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Abstract

The Schrödinger map equation is a geometric Schrödinger model, closely associated to the harmonic heat flow and to the wave map equation. The aim of these notes is to describe recent and ongoing work on this model, as well as a number of related open problems.

1. Geometric pde's

The Schrödinger map equation belongs to a larger class of nonlinear pde which are often referred to as geometric pde's. The key difference, compared to the usual semilinear pde models, one instead considers evolutions where the state space consists of functions which take values into a Riemannian manifold (M, g). Before focusing on Schrödinger maps, it is useful to describe several related models.

1.1. Harmonic maps in $\mathbb{R}^n$

These are maps

$$\phi : \mathbb{R}^n \to (M, g)$$

The derivatives of ϕ are tangent vectors,

$$\partial_j \phi(x) \in T_{\phi(x)} M.$$

Inspired by the Lagrangian interpretation of the Laplace equations, to such maps we associate the elliptic Lagrangian

$$L^e(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} \langle \partial_j \phi, \partial_j \phi \rangle_g \ \ dx, \tag{1.1}$$

The associated Euler-Lagrange equation is called the **harmonic map** equation, and is similar to the Laplace equation, namely

$$D_j \partial_j \phi = 0 \tag{1.2}$$

However, since $\partial_j \phi(x) \in T_{\phi(x)} M$, here, instead of the standard differentiation operator ∂_j, we use the covariant differentiation operator D_j induced by the map ϕ. A

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consequence of this is that the above equation is no longer a linear equation; instead it becomes a semilinear elliptic equation.

1.2. The wave map equation

Replacing $\mathbb{R}^n$ with the Minkowski space $(\mathbb{M}^{n+1}, m)$, we can consider a similar Lagrangian to the above one,

$$L^m(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} \langle \partial^\alpha \phi, \partial_\alpha \phi\rangle_g \; dx,$$

with the key difference that the index α is now lifted with respect to the Minkowski metric. Thus L^m is no longer positive definite. The associated Euler-Lagrange equation is called the **wave map** equation, and has the form

$$D^\alpha \partial_\alpha \phi = 0, \qquad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1. \tag{1.3}$$

This is a semilinear wave equation, for which the initial position and velocity are maps

$$\phi_0 : \mathbb{R}^n \to M, \qquad \phi_1 : \mathbb{R}^n \to T_{\phi_0} M$$

A feature which is common with the linear wave equation is the conservation of the energy and momentum,

$$E(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_x \phi|^2 + |\partial_t \phi|^2 \; dx, \qquad M_i(\phi) = \int_{\mathbb{R}^n} \partial_i \phi \cdot \partial_t \phi \; dx.$$

1.3. The harmonic heat flow

This the gradient flow associated to the elliptic Lagrangian L^e in (1.1), and has the form

$$\partial_t \phi - D_j \partial_j \phi = 0, \qquad \phi(0) = \phi_0 : \mathbb{R}^n \to M \tag{1.4}$$

This is a semilinear parabolic equation for which L^e is a Lyapunov functional,

$$\frac{d}{dt} L^e(\phi) = -\int_{\mathbb{R}^n} \langle D_i \partial_i \phi, D_j \partial_j \phi\rangle_g dx$$

1.4. Schrödinger maps

The situation is slightly more complicated if one tries to obtain the geometric analogue of the Schrödinger equation. For that to make sense in the above context, we need a complex structure on the tangent space TM. Thus the natural setting is to have a Kahler manifold (M, g, J, ω) as a target. The elliptic Lagrangian above now plays the role of the Hamiltonian,

$$H(\phi) = \int_{\mathbb{R}^n} |\nabla \phi|_g^2 dx$$

The complex structure J on M induces a symplectic form on the phase space, namely

$$\omega(u, v) = \int_{\mathbb{R}^n} \langle u, Jv\rangle_g dx, \qquad u, v \in T_\phi M$$

The corresponding Hamilton flow is the **Schrödinger map** equation

$$\phi_t = J D^j \partial_j \phi, \qquad \phi(0) = \phi_0 \tag{1.5}$$

where J is the complex structure on TM. Properly interpreted this is a semilinear Schrödinger equation. We remark however that it can no longer be viewed as a holomorphic extension of the harmonic heat flow equation; indeed, the two flows no longer commute in general. The equation (1.5) admits one conserved quantity which is the energy (Hamiltonian). However, in general there seems to be no direct counterpart of the conservation of mass and momentum; see however [2].

1.5. Scaling and criticality

The harmonic map equation is invariant with respect to the dimensionless scaling

$$\phi(x) \to \phi(\lambda x)$$

and also with respect to isometries of the base space $\mathbb{R}^n$. Depending on the geometry of the target manifold (M, g) one may have further symmetries for this equation.

The homogeneous Sobolev norm which is invariant with respect to the above scaling is $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$; this is called the **critical Sobolev space**. In two dimensions this coincides with the energy space $\dot{H}^1$ defined by the Lagrangian. We call the two dimensional case **energy critical**. This is where it is natural to look for finite energy harmonic maps, and study their regularity.

In one dimension the harmonic map equation becomes a second order ode; in fact, it is exactly the equation for the geodesics on M.

In higher dimensions the critical Sobolev space has a higher index than the energy norm. This is where the energy no longer plays a significant role in the study of the solutions. We call these problems **energy supercritical**.

The scaling for both the harmonic heat flow and for Schrödinger maps is similar to the above one with an added time component,

$$\phi(t, x) \to \phi(\lambda^2 t, \lambda x)$$

This is natural since harmonic maps can be viewed as steady states for both flows. Here the Sobolev space $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ serves as the scale invariant initial data space. The two dimensional case is still energy critical. The one dimensional case is energy subcritical, while dimension three and higher is energy supercritical.

For wave-maps we still have a dimensionless scaling law,

$$\phi(t, x) \to \phi(\lambda t, \lambda x)$$

and the harmonic maps are also the steady states. Here we have a richer family of symmetries given by the isometries of the Minkowski space, namely the Lorentz group. Via the Lorentz group, the harmonic maps yield a richer family of wave maps which travel in time with a constant velocity of size less than one and a fixed profile; we call these solutions **solitons**.

The scale invariant initial data space for the wave map equation is $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$. As above, the two dimensional case is still energy critical, the one dimensional case is energy subcritical, while dimension three and higher is energy supercritical.

2. The extrinsic formulation

The invariant formulation of the geometric equations in the previous section is very short and efficient, but not so revealing for the pde analysis. Here we explore different ways of understanding the structure of the equations. Rather than working with a general target manifold, we will eventually restrict ourselves to the two most interesting special targets: the sphere $\mathbb{S}^2$ and the hyperbolic space $\mathbb{H}^2$.

In what follows we think of the target manifold as being embedded into a higher dimensional flat space, namely

$$\mathbb{S}^2 \subset (\mathbb{R}^3, e), \qquad \mathbb{M}^2 \subset (\mathbb{R}^{2+1}, m)$$

where e and m stand for the Euclidean, respectively the Minkowski metric. In the latter case one can use either sheet of the two sheeted hyperboloid $\phi_0^2 = \phi_1^2 + \phi_2^2$.

2.1. The case of the sphere $\mathbb{S}^2$

This case serves as a model case of a positively curved target. The equations take a particularly simple form, namely

$$-\Delta\phi = \phi|\nabla\phi|^2$$

for harmonic maps,

$$(\partial_t - \Delta)\phi = \phi|\nabla\phi|^2$$

for the heat flow,

$$\Box\phi = \phi(\partial^\alpha\phi \cdot \partial_\alpha\phi)$$

for wave maps. Furthermore, the sphere $\mathbb{S}^2$ is also a Riemann surface so we can also consider Schrödinger maps into $\mathbb{S}^2$,

$$\partial_t\phi = \phi \times \Delta\phi$$

where the cross-product by ϕ attains the double goal of removing the normal component of $\Delta\phi$, and of rotating its tangential component by $\frac{\pi}{2}$.

In the case of two space dimensions, a key observation is that there exist nontrivial finite energy harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^2$. By a theorem of Helein, all such maps are smooth. To describe them better we observe that the energy space $\dot{H}^1 = \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ splits into connected components associated to homotopy classes.

Then it is natural to consider energy minimizers in each homotopy class. These turn out to be unique modulo symmetries, namely scaling, isometries in the base and isometries of the target. Using polar coordinates in the plane and on the sphere, a representative of each class can be described in the form

$$Q_k(r,\theta) = (2\tan^{-1}(r^k), k\theta), \qquad k \geq 1$$

where k indexes the homotopy class. Note that the $k = 0$ minimizers are the constant functions. One can also take $k < 0$; that is equivalent to positive k seen in the mirror.

The maps above have additional symmetries. Precisely, the belong to the so-called k-equivariant class, which comprises maps of the form

$$\phi(r,\theta) = (f(r), k\theta + g(r))$$

Often it is helpful to consider some of the more difficult problems restricted to the equivariant class.

2.2. The case of the hyperbolic space $\mathbb{H}^m$

The equations look very much as in the case of the sphere,

$$-\Delta\phi = \phi|\nabla\phi|_m^2$$

for harmonic maps, respectively

$$(\partial_t - \Delta)\phi = \phi|\nabla\phi|_m^2$$

for the heat flow,

$$\Box\phi = \phi(\partial^\alpha\phi \cdot \partial_\alpha\phi)_m$$

for wave maps and

$$\partial_t\phi = \phi \times_m \Delta\phi$$

for Schrödinger maps. The one key difference is that all vector norms, inner products and cross products are taken with respect to the Minkowski metric.

A key difference, when compared to the sphere case, is that the state space is connected, and that no finite energy harmonic maps.

3. The frame method

As written in the extrinsic formulation, it is not immediately apparent that the Schrödinger map equation is a Schrödinger type equation, much less a semilinear one.

In order to interpret it as a semilinear Schrödinger system, it is very convenient to use the frame method. We describe it in the case of the spherical target, but the method applies without almost any changes to the hyperbolic space. The same description applies in $\mathbb{R}^n$, which we use as the domain in this section. As a target we use the sphere; the case of a hyperbolic space target is nearly identical.

For each $(x,t) \in \mathbb{R}^{n+1}$ choose an orthonormal frame (v,w) in $T_{\phi(x,t)}\mathbb{S}^2$. We represent the derivatives of our map ϕ in this frame by the *differentiated fields*

$$\psi_j = \partial_j\phi \cdot v + i\partial_j\phi \cdot w, \qquad j = 1, \cdots, n, n+1$$

where the index $n+1$ corresponds to time.

In turn, the of the frame is described by the real *connection coefficients*

$$A_j = \partial_j v \cdot w$$

which define the connection $\mathbf{D}_m = \partial_m + iA_m$. Then ψ_m satisfy the compatibility conditions

$$\mathbf{D}_l\psi_m = \mathbf{D}_m\psi_l.$$

The curvature of the connection is given by

$$\mathbf{D}_l\mathbf{D}_m - \mathbf{D}_m\mathbf{D}_l = i(\partial_l A_m - \partial_m A_l) = i\Im(\psi_l\overline{\psi_m}).$$

Then the idea is to replace the evolution of ϕ with the evolution of its derivatives ψ_m. The Schrödinger map equation becomes

$$\psi_{n+1} = i\mathbf{D}_l\psi_l.$$

Differentiating we obtain

$$\mathbf{D}_{n+1}\psi_m = i\mathbf{D}_l\mathbf{D}_l\psi_m + \Im(\psi_l\overline{\psi_m})\psi_l$$

and expanding

$$(i\partial_t + \Delta_x)\psi_m = -2iA_l\partial_l\psi_m + \Big(A_{n+1} + (A_l^2 - i\partial_l A_l)\Big)\psi_m - i\psi_l\Im(\overline{\psi}_l\psi_m). \tag{3.1}$$

This is coupled with the curl system for the A_j's

$$\partial_l A_m - \partial_m A_l = \Im(\psi_l\overline{\psi_m}). \tag{3.2}$$

We remark that the original map ϕ has completely disappeared from the equations. This is very convenient, and is a consequence of the fact that the target has constant curvature.

As written above, the system for $\{\psi_k\}$ is invariant with respect to the gauge transformation

$$\psi_m \to e^{i\theta}\psi_m, \qquad A_m \to A_m + \partial_m\theta.$$

To solve the equation we need to make a unique choice for the A's, i.e. to fix the gauge. The aim is to do it in such a way so that the right hand side of (3.1) is perturbative, at least in the small data case.

To best describe the issues which arise, we begin with a model problem, namely the cubic NLS,

$$(i\partial_t - \Delta)u = \pm\psi|\psi|^2$$

For small data this can be solved perturbatively via the Strichartz estimates. Next we consider several gauge choices.

3.1. The extrinsic gauge.

Here (v, w) are a fixed frame on the sphere. While no such frame exists globally, one may still attempt to use one locally. Then the equation for ψ takes the schematic form

$$(i\partial_t - \Delta)\psi = \psi \cdot \nabla\psi \pm \psi|\psi|^2$$

Unfortunately the first term on the right is nonperturbative.

3.2. The Coulomb gauge.

Here we complement the elliptic system (3.2) with a divergence relation

$$\partial_j A_j = 0$$

Then the system roughly takes the form

$$(i\partial_t - \Delta)\psi = |\nabla|^{-1}|\psi|^2 \cdot \nabla\psi \pm \psi|\psi|^2$$

One obvious gain is that the quadratic term we had before is now replaced by a cubic term, which should have better decay. Indeed, such a term is perturbative in high dimension $n \geq 4$. However, in low dimensions the high $\times$ high $\to$ low interaction in the expression $|\nabla|^{-1}|\psi|^2$ becomes to degenerate. An exception to this is the special case of equivariant maps. There the extra structure renders the nonlinearity manageable.

3.3. The Caloric gauge.

This was introduced by Tao [9] and is considerably more involved. It is constructed in several steps:

- At each time t solve the harmonic heat flow with $\phi(t)$ as the initial data,
$$\partial_s\phi - \Delta_x\phi = \phi|\partial_x\phi|^2 \qquad \phi(0,t,x) = \phi(t,x).$$
Assuming $\phi(t)$ is not too large, in the limit $s \to \infty$ the map converges to a constant, $\phi \to P$ which is independent of t.

- Choose (v_∞, w_∞) at $s = \infty$ as an arbitrary orthonormal base in $T_P\mathbb{S}^2$, and pull back along the heat flow using parallel transport,
$$w \cdot \partial_s v = 0 \Leftrightarrow A_0 = 0$$

- Derive a heat equation for ψ_m,
$$(\partial_s - \Delta_x)\psi_m = 2iA_l\partial_l\psi_m - (A_l^2 - i\partial_l A_l)\psi_m + i\Im(\psi_m\overline{\psi_l})\psi_l.$$

- To recover the coefficients A_m at $s = 0$ we integrate in s in the relation
$$\partial_s A_m = \Im(\psi_0\overline{\psi_m}) = \Im\Big((\partial_l\psi_l + iA_l\psi_l)\overline{\psi_m}\Big)$$

The gain achieved by going through all this extra work is that the ψ system now takes the form
$$(i\partial_t - \Delta)\psi = B(\psi,\bar{\psi})\cdot\nabla\psi \pm \psi|\psi|^2$$
where B is a translation invariant bilinear form whose symbol has size
$$B(\xi,\eta) \sim \frac{\xi+\eta}{\xi^2+\eta^2}$$
Comparing this with the Coulomb gauge, one sees that this symbol improves exactly in the case of the high $\times$ high $\to$ low interaction.

4. The small data problem

The main result we have so far for the small data problem is as follows:

Theorem 1 (Bejenaru-Ionescu-Kenig-T., '08, [1])**.** *Consider the Schrödinger map equation in $\mathbb{R}^n$ with target either $\mathbb{S}^2$ or $\mathbb{H}^2$. Then global well-posedness, regularity and scattering holds for any initial data u_0 with small critical Sobolev norm $\dot{H}^{\frac{n}{2}}$.*

Earlier results were obtained by Nahmod-Stephanov-Uhlenbeck, Kenig-Nahmod, Bejenaru, Ionescu-Kenig, Bejenaru-Ionescu-Kenig. The most interesting but also the most difficult case is $n = 2$ (the energy critical problem).

Our approach is based on the frame method, using the caloric gauge. The key point is to prove that the nonlinearity is perturbative in the equation for the differentiated fields ψ_k. However this is easier said than done.

In general, cubic nonlinearities in $2+1$ dimensional NLS problems are amenable via Strichartz type estimates. In this particular case, however, one faces an unfavorable balance of frequencies, which needs to be compensated for in the trilinear estimates. This is often achieved by using lateral Strichartz estimates, with respect to suitable frames. In the problem at hand, the correct frequency balance would be indeed gained by twice matching lateral energy $L^\infty L^2$ bounds with lateral $L^2 L^\infty$ Pecher type bounds. Unfortunately the latter is exactly the forbidden Strichartz endpoint in two space dimension. To remedy this, the idea is to replace the single lateral $L^2 L^\infty$ Pecher space with a more robust $\sum L^2 L^\infty$ norm. This is somewhat reminiscent of ideas previously used in the study of wave maps in low dimension.

5. The large data problem

For the purpose of this section we assume that we are in two space dimensions, i.e. the energy critical case. The reason is that in this case the energy is a meaningful invariant object which can be used in the description of the global behavior of solutions.

We begin with the case of the $\mathbb{H}^2$ target, where there are no finite energy harmonic maps, and no other known obstructions to global well-posedness. This is the geometric version of a defocusing problem. Then we have

Conjecture 2 (Defocusing Conjecture)**.** *Consider the Schröedinger map problem in two space dimensions, with values in $\mathbb{H}^2$. For this problem, global well-posedness and scattering holds for all finite energy data.*

In the case of the $\mathbb{S}^2$ target, the harmonic maps provide an obvious obstruction to a large data result. In addition, scattering can only occur for solutions in the zero homotopy class. The smallest nontrivial soliton, on the other hand, is the stereographic projection which belongs to the homotopy one class. In order to emulate such a soliton in the zero homotopy class, one needs to wrap the sphere and then unwrap it; this requires twice the soliton energy. Thus the natural conjecture is:

Conjecture 3 (Strong Threshold Conjecture)**.** *Consider the Schröedinger map problem in two space dimensions, with values in $\mathbb{S}^2$. For this problem, global well-posedness and scattering holds for all zero homotopy data which satisfies $E(\phi) < 2E(Q_1)$.*

We note that the corresponding results for wave maps have been proved recently see Sterbenz-Tataru [7],[8] (any target manifold), Tao [10] ($\mathbb{H}^n$ target) and Krieger-Sterbenz [5]($\mathbb{H}^2$ target). Both conjectures are still open for Schrödinger maps. However, the equivariant case has recently been studied.

Theorem 4 (Bejenaru-Kenig-Ionescu-Tataru, in preparation)**.** *Consider the Schröedinger map problem in two space dimensions, with values in $\mathbb{S}^2$. For this problem, global well-posedness and scattering holds in the* 1*-equivariant class for all finite energy data.*

Theorem 5 (Bejenaru-Kenig-Ionescu-Tataru [2])**.** *Consider the Schröedinger map problem in two space dimensions, with values in $\mathbb{S}^2$. For this problem, global well-posedness and scattering holds in the* 1*-equivariant class for all zero homotopy data which satisfies $E(\phi) < E(Q_1)$.*

The proof uses the Kenig-Merle method, which involves

- proving that if the result does not hold then minimal energy blow-up solutions exist and

- eliminating the minimal energy blow-up solutions via mass and momentum Morawetz type estimates.

Key difficulties in the proof:

- Gauge formulation of the problem: via the Coulomb gauge one obtains two coupled NLS type equations, and the coupling needs to survive in the concentration compactness argument.

- Morawetz (momentum) estimates are harder, and only yield local energy decay in a restricted regime; in particular we cannot reach the conjectured $2E(Q_1)$ threshold for $\mathbb{S}^2$ targets.

6. Near soliton behavior

In this section we consider the behavior of solutions with energy above the ground state threshold. For clarity we discuss only the simplest such problem, which is still wide open. Thus, we consider the case of the $\mathbb{S}^2$ target and solutions in the homotopy one class, which have energy just above the soliton energy,

$$E(Q_1) \leq E(\phi) < E(Q_1) + \epsilon \tag{6.1}$$

We note that if $E(Q_1) = E(\phi)$ then ϕ must belong to the class $\mathcal{Q}_1$ of ground states obtained from Q_1 via symmetries. We also remark that energy considerations show that any such state ϕ must satisfy

$$\text{dist}(\phi, \mathcal{Q}_1) \lesssim \epsilon.$$

Thus the family $\mathcal{Q}_1$ is orbitally stable. Unfortunately this does not say as much as one might want since the group of symmetries is noncompact. Thus we have the following

Open problem 6. *For Schrödinger maps from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ which have homotopy one and satisfy* (6.1)*, understand the possible global dynamics for the flow.*

The key element in this is understanding the motion of solutions along the Q_1 family. Possible issues to consider are

- Can finite time blow-up occur ? If so, what are the possible rates ?

- For global solutions, what is the asymptotic behavior at infinity (if any) ?

- Can solutions drift away to spatial infinity in finite time ? In infinite time ?

- Are there any breather type solutions in this class ?

While in such generality the above problem seems out of reach for now, some partial results have been obtained for equivariant solutions. An advantage of working in the equivariant class is that the dimension of the symmetry group is reduced to two, namely scaling and horizontal rotations. The first is noncompact, but the second is compact. Thus we can parametrize the ground states as

$$\mathcal{Q}_1^{eq} = \{Q_{\alpha,\lambda};\ \lambda \in \mathbb{R}^+, \alpha \in \mathbb{S}^1\}$$

The equivariant solutions are represented as

$$\phi(t) = Q_{\alpha(t),\lambda(t)} + O_{\dot{H}^1}(\epsilon)$$

and the question is to understand the behavior of the functions $\alpha(t)$ and $\lambda(t)$.

In chronological order, the results we have so far are as follows:

Theorem 7 (Gustafson-Nakanishi-Tsai [4]). *$\mathcal{Q}_k$ ground states are stable in the k equivariant class for $k \geq 3$.*

We remark that this result is very different from the wave-map picture. Also, it seems somewhat unlikely that the result will survive outside the equivariant class.

Theorem 8 (Bejenaru-Tataru ($k = 1$, [3]) ($k = 2$, in progress)). *a) $\mathcal{Q}_1$ ground states are unstable in the energy norm $\dot{H}^1$.*

b) $\mathcal{Q}_1$ ground states are stable in the one equivariant class with respect to a stronger topology X satisfying

$$H^1 \subset X \subset \dot{H}^1$$

A key role in this analysis is played by the linearized equation near Q_1 expressed in a suitable gauge. This is a linear Schrödinger equation governed by an explicit operator

$$H = -\Delta + V, \qquad V(r) = \frac{1}{r^2} - \frac{8}{(1+r^2)^2}.$$

A key difficulty is that H has a zero resonance

$$\phi_0 = r\partial_r Q_1 = \frac{2r}{1+r^2}$$

which corresponds to motion along the soliton family.

This is unlike what happens in higher equivariance classes $k \geq 3$ where the analogue of ϕ_0 is not only an eigenvalue but also belongs to H^{-1}. This allows one to define a corresponding orthogonal projection for functions in $\dot{H}^1$ and opens the door to a more standard perturbation theory.

The proof of the above result requires developing a complete spectral resolution for the operator H. In addition, the parameter $\lambda(t)$ is the main nonperturbative parameter in this analysis, so one in effect needs to work with a linear evolution of the form

$$(i\partial_t + H_{\lambda(t)})\psi = f$$

with a nontrivial dependence of λ on t.

Finally, the last result we mention is

Theorem 9 (Merle-Raphael-Rodnianski [6]). *Near $\mathcal{Q}_1$ blow-up solutions exist.*

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