \quad T_i^a := \int_{\partial \mathcal{S}_i} (x - h_{i,\varepsilon}) \wedge \Sigma(u^\flat, p^\flat) n \, d\mathcal{H}^2.$$

The choice of the index “a” is for Archimedes, because, as one proceeds in the usual computation of gravity buoyancy, see [11, (4.18)] or [39, p. 105], one may use integration by parts inside the filaments to arrive at

$$F_i^a = \int_{\mathcal{S}_i} (\Delta u^\flat + \nabla p^\flat) \, dx \quad \text{and} \quad T_i^a = \int_{\mathcal{S}_i} (x - h_{i,\varepsilon}) \wedge (\Delta u^\flat + \nabla p^\flat) \, dx. \tag{188}$$

– Finally let us gather the inertia of the N filaments into the $6N \times 6N$ block diagonal matrix $\mathcal{M}$ whose 6×6 blocks are

$$\mathcal{M} := (m_i \operatorname{Id}_3, \mathcal{J}_i)_{1\leqslant i\leqslant N}. \tag{189}$$

We can now state the main result of this subsection.

Proposition 6.1. — *As long as the filaments are separated, the Newton equations* (22c)–(22d) *are equivalent to the following compressed form:*

$$\varepsilon^2 \frac{d}{dt}(\mathcal{M}\,\mathrm{Y}) = -\mathcal{K}(\mathrm{Y}-\mathrm{Y}^\flat) + \mathfrak{f}^a, \tag{190}$$

Let us highlight that $\mathcal{M}$, $\mathcal{K}$, $\mathrm{Y}^\flat$ and $\mathfrak{f}^a$ depend on the positions of the filaments, that is to the solution Y through its time antiderivative, so that the ODE (190) is quasilinear.

Proof. — Since the left hand sides of (22c)–(22d) clearly correspond to the left hand side of (190) according to the definitions (183) and (189), it is sufficient to consider the right hand sides of (22c)–(22d), which we decompose into

$$\int_{\partial \mathcal{S}_i} \Sigma(u,p)n\, d\mathcal{H}^2 = F_i^a + \int_{\partial \mathcal{S}_i} \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2, \tag{191a}$$

$$\int_{\partial \mathcal{S}_i} (x-h_{i,\varepsilon}) \wedge \Sigma(u,p)n\, d\mathcal{H}^2 = T_i^a + \int_{\partial \mathcal{S}_i} (x-h_{i,\varepsilon}) \wedge \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2. \tag{191b}$$

The second terms in the right hand sides of (191) can be computed as follows. For $1 \leqslant i \leqslant N$, by (71b),

$$\begin{aligned}\begin{pmatrix} \int_{\partial \mathcal{S}_i} \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2 \\ \int_{\partial \mathcal{S}_i} (x-h_{i,\varepsilon}) \wedge \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2 \end{pmatrix} &= \Big(\int_{\bigcup_{j=1}^N \partial \mathcal{S}_j} \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n \cdot V_{i,\alpha}\, d\mathcal{H}^2 \Big)_{1 \leqslant \alpha \leqslant 6} \\ &= \Big(\int_{\bigcup_{j=1}^N \partial \mathcal{S}_j} \Sigma(V_{i,\alpha}, P_{i,\alpha})n \cdot u^\mathfrak{p}\, d\mathcal{H}^2 \Big)_{1 \leqslant \alpha \leqslant 6},\end{aligned}$$

by Lorentz's reciprocity theorem, using that $u^\mathfrak{p}$ and $V_{i,\alpha}$ are both solutions of the steady Stokes system in $\mathcal{F}$, see (22f) and (71). Then, using the boundary condition (22g)–(22h) and the definition (26) of $v_{j,\beta}$, we have that in $\mathcal{S}_j$,

$$u^\mathfrak{p} = \Big(\sum_{1 \leqslant \beta \leqslant 6} \mathrm{Y}_{j,\beta}\; v_{j,\beta} \Big) - u^\flat. \tag{192}$$

We deduce that, for $1 \leqslant \alpha \leqslant 6$,

$$\begin{aligned}\int_{\bigcup_{j=1}^N \partial \mathcal{S}_j} \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n \cdot V_{i,\alpha}\, d\mathcal{H}^2 = \sum_{1 \leqslant j \leqslant N} \sum_{1 \leqslant \beta \leqslant 6} \mathrm{Y}_{j,\beta} \int_{\partial \mathcal{S}_j} \Sigma(V_{i,\alpha}, P_{i,\alpha})n \cdot v_{j,\beta}\, d\mathcal{H}^2 \\ - \int_{\bigcup_{j=1}^N \partial \mathcal{S}_j} \Sigma(V_{i,\alpha}, P_{i,\alpha})n \cdot u^\flat\, d\mathcal{H}^2.\end{aligned}$$

Thus,

$$\left(\begin{pmatrix} \int_{\partial \mathcal{S}_i} \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2 \\ \int_{\partial \mathcal{S}_i} (x-h_{i,\varepsilon}) \wedge \Sigma(u^\mathfrak{p}, p^\mathfrak{p})n\, d\mathcal{H}^2 \end{pmatrix} \right)_{1 \leqslant i \leqslant N} = -\mathcal{K}\,\mathrm{Y} + \mathfrak{f}^\flat. \tag{193}$$

Thus combining (193), (184) and (191) we find (190). □

One difficulty associated with the equation (190) is the factor ε^2 in front of the left hand side which makes the asymptotic analysis of this ordinary differential system belong to the class of singular perturbations, i.e., degeneracy at the main order.

However the matrix $\mathcal{K}$ is positive definite symmetric which guarantees that the effect of the associated term is to damp the velocities when time proceeds, or more exactly that they relax to the Faxén velocities. Indeed to tackle the asymptotic behaviour of the solutions to (190) one key point is the behaviour of Stokes' resistance matrix $\mathcal{K}$ with respect to ε, that is to quantify the damping effect in the limit of zero thickness.

6.2. Modulated energy and lifetime. — To estimate the relaxation of the exact solution Y of (190) to the time-dependent vector $\mathrm{Y}^\flat$ for small ε, we consider the modulated energy:

$$E := \frac{1}{2}\,(\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{M}(\mathrm{Y} - \mathrm{Y}^\flat). \tag{194}$$

Thanks to the assumptions on the inertia of the filaments in Section 2, the matrix $\mathcal{M}$ defined in (189) is symmetric positive definite uniformly in ε, and it is also uniformly bounded. Thus the modulated energy E is ε-uniformly equivalent to $|\,\mathrm{Y} - \mathrm{Y}^\flat\,|^2$.

As mentioned in Proposition 2.2, for each ε there is a positive time interval during which the filaments remain separated. Below we will perform some computations which are valid as long as the filaments remain well separated uniformly with respect to ε. By a bootstrap argument in Section 6.3 we then derive uniform estimates of this time with respect to ε and show that it extends until $\widehat{T}$ in the sense of Theorem 3.4.

More precisely, for $d > 0$, we define

$$T_{\varepsilon,d} := \inf\Big\{t \geqslant 0 : d_{\min}(t) > d, Z \leqslant \frac{c_d}{C_d \varepsilon^2\,|\log\varepsilon|}\Big\}, \tag{195}$$

where $Z := \sqrt{E}$ and $d_{\min}$ is the minimal distance between the centerlines as defined in (65). Since $\min_{i\neq j} \operatorname{dist}(\mathcal{S}_i, \mathcal{S}_j) \geqslant d_{\min} - C\varepsilon$, Proposition 2.2 implies that for $\varepsilon \leqslant \varepsilon_0(d)$ we have $T_{\varepsilon,d} < T_\varepsilon^{\max}$ and thus that the dynamics is well-posed on $(0, T_{\varepsilon,d})$. Note that $Z = \sqrt{E} \leqslant C|\,\mathrm{Y} - \mathrm{Y}^\flat\,|$. Thus, since Z is continuous, decreasing the value of $\varepsilon_0(d)$ if necessary, we have for all $\varepsilon \in (0, \varepsilon_0)$ that $T_{\varepsilon,d} > 0$. In the following we consider only $t < T_{\varepsilon,d}$.

Proposition 6.2. — *For all $d > 0$, there exists $C(d) > 0$ and $\varepsilon_0(d) > 0$ independent of ε such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $t \in (0, T_{\varepsilon,d})$,*

$$\big|\mathrm{Y}(t) - \mathrm{Y}^\flat(t)\big| \leqslant \big|\mathrm{Y}(0) - \mathrm{Y}^\flat(0)\big| e^{-c_d t/\varepsilon^2|\log\varepsilon|} + C_d \varepsilon^2 |\log\varepsilon|. \tag{196}$$

To prove Proposition 6.2 we will use the following lemma.

Lemma 6.3. — *For all $d > 0$, there exists $C(d) > 0$ and $\varepsilon_0(d) > 0$ independent of ε such that if the minimal distance $d_{\min}$ between the centerlines satisfies $d_{\min} \geqslant d$ and $\varepsilon \in (0, \varepsilon_0)$, we have the following estimate:*

$$|(\mathrm{Y}^\flat)'| \leqslant C(1 + |\mathrm{Y}|). \tag{197}$$

Proof of Lemma 6.3. — First, by (184),

$$(\mathrm{Y}^\flat)' = -\mathcal{K}^{-1}\mathcal{K}'\mathcal{K}^{-1}\,\mathfrak{f}^\flat + \mathcal{K}^{-1}\,(\mathfrak{f}^\flat)'. \tag{198}$$

Let $\overline{u}^\flat$ the solution to

$$-\Delta\overline{u}^\flat + \nabla\overline{p}^\flat = 0 \quad \text{and} \quad \operatorname{div}\overline{u}^\flat = 0 \quad \text{in } \mathcal{F}$$
$$\overline{u}^\flat = u^\flat \quad \text{in} \bigcup_{1\leqslant i\leqslant N} \mathcal{S}_i,$$

and let

$$\boldsymbol{v} := (V_{i,\alpha})_{1\leqslant i\leqslant N, 1\leqslant\alpha\leqslant 6},$$

where we recall that the $V_{i,\alpha}$ are the unique solutions to the steady Stokes equations associated with the rigid velocities in the filaments, see (71). Recalling the definition of $\mathfrak{f}^\flat$ in (76)–(77)–(78)–(79) we obtain by an integration by parts that

$$\mathfrak{f}^\flat = \Big((D(V_{i,\alpha}), D(\overline{u}^\flat))_{L^2(\mathcal{F})}\Big)_{1\leqslant i\leqslant N, 1\leqslant\alpha\leqslant 6},$$

where we recall that the notation $\big(\cdot,\cdot\big)_{L^2(\mathcal{F})}$ stands for the inner product in $L^2(\mathcal{F})$. By (75) and (76), we deduce that

$$|(\mathrm{Y}^\flat)'| \leqslant C\,|\log\varepsilon|^2\,|\mathcal{K}'|\|\overline{u}^\flat\|_{\dot H^1(\mathcal{F})}\|\boldsymbol{v}\|_{\dot H^1(\mathcal{F})} + C\,|\log\varepsilon|\,|(D(\boldsymbol{v}), D(\overline{u}^\flat))'_{L^2(\mathcal{F})}|. \tag{199}$$

Moreover, by (73) and an integration by parts, we have that

$$\mathcal{K} = \Big((D(V_{i,\alpha}), D(V_{j,\beta}))_{L^2(\mathcal{F})}\Big)_{1\leqslant i,j\leqslant N, 1\leqslant\alpha,\beta\leqslant 6}.$$

Thus, with the convention that the terms containing $\boldsymbol{v}$ are the sum for $1\leqslant i\leqslant N$ and $1\leqslant\alpha\leqslant 6$, of corresponding terms for $V_{i,\alpha}$, and by Theorem 4.1, we have

$$\|\boldsymbol{v}\|_{\dot H^1(\mathcal{F})} + \|\overline{u}^\flat\|_{\dot H^1(\mathcal{F})} \leqslant C\,|\log\varepsilon|^{-1/2},$$

so that we arrive at

$$|(\mathrm{Y}^\flat)'| \leqslant C\,|\log\varepsilon|\,\Big(\big|\big(D(\boldsymbol{v}), D(\boldsymbol{v})\big)'_{L^2(\mathcal{F})}\big| + \big|\big(D(\boldsymbol{v}), D(\overline{u}^\flat)\big)'_{L^2(\mathcal{F})}\big|\Big). \tag{200}$$

We decompose the time derivative in the terms on the right hand side of (200) into several contributions. To this end, we introduce the operator

$$G\colon W^{2,\infty}_\sigma(\mathbb{R}^3)\times W^{2,\infty}_\sigma(\mathbb{R}^3) \longrightarrow \mathbb{R},$$

defined by

$$G(\varphi_1,\varphi_2) := (D(\psi_1), D(\psi_2))_{L^2(\mathcal{F})},$$

where ψ_k, for $k=1,2$, is the solution in $\dot H^1(\mathcal{F})$ to the problem

$$-\Delta\psi_k + \nabla p_k = 0, \quad \operatorname{div}\psi_k = 0 \quad \text{in } \mathcal{F},$$
$$\psi_k = \varphi_i \quad \text{in} \bigcup_i \mathcal{S}_i.$$

Then, for any $1\leqslant i\leqslant N$, for any $1\leqslant\alpha\leqslant 6$,

$$(D(V_{i,\alpha}), D(\overline{u}^\flat))_{L^2(\mathcal{F})} = G(v_{i,\alpha}, u^\flat).$$

Note that the operator G as well as $v_{i,\alpha}$ implicitly depends on the positions and orientations of the particles. Consequently,

$$(D(V_{i,\alpha}), D(\overline{u}^\flat))'_{L^2(\mathcal{F})} = \mathrm{Y}\cdot\nabla_{h,Q}G(v_{i,\alpha}, u^\flat) + G(\mathrm{Y}\cdot\nabla_{h,Q}v_{i,\alpha}, u^\flat) + G(v_{i,\alpha}, \partial_t u^\flat).$$

Theorem 4.1 implies

$$|G(\varphi_1, \varphi_2)| \leqslant C\, |\log \varepsilon|^{-1}\, \|\varphi_1\|_{W^{1,\infty}} \|\varphi_2\|_{W^{1,\infty}}.$$

Thus, taking into account the regularity assumptions on the background flow $u^\flat$, see (17), we arrive at

$$|(D(V_{i,\alpha}), D(\overline{u}^\flat))'_{L^2(\mathcal{F})}| \leqslant |\mathrm{Y}||\nabla_{h,Q} G(v_{i,\alpha}, u^\flat)| + C\, |\log \varepsilon|^{-1}\, (1 + |\mathrm{Y}|).$$

Analogous considerations hold for the term $\big|\big(D(\boldsymbol{v}), D(\boldsymbol{v})\big)'_{L^2(\mathcal{F})}\big|$. Proposition 5.1 applied to both $(V_{i,\alpha}, \overline{u}^\flat)$ and to $(V_{i,\alpha}, V_{j,\beta})$ leads to the result. $\square$

With the result of Lemma 6.3 in hands we can now start the proof of Proposition 6.2.

Proof of Proposition 6.2. — We first recast (190) as

$$\varepsilon^2(\mathcal{M}(\mathrm{Y} - \mathrm{Y}^\flat))' = -\mathcal{K}(\mathrm{Y} - \mathrm{Y}^\flat) + \mathfrak{f}^a - \varepsilon^2(\mathcal{M}\,\mathrm{Y}^\flat)',$$

and then take the inner product with $\mathrm{Y} - \mathrm{Y}^\flat$, with the observation that

$$(\mathrm{Y} - \mathrm{Y}^\flat) \cdot \big(\mathcal{M}(\mathrm{Y} - \mathrm{Y}^\flat)\big)' = E' + \frac{1}{2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{M}'(\mathrm{Y} - \mathrm{Y}^\flat),$$

so that

$$\begin{aligned} E' = -\varepsilon^{-2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{K}(\mathrm{Y} - \mathrm{Y}^\flat) + \varepsilon^{-2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathfrak{f}^a \\ - (\mathcal{M}\,\mathrm{Y}^\flat)' \cdot (\mathrm{Y} - \mathrm{Y}^\flat) - \frac{1}{2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{M}'(\mathrm{Y} - \mathrm{Y}^\flat). \end{aligned}$$

Recalling the definition of $\mathcal{M}$ in (189), we arrive at the following formula for the time derivative E' of the modulated energy:

$$\begin{aligned} E' = -\varepsilon^{-2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{K}(\mathrm{Y} - \mathrm{Y}^\flat) + \varepsilon^{-2}\, (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathfrak{f}^a \\ - (\mathrm{Y} - \mathrm{Y}^\flat) \cdot \mathcal{M}(\mathrm{Y}^\flat)' - (\omega - \omega^\flat) \cdot \mathcal{J}'\omega^\flat - \frac{1}{2}(\omega - \omega^\flat) \cdot \mathcal{J}'(\omega - \omega^\flat), \end{aligned} \tag{201}$$

where $\mathcal{J}$ is the $3N \times 3N$ block diagonal matrix whose 3×3 blocks are $\mathcal{J}_i$ for $1 \leqslant i \leqslant N$.

By Corollary 4.2, the first term of the right hand side of (201) can be bounded by $-c\varepsilon^{-2}\, |\log \varepsilon|^{-1}\, E$ for some constant c which is positive and uniform with respect to ε, and will possibly change from line to line, while still satisfying these properties.

By (188), since the background flow is assumed to be smooth, the term $\mathfrak{f}^a$ can be bounded by ε^2. Therefore the second term of the right hand side of (201) can be bounded by $C\sqrt{E}$, where the constant C is also positive and uniform with respect to ε, and will also possibly change from line to line, while still satisfying these properties.

Similarly the last three terms of the right hand side of (201) can be respectively bounded by $C\sqrt{E}|(\mathrm{Y}^\flat)'|$, $C\sqrt{E}|\mathcal{J}'|$ and $CE|\mathcal{J}'|$.

Thus

$$E' + c\varepsilon^{-2}|\log \varepsilon|^{-1}\, E \leqslant C\sqrt{E}\big(1 + |(\mathrm{Y}^\flat)'| + |\mathcal{J}'|\big) + CE|\mathcal{J}'|. \tag{202}$$

Regarding $\mathcal{J}'$, we use (22e) and $|\mathcal{J}_{0,i}| \leqslant C$ to deduce

$$|\mathcal{J}'| \leqslant C|Q'| \leqslant C|\omega| \leqslant C|\mathrm{Y}|. \tag{203}$$

Combining this with the estimate for $(\mathrm{Y}^\flat)'$ from Lemma 6.3 in the energy estimate (202) yields

$$E' \leqslant -\frac{c}{\varepsilon^2\,|\log\varepsilon|}\,E + C\sqrt{E}\,(1+|\mathrm{Y}|) + CE|\mathrm{Y}|. \tag{204}$$

Since $|\mathrm{Y}| \leqslant C(\sqrt{E}+1)$, and using the uniform bound on $\mathrm{Y}^\flat$ from (185), we arrive at

$$E' \leqslant -\frac{c}{\varepsilon^2\,|\log\varepsilon|}\,E + C\sqrt{E}\,(1+E) \tag{205}$$

and for $Z=\sqrt{E}$

$$Z' \leqslant -\frac{c}{\varepsilon^2\,|\log\varepsilon|}\,Z + C(1+Z^2). \tag{206}$$

Recall that the constants depend on the minimal distance between the particles $d_{\min}(t)$ (see (65)). More precisely, if $d_{\min}(t)\geqslant d$, then

$$Z' \leqslant -\frac{c_d}{\varepsilon^2\,|\log\varepsilon|}\,Z + C_d(1+Z^2), \tag{207}$$

for all $\varepsilon \leqslant \varepsilon_0(d)$.

By definition of $T_{\varepsilon,d}$ in (195), we find that on $(0,T_{\varepsilon,d})$

$$Z' \leqslant -\frac{c_d}{2\varepsilon^2\,|\log\varepsilon|}\,Z + C_d. \tag{208}$$

By Gronwall's inequality, and recalling that the modulated energy E is ε-uniformly equivalent to $|\,\mathrm{Y}-\mathrm{Y}^\flat\,|^2$, we obtain that (196) holds for all $t\in(0,T_{\varepsilon,d})$, up to an adaptation of the constants c_d and C_d. $\square$

Below, in Section 6.3, we will prove the following result on the asymptotic behaviour of $T_{\varepsilon,d}$ as ε converges to 0.

To this end, let us recall the definition $\widehat{d}_{\min} := \inf_{i\neq j}\operatorname{dist}(\widehat{\mathcal{C}}_i,\widehat{\mathcal{C}}_j)$ from (41) and that $\widehat{T}$ from (42) is the maximal time for which $\widehat{d}_{\min}$ stays positive.

Proposition 6.4. — *There is $\varepsilon_0>0$ small enough which depends only on the reference filaments, $u^\flat$ and $\min_{t\leqslant T}\widehat{d}_{\min}(t)$, such that for all T in $(0,\widehat{T})$, for $d=\frac{1}{4}\min_{t\leqslant T}\widehat{d}_{\min}(t)$ and for all $\varepsilon<\varepsilon_0$*

$$T_{\varepsilon,d} > T. \tag{209}$$

Let us already observe that combining Proposition 6.4 and Proposition 6.2 we obtain the following result.

Theorem 6.5. — *Under the same assumptions as in Theorem 3.4 we have on the one hand the estimate (52) on the lifetime and on the other hand, for all $T<\widehat{T}$ there exists C depending only on $u^\flat$, on the reference filaments $\overline{\mathcal{S}}_i$, on $\inf_{t\in[0,T]}\widehat{d}_{\min}(t)$ and on the initial velocities, and there exists $\varepsilon_0>0$ depending in addition on T such that for all ε in $(0,\varepsilon_0)$ and all t in $[0,T]$ the difference between the solution (h',ω) to (22) and the "Faxén's" velocities $(v^\flat,\omega^\flat)$ defined in (184) satisfies*

$$\big|(h',\omega)(t)-(v^\flat,\omega^\flat)(t)\big| \leqslant \big|(h',\omega)(0)-(v^\flat,\omega^\flat)(0)\big|\,e^{-Ct/\varepsilon^2|\log\varepsilon|} + C\varepsilon^2|\log\varepsilon|. \tag{210}$$

Indeed, as we emphasized in Section 3.4, this theorem provides a more precise approximation than Theorem 3.4, at the expense of ε-dependent positions and implicit forces.

6.3. Proof of the part of Theorem 3.4 which concerns the filaments. — We now turn to the proofs of (53) and of (55). In particular we are going to prove the convergence of the filament positions given by the time dependent vector Y defined in (183) to the limit dynamics for which we use the notation

$$\widehat{\mathrm{Y}}(t) := \widehat{\mathcal{K}}^{-1}(\widehat{h}(t), \widehat{Q}(t))\widehat{\mathfrak{f}}^{\flat}(t, \widehat{h}(t), \widehat{Q}(t)), \tag{211}$$

where $\widehat{\mathcal{K}}$ is the $6N \times 6N$ matrix whose 6×6 diagonal blocks are the $\widehat{\mathcal{K}}_{i,i}$, for $1 \leqslant i \leqslant N$, are defined in (34), and proved to be invertible in Lemma 3.1, and $\widehat{\mathfrak{f}}^{\flat}$ is the vector in $\mathbb{R}^{6N}$ which gathers the vectors $\widehat{\mathfrak{f}}_i^{\flat}$, for $1 \leqslant i \leqslant N$, defined in (35). It follows from (39) that

$$\widehat{\mathrm{Y}}(t) := (\widehat{\mathrm{v}}_i(t), \widehat{\omega}_i(t))_{1\leqslant i\leqslant N}. \tag{212}$$

From the estimates in the previous subsection we already know that the velocities Y and $\mathrm{Y}^{\flat}$ are close as long as the filaments are well separated. We now introduce

$$\widetilde{\mathrm{Y}}(t) := \widehat{\mathcal{K}}^{-1}(h_\varepsilon(t), Q_\varepsilon(t))\widehat{\mathfrak{f}}^{\flat}(t, h_\varepsilon(t), Q_\varepsilon(t)). \tag{213}$$

The velocities $\widetilde{\mathrm{Y}}$ correspond to the limit dynamics but with the positions of the filaments given by the ε-dynamics rather than the limit dynamics. In this sense, $\widetilde{\mathrm{Y}}$ can be seen as intermediate between $\mathrm{Y}^{\flat}$ and $\widehat{\mathrm{Y}}$. Next lemma takes benefit from the previous estimates of $\mathrm{Y}-\mathrm{Y}^{\flat}$ to establish some estimates of $\widetilde{\mathrm{Y}}-\mathrm{Y}^{\flat}$ as long as the filaments are well separated.

Lemma 6.6. — *For all $d > 0$ there exists a constant $C(d) > 0$ and $\varepsilon_0(d) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $d_{\min} \geqslant d$,*

$$|\widetilde{\mathrm{Y}} - \mathrm{Y}^{\flat}| \leqslant C\, |\log\varepsilon|^{-1/2}\, \|u^{\flat}\|_{W^{1,\infty}(\mathbb{R}^3)}. \tag{214}$$

Proof. — Recalling the definition of $\mathrm{Y}^{\flat}$ in (184) and the one of $\widetilde{\mathrm{Y}}$ above, we observe that

$$\mathrm{Y}^{\flat} - \widetilde{\mathrm{Y}} = \mathcal{K}^{-1}(\mathfrak{f}^{\flat} - |\log\varepsilon|^{-1}\widehat{\mathfrak{f}}) - \mathcal{K}^{-1}(\mathcal{K} - |\log\varepsilon|^{-1}\widehat{\mathcal{K}})\widehat{\mathcal{K}}^{-1}\widehat{\mathfrak{f}},$$

where $\widehat{K}$ and $\widehat{f}$ should be understood as being evaluated at $(h_\varepsilon, Q_\varepsilon)$. Combining (74), (75), (80) and observing that $|\widehat{\mathcal{K}}^{-1}|+|\widehat{\mathfrak{f}}| \leqslant C$, we conclude the proof of Lemma 6.6. $\square$

We now turn to the proof that (53) and (55) holds on $[0, T_{\varepsilon,d}]$, where $T_{\varepsilon,d}$ is defined by (195), that is in particular to to the estimate of $\mathrm{Y} - \widehat{\mathrm{Y}}$.

Proposition 6.7. — *For all $d > 0$, there exists $C(d) > 0$ and $\varepsilon_0(d) > 0$ independent of ε such that for all $\varepsilon \in (0, \varepsilon_0)$, the estimates* (53) *and* (55) *hold on* $[0, T_{\varepsilon,d}]$.

Proof. — First we recall that the coefficients of (40) are smooth and globally Lipschitz, so that recalling (211) and (212), we infer that

$$|\widetilde{\mathrm{Y}} - \widehat{\mathrm{Y}}| \leqslant C|(h_\varepsilon, Q_\varepsilon) - (\widehat{h}, \widehat{Q})|. \tag{215}$$

Recalling (10) and (8), we also have that the initial data $(h(0), Q(0))$ and $(\widehat{h}(0), \widehat{Q}(0))$ satisfy

$$|(h(0), Q(0)) - (\widehat{h}(0), \widehat{Q}(0))| \leqslant C\varepsilon, \tag{216}$$

Thus, using (22a), (22b) and (39), by a combination of (196), (215) and Lemma 6.6 we obtain that for all $t \leqslant T_{\varepsilon,d}$,

$$\begin{aligned}
&|(h_\varepsilon(t), Q_\varepsilon(t)) - (\widehat{h}(t), \widehat{Q}(t))| \\
&\qquad \leqslant C\varepsilon + \int_0^t |\mathrm{Y} - \mathrm{Y}^\flat| + |\mathrm{Y}^\flat - \widetilde{\mathrm{Y}}| + |\widetilde{\mathrm{Y}} - \widehat{\mathrm{Y}}|\, ds \\
&\qquad \leqslant C\varepsilon + \int_0^t \big(|\mathrm{Y}(0) - \mathrm{Y}^\flat(0)| e^{-c_d s/\varepsilon^2 |\log\varepsilon|} + C_d \varepsilon^2 |\log\varepsilon| + C_d |\log\varepsilon|^{-1/2}\big)\, ds \\
&\qquad\qquad + C\int_0^t |(h_\varepsilon(s), Q_\varepsilon(s)) - (\widehat{h}(s), \widehat{Q}(s))|\, ds \\
&\qquad \leqslant C\varepsilon + C_d |\log\varepsilon|^{-1/2} t + C\int_0^t |(h_\varepsilon(s), Q_\varepsilon(s)) - (\widehat{h}(s), \widehat{Q}(s))|\, ds.
\end{aligned}$$

Here, we used the bound (185) on $\mathrm{Y}^\flat$ and the constant in the last line depends on the initial velocities $\mathrm{Y}(0)$. By Gronwall's estimate, we deduce that (53) holds for all $t \leqslant T_{\varepsilon,d}$. We also note that, bookkeeping the computations above, this allows to prove the following bound on the velocities:

$$|\mathrm{Y} - \widehat{\mathrm{Y}}| \leqslant C_d e^{-c_d t/\varepsilon^2 |\log\varepsilon|} |\mathrm{Y}(0) - \mathrm{Y}^\flat(0)| + C_d |\log\varepsilon|^{-1/2} + C_d |\log\varepsilon|^{-1/2} t e^{Ct}. \tag{217}$$

Thus, estimating

$$|\mathrm{Y}(0) - \mathrm{Y}^\flat(0)| \leqslant |\mathrm{Y}(0) - \widehat{\mathrm{Y}}(0)| + |\widehat{\mathrm{Y}}(0) - \widetilde{\mathrm{Y}}(0)| + |\widetilde{\mathrm{Y}}(0) - \mathrm{Y}^\flat(0)|, \tag{218}$$

and applying again (215) and Lemma 6.6 yields (55) on $[0, T_{\varepsilon,d}]$. □

We now turn to the proof of Proposition 6.4.

Proof of Proposition 6.4. — Let $T < \widehat{T}$ and $d = \frac{1}{4} \min_{t \leqslant T} \widehat{d}_{\min}(t)$. We first observe that (196) implies $T_{\varepsilon,d} = \infty$ or $d_{\min}(T_{\varepsilon,d}) = d$. Moreover, by (53), we have on $(0, T_{\varepsilon,d})$,

$$d_{\min}(t) \geqslant \widehat{d}_{\min}(t) - C\varepsilon - C_d(\varepsilon + |\log\varepsilon|^{-1/2} t) e^{Ct}, \tag{219}$$

where the term $C\varepsilon$ accounts for the filaments' thickness. Thus, the choice $d = \frac{1}{4} \min_{t \leqslant T} \widehat{d}_{\min}(t)$ implies for ε sufficiently small (depending on d, T, $u^\flat$ and the reference filaments), $d_{\min}(t) \geqslant 2d$ for all $t \leqslant \min\{T_{\varepsilon,d}, T\}$. Since $T_{\varepsilon,d} = \infty$ or $d_{\min}(T_{\varepsilon,d}) = d$ this implies $T_{\varepsilon,d} > T$. This concludes the proof of (53) and (55) on $[0, T]$. This completes the proof of Proposition 6.4. □

7. Asymptotic behaviour of the fluid

This section is devoted to the proof of the part of Theorem 3.4 devoted to the asymptotic behaviour of the fluid, that is to the proof of (54), together with the proof of (56). To this aim we first decompose $u^{\mathfrak{p}}$ into

$$u^{\mathfrak{p}} = \sum_{1\leqslant i\leqslant N} u_i^{\mathfrak{p}} \quad \text{and} \quad p^{\mathfrak{p}} = \sum_{1\leqslant i\leqslant N} p_i^{\mathfrak{p}}, \tag{220}$$

where

$$-\Delta u_i^{\mathfrak{p}} + \nabla p_i^{\mathfrak{p}} = 0 \quad \text{and} \quad \operatorname{div} u_i^{\mathfrak{p}} = 0 \quad \text{in } \mathcal{F}(t), \tag{221a}$$

$$u_i^{\mathfrak{p}} = v^{\mathcal{S}_i} - u^{\flat} \quad \text{for } x \in \mathcal{S}_i(t), \tag{221b}$$

$$u_i^{\mathfrak{p}} = 0 \quad \text{for } x \in \mathcal{S}_j(t), \quad \text{for } j \neq i. \tag{221c}$$

Then, we apply Theorem 4.1 to $u_i^{\mathfrak{p}}$, for each i, recalling that in $\mathcal{S}_i$,

$$v^{\mathcal{S}_i} = \sum_{1\leqslant \beta\leqslant 6} \mathrm{Y}_{i,\beta}\, v_{i,\beta},$$

to obtain the following proposition.

Proposition 7.1. — *Let $d > 0$. Then there exists $\varepsilon_0(d) > 0$ and $C(d) < \infty$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, T_\varepsilon^{\max}]$ with $d_{\min}(t) \geqslant d$*

$$\begin{aligned}\|u^{\mathfrak{p}}(t,\cdot) - |\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i(t)}[v^{\mathcal{S}_i(t)} - u^{\flat}(t,\cdot)]\|_{\dot H^1(\mathbb{R}^3 \smallsetminus \bigcup_i \mathcal{S}_i(t))} \\ \leqslant C\,|\log\varepsilon|^{-1} \big(|\mathrm{Y}(t)| + \|u^{\flat}(t,\cdot)\|_{W^{1,\infty}}\big).\end{aligned} \tag{222}$$

Moreover, for $1 \leqslant q < 3/2$,

$$\begin{aligned}\|u^{\mathfrak{p}}(t,\cdot) - |\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i(t)}[v^{\mathcal{S}_i(t)} - u^{\flat}(t,\cdot)]\|_{W^{1,q}_{\mathrm{loc}}} \\ \leqslant C\,|\log\varepsilon|^{-3/2} \big(|\mathrm{Y}(t)| + \|u^{\flat}(t,\cdot)\|_{W^{1,\infty}}\big).\end{aligned} \tag{223}$$

Furthermore, for $1 \leqslant p < 3$,

$$\|u^{\mathfrak{p}} - |\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i}[v^{\mathcal{S}_i} - u^{\flat}]\|_{L^p_{\mathrm{loc}}} \leqslant C\,|\log\varepsilon|^{-3/2} \big(|\mathrm{Y}| + \|u^{\flat}\|_{W^{1,\infty}}\big), \tag{224}$$

and, for $3 \leqslant p < 6$

$$\begin{aligned}\|u^{\mathfrak{p}} - |\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i}[v^{\mathcal{S}_i} - u^{\flat}]\|_{L^p_{\mathrm{loc}}} \\ \leqslant C\,|\log\varepsilon|^{-1+3/p-1/2-\delta} \big(|\mathrm{Y}(t)| + \|u^{\flat}\|_{W^{1,\infty}}\big).\end{aligned} \tag{225}$$

Proof. — Estimates (222) and (223) follow immediately from Theorem 4.1. Moreover, (224) follows from (223) and Sobolev embedding.

Concerning (225) for $3 \leqslant p < 6$, we combine (69) and (70) instead of just relying on (70). More precisely, combining (69) and (70) with Hölder' inequality yields for all $3/2 \leqslant q < 2$ and all $\delta > 0$

$$\|u^{\mathfrak{p}} - |\log\varepsilon|^{-1} \sum_{1\leqslant i\leqslant N} U_{\mathcal{C}_i}[v^{\mathcal{S}_i} - u^{\flat}]\|_{W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3 \smallsetminus \bigcup_i \mathcal{S}_i)} \tag{226}$$

$$\leqslant C\,|\log\varepsilon|^{-1-3/p+1/2+\delta} \big(|\mathrm{Y}(t)| + \|u^{\flat}\|_{W^{1,\infty}}\big). \tag{227}$$

Since the H^1-estimate in (69) excludes the sets occupied by the filaments, we had to exclude it in the above estimate. However, the pointwise estimates (67) imply that for all $q<2$

$$\|u^{\mathfrak{p}}-|\log\varepsilon|^{-1}\sum_{1\leqslant i\leqslant N}U_{\mathcal{C}_i}[v^{\mathcal{S}_i}-u^\flat]\|_{L^p(\bigcup_i\mathcal{S}_i)} \leqslant |\log\varepsilon|^{-1}\,\varepsilon^{2-q}\big(|\mathrm{Y}(t)|+\|u^\flat\|_{W^{1,\infty}}\big). \tag{228}$$

Combining (226) and (228) yields (225). ☐

In order to conclude that (54) and (56) hold true, we show the following result.

Lemma 7.2. — *Let $1\leqslant i\leqslant N$ and denote*

$$w:=U_{\mathcal{C}_i}[v^{\mathcal{S}_i}-u^\flat]-U_{\widehat{\mathcal{C}}_i}[\widehat{v}^{\mathcal{S}_i}(t,\cdot)-u^\flat(t,\cdot)]. \tag{229}$$

Then, there exists $\varepsilon_0>0$ such that for all ε in $(0,\varepsilon_0)$, for all $t\in[0,T_\varepsilon^{\max}]$,

– for all $1\leqslant p<2$, and all compact subset K of $\mathbb{R}^3$, there exists C in $(0,+\infty)$ such that

$$\|w\|_{L^p(K)} \leqslant C|(Q_{\varepsilon,i},h_{i,\varepsilon}+\overline{h}_{i,\varepsilon})-(\widehat{Q}_i,\widehat{h}_i)|\big(|\mathrm{Y}|+\|u^\flat\|_{W^{1,\infty}}\big)+C|(v,\omega)-(\widehat{v},\widehat{\omega})|. \tag{230}$$

– for $2\leqslant p<6$, and all compact subset K of $\mathbb{R}^3$, there exists C in $(0,+\infty)$ such that

$$\|w\|_{L^p(K)}\leqslant C|(Q_{\varepsilon,i},h_{i,\varepsilon}+\overline{h}_{i,\varepsilon})-(\widehat{Q}_i,\widehat{h}_i)|^{3/p-1/2-\delta}\big(|\mathrm{Y}|+|\widehat{\mathrm{Y}}|+\|u^\flat\|_{W^{1,\infty}}\big) + C|(v,\omega)-(\widehat{v},\widehat{\omega})|. \tag{231}$$

Before proving Lemma 7.2, we show how to deduce (54) and (56).

Proof of (54) *and* (56). — Let $T<\widehat{T}$ and $d=\frac14\min_{t\in[0,T]}\widehat{d}_{\min}(t)$. Then, by Proposition 6.4, for ε sufficiently small, we have $T_\varepsilon^{\max}\geqslant T$ and $d_{\min}\geqslant d$ on $[0,T]$. Therefore, Proposition 7.1 and Lemma 7.2, as well as (53) and (55) yields (54) and (56). ☐

Proof of Lemma 7.2. — We first observe that w satisfies

$$w(x)=\frac12\int_{\mathcal{C}_i}S(x-y)k(y)(v^{\mathcal{S}_i}-u^\flat)(y)\,d\mathcal{H}^1(y) -\frac12\int_{\widehat{\mathcal{C}}_i}S(x-y)k(y)(\widehat{v}^{\mathcal{S}_i}-u^\flat)(y)\,d\mathcal{H}^1(y)=:w_1+w_2,$$

where

$$w_1(x):=\frac12\int_{\mathcal{C}_i}S(x-y)k(y)(v^{\mathcal{S}_i}-u^\flat)(y)\,d\mathcal{H}^1(y) -\frac12\int_{\widehat{\mathcal{C}}_i}S(x-y)Qk(\phi(y))(v^{\mathcal{S}_i}-u^\flat)(\phi(y))\,d\mathcal{H}^1(y),$$

and

$$w_2(x):=\frac12\int_{\widehat{\mathcal{C}}_i}S(x-y)\Big(Qk(\phi(y))(v^{\mathcal{S}_i}-u^\flat)(\phi(y))-k(y)(\widehat{v}^{\mathcal{S}_i}-u^\flat)(y)\Big)\,d\mathcal{H}^1(y),$$

where

$$Q:=Q_{i,\varepsilon}\widehat{Q}_i^T \text{ and } \phi(x):=Q(x-\widehat{h}_i)+h_{i,\varepsilon}+\overline{h}_{i,\varepsilon}$$

is the rigid body motion that transforms $\widehat{\mathcal{C}}_i$ to $\mathcal{C}_i$. Then, using $S(Q^T x) = Q^T S(x) Q$, we find

$$\begin{aligned} w_1(x) = \frac{1}{2}\int_{\mathcal{C}_i} & S(x-y)k(y)(v^{\mathcal{S}_i} - u^\flat)(y)\, d\mathcal{H}^1(y) \\ & - \frac{1}{2} Q^T \int_{\mathcal{C}_i} S(\phi(x)-y)k(y)(v^{\mathcal{S}_i} - u^\flat)(y)\, d\mathcal{H}^1(y) \\ = {} & U_{\mathcal{C}_i}[v^{\mathcal{S}_i} - u^\flat](x) - Q^T U_{\mathcal{C}_i}[v^{\mathcal{S}_i} - u^\flat](\phi(x)). \end{aligned}$$

Using the fundamental theorem of calculus, we observe that for any $\psi \in W^{1,s}_{\mathrm{loc}}$, and for s in $[1,\infty]$,

$$\|\psi - \psi\circ\phi\|_{L^s_{\mathrm{loc}}} \leqslant C|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)| \|\nabla\psi\|_{L^s_{\mathrm{loc}}}. \tag{232}$$

Hence, for $p < 2$, by recalling (88), we infer that

$$\|w_1\|_{L^p_{\mathrm{loc}}} \leqslant C|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)| \|v^{\mathcal{S}_i} - u^\flat\|_{L^\infty(\mathcal{C}_i)}. \tag{233}$$

On the other hand, pointwise bounds analogous to (66) imply for all $p < \infty$

$$\begin{aligned} & \|w_2\|_{L^p_{\mathrm{loc}}} \\ & \leqslant C|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)| \Big(\|v^{\mathcal{S}_i} - u^\flat\|_{L^\infty(\mathcal{C}_i)} + \|\nabla(v^{\mathcal{S}_i} - u^\flat)\|_{L^\infty(\mathbb{R}^3)} \Big) \\ & \qquad + C|(v,\omega) - (\widehat{v}, \widehat{\omega})|. \end{aligned} \tag{234}$$

Combining (233) and (234) yields (230).

Finally, we give the proof of (231). Notice that in the case $|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)| \geqslant 1$, the estimate follows from (88) and the Sobolev embedding by estimating the two terms separately. Thus, since (234) holds for all $p < \infty$, it suffices to show for $2 \leqslant p < 6$ and for $|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)| \leqslant 1$,

$$\|w_1\|_{L^p_{\mathrm{loc}}} \leqslant C|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)|^{3/p - 1/2 - \delta} \|v^{\mathcal{S}_i} - u^\flat\|_{L^\infty(\mathcal{C}_i)} \tag{235}$$

With ψ as above, the critical Sobolev inequality yields for $s < 3$ and $s^\star = (3s)/(3-s)$

$$\|\psi - \psi\circ\phi\|_{L^{s^\star}_{\mathrm{loc}}} \leqslant \|\nabla\psi\|_{L^s_{\mathrm{loc}}}. \tag{236}$$

Therefore, for any $p < 6$, $s < 2$, $\theta \in [0,1]$ such that

$$\frac{1}{p} = \frac{\theta}{s} + \frac{1-\theta}{s^\star}, \tag{237}$$

Hölder's inequality, (232) and (236) yield

$$\|\psi - \psi\circ\phi\|_{L^p_{\mathrm{loc}}} \leqslant |(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)|^\theta \|\nabla\psi\|_{L^s_{\mathrm{loc}}}. \tag{238}$$

Elementary calculations show that for all $2 \leqslant p < 6$ and any $\delta > 0$ we can choose $s < 2$, such that (237) holds with

$$\theta = \frac{3}{p} - \frac{1}{2} - \delta. \tag{239}$$

Hence,

$$\|w_1\|_{L^p_{\mathrm{loc}}} \leqslant C|(Q_{\varepsilon,i}, h_{i,\varepsilon} + \overline{h}_{i,\varepsilon}) - (\widehat{Q}_i, \widehat{h}_i)|^{3/p - 1/2 - \delta} \|v^{\mathcal{S}_i} - u^\flat\|_{L^\infty(\mathcal{C}_i)}. \tag{240}$$

This concludes the proof of Lemma 7.2. □

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