

Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Biarritz, 3–7 juin 2012

Kenji Nakanishi

Global dynamics beyond the ground state energy for nonlinear dispersive equations

J. É. D. P. (2012), Exposé n° VIII, 6 p.

<http://jedp.cedram.org/item?id=JEDP_2012____A8_0>

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

Global dynamics beyond the ground state energy for nonlinear dispersive equations

Kenji Nakanishi

Abstract

This is a brief introduction to the joint work with Wilhelm Schlag and Joachim Krieger on the global dynamics for nonlinear dispersive equations with unstable ground states. We prove that the center-stable and the center-unstable manifolds of the ground state solitons separate the energy space by the dynamical property into the scattering and the blow-up regions, respectively in positive time and in negative time. The transverse intersection of the two manifolds yields nine sets of global dynamics, which include stable transition from blow-up to scattering and vice versa.

1. Introduction

The ultimate goal of this study is to classify and predict the global behavior of general solutions for nonlinear dispersive equations. Given the variety and the complexity of their dynamics, this is obviously too ambitious in the full generality, but we have obtained some partial answers, either under an energy restriction which is slightly above the ground state, or for those solutions getting close to the ground state at some time. Specifically, we can treat the nonlinear Klein-Gordon equation (NLKG)

$$\ddot{u} - \Delta u + u = |u|^{p-1}u, \quad u(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{R}, \quad 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \quad (1.1)$$

the nonlinear Schrödinger equation (NLS)

$$i\dot{v} - \Delta v = |v|^{p-1}v, \quad v(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{C}, \quad 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \quad (1.2)$$

and the nonlinear critical wave equation (NLW)

$$\ddot{w} - \Delta w = |w|^{4/(d-2)}w, \quad w(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{R}, \quad d \geq 3. \quad (1.3)$$

MSC 2000: 35L70, 35Q55.

Keywords: nonlinear wave equations, nonlinear Schrödinger equation, nonlinear Klein-Gordon equation, solitons, scattering theory, blow-up, invariant manifolds.

The main and common properties of them are that the nonlinearity is attractive and controlable in the scaling by the energy (\dot{H}^1) and the mass (L^2), and that they have unstable ground states, i.e., the nontrivial stationary or standing-wave solution with the least energy and positive profiles, denoted respectively by

$$u(t, x) = Q(x), \quad w(t, x) = W(x), \quad v(t, x) = e^{-it\omega}Q(x) \quad (\omega > 0). \quad (1.4)$$

The conserved energy is defined respectively by

$$\begin{aligned} E(u) &= \int_{\mathbb{R}^d} \frac{|\dot{u}|^2 + |\nabla u|^2 + |u|^2}{2} - \frac{|u|^{p+1}}{p+1} dx, \\ E(v) &= \int_{\mathbb{R}^d} \frac{|\nabla v|^2 + \omega|v|^2}{2} - \frac{|v|^{p+1}}{p+1} dx, \quad E(w) = \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} dx. \end{aligned} \quad (1.5)$$

Hence the energy space is naturally defined by $(u, \dot{u}) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$, or $(w, \dot{w}) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, in which we investigate the global dynamics.

The existence as well as the instability of the ground state is well known for the above equations. $Q(x)$ decays exponentially as $|x| \rightarrow \infty$, while $W(x)$ decays algebraically. The Lorentz or Galilei invariance of the equation generates a class of solitons with various traveling velocity from any stationary or standing wave solution.

Besides those special solutions, it is known that these equations have, at least, the following two different types of solutions.

1. Scattering: They are asymptotic to some solutions of the free equation (i.e. the equation without the nonlinearity) as $t \rightarrow \infty$ in the energy space.
2. Blow-up: The energy norm diverges or the energy concentrates in finite time (the latter happens only for the critical equation w above). More precisely, it can be characterized by a finite maximal existence time of the local solution which is strongly continuous in the energy space.

The first group of results concern those solutions with energy at most slightly above the ground state. We can completely classify the global dynamics, which shows in particular that even if the scattering, blowup and the solitons are mixed together, the dynamics can stay reasonably ordered, thanks to the global dispersion of the equations.

More precisely, our results state that all initial data under the energy restriction are split into 9 sets, according to the global dynamics, which include scattering, solitons and blow-up, as well as transitions among them from $t < 0$ to $t > 0$. The classification is given by center-stable and center-unstable manifolds of the ground state, which we can construct globally, as the hypersurfaces separating scattering and blowup solutions, respectively for $t > 0$ and for $t < 0$.

For simplicity, the full statement is given below in the model case: radial, cubic and 3D NLKG [3]. The other cases [6, 1, 5, 2, 7] can be understood as extension of it. We introduce minimal amount of notation to state the main results.

For each $\sigma = \pm$, let $\mathcal{S}_\sigma(0), \mathcal{B}_\sigma, \mathcal{S}_\sigma(\pm Q)$ be the totality of the initial data $(u(0), \dot{u}(0)) \in \mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, for which the corresponding solution u of NLKG satisfies

$$\begin{aligned} (u(0), \dot{u}(0)) \in \mathcal{S}_\sigma(0) &\implies u \text{ scatters as } \sigma t \rightarrow \infty, \\ (u(0), \dot{u}(0)) \in \mathcal{B}_\sigma &\implies u \text{ blows up in } \sigma t > 0, \\ (u(0), \dot{u}(0)) \in \mathcal{S}_\sigma(\pm Q) &\implies u \mp Q \text{ scatters as } \sigma t \rightarrow \infty. \end{aligned} \quad (1.6)$$

For each subset $X \subset \mathcal{H}$ and $\varepsilon > 0$, the energy restriction is denoted by

$$X^\varepsilon = \{\varphi \in X \mid E(\varphi) < E(Q) + \varepsilon^2\}. \quad (1.7)$$

For any solution u of NLKG in \mathcal{H} , let $I(u) \subset \mathbb{R}$ be the maximal existence interval. Denote the ground state of the linearized operator and its orthogonal projection by $L_+ := -\Delta + 1 - 3Q^2$, $L_+\rho = -k^2\rho$, $\rho = \rho(|x|) > 0$, $\|\rho\|_2 = 1$, $P_\pm := 1 - \rho\langle \rho |$.

Théorème 1. *Let $d = p = 3$ for (NLKG). Then there exists $\varepsilon > 0$ such that the following holds. We have*

$$\mathcal{H}^\varepsilon = \mathcal{S}_\sigma(0)^\varepsilon \cup \mathcal{B}_\sigma^\varepsilon \cup \mathcal{S}_\sigma(Q)^\varepsilon \cup \mathcal{S}_\sigma(-Q)^\varepsilon \quad (\text{disjoint union}), \quad (1.8)$$

for $\sigma = \pm$. Each of the 14 intersections

$$\begin{aligned} \mathcal{S}_+(0)^\varepsilon \cap \mathcal{S}_-(0)^\varepsilon, \quad \mathcal{S}_+(0)^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{S}_+(0)^\varepsilon \cap \mathcal{S}_-(\pm Q)^\varepsilon, \\ \mathcal{B}_+^\varepsilon \cap \mathcal{S}_-(0)^\varepsilon, \quad \mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{B}_+^\varepsilon \cap \mathcal{S}_-(\pm Q)^\varepsilon, \\ \mathcal{S}_+(\pm Q)^\varepsilon \cap \mathcal{S}_-(0)^\varepsilon, \quad \mathcal{S}_+(\pm Q)^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{S}_+(\pm Q)^\varepsilon \cap \mathcal{S}_-(\pm Q)^\varepsilon \end{aligned} \quad (1.9)$$

contains infinitely many orbits, while $\mathcal{S}_-(\pm Q)^\varepsilon \cap \mathcal{S}_+(\mp Q)^\varepsilon$ are empty. $\mathcal{S}_\sigma(0)^\varepsilon$ and $\mathcal{B}_\sigma^\varepsilon$ are unbounded connected open sets. $\mathcal{S}_\sigma(\pm Q)^\varepsilon$ are unbounded, connected and smooth manifold of codimension 1. $\mathcal{S}_+(\pm Q)^\varepsilon \cap \mathcal{S}_-(\pm Q)^\varepsilon$ are bounded, connected and smooth manifold of codimension 2 within $O(\varepsilon)$ distance from $\pm Q$.

The key ingredient in the proof is the following result, which we call the one-pass theorem. It precludes orbits departing from and returning to a small neighborhood of the ground state. Once it is established, the analysis is essentially decomposed into a scattering region, a blowup region and the small neighborhood of the ground state. In each of them, some modification of the preceding works suffices.

Théorème 2. *Let $d = p = 3$ and $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Then there are small $\varepsilon > 0$, $O(\varepsilon)$ -neighborhoods U_\pm of $\pm Q$, a continuous function $\mathfrak{S} : \mathcal{H}^\varepsilon \setminus (U_+ \cup U_-) \rightarrow \{\pm 1\}$ such that the following hold. For any solution u of NLKG in \mathcal{H}^ε , let*

$$\begin{aligned} I_{\mathcal{S}(0)}(u) &= \{t \in I(u) \mid \mathfrak{S}((u(t), \dot{u}(t))) = +1\}, \\ I_{\mathcal{B}}(u) &= \{t \in I(u) \mid \mathfrak{S}((u(t), \dot{u}(t))) = -1\}, \\ I_{\mathcal{S}(\pm Q)}(u) &= \{t \in I(u) \mid (u(t), \dot{u}(t)) \in U_\pm\}. \end{aligned} \quad (1.10)$$

Then either $I_{\mathcal{S}(Q)}(u)$ or $I_{\mathcal{S}(-Q)}(u)$ is empty, while the other is connected. $I_{\mathcal{S}(0)}(u)$ and $I_{\mathcal{B}}(u)$ together have at most two connected components. $I_{\mathcal{S}(0)}(u)$ consists of unbounded intervals, while $I_{\mathcal{B}}(u)$ consists of bounded ones. For each $X = \mathcal{S}(0), \mathcal{B}, \mathcal{S}(\pm Q)$, we have $(u(0), \dot{u}(0)) \in X_+$ iff $\exists T \in I(u)$ such that $(T, \infty) \cap I(u) \subset I_X(u)$, and the same property for the negative time direction. For those $(\varphi, \psi) \in \mathcal{H}$ close to $(\pm Q, 0)$, we have

$$\mathfrak{S}((\varphi, \psi)) = \text{sign} \langle Q \mp \varphi | \rho \rangle. \quad (1.11)$$

For (φ, ψ) away from $(\pm Q, 0)$, we have

$$\mathfrak{S}((\varphi, \psi)) = \text{sign} \int_{\mathbb{R}^3} [|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4] dx. \quad (1.12)$$

Moreover, \mathfrak{S} is uniquely determined by the above two formulas.

The one-pass theorem is proved by combining the hyperbolic dynamics of the linearized equation near but slightly away from the ground state, and the variational analysis far from the ground state, together with a virial identity localized in space-time for an almost homoclinic orbit.

Despite the lengthy statements, the proofs are not technically complicated at all, which are based on numerous classical and recent ideas. The interested reader is invited to look first into the above simplest setting, either in the paper [3] or in the book [4].

2. Dynamics in higher energy

The study beyond the constraint $E(\vec{u}) < E(Q) + \varepsilon^2$ is still to be developed, which seems to require a lot of new ideas. However, we can say a few things without the constraint, mainly around the center-(un)stable manifolds. Again for simplicity, we restrict ourselves to the cubic NLKG in the radial 3D case. The main idea for the proof is that the dynamics around the center-stable manifold can essentially be reduced to the case with energy slightly above the ground state, by detaching radiating high energy.

The first theorem extends the center-stable manifold globally in the energy space, with the scattering property.

Théorème 3. $\mathcal{S}_+(Q)$ is an invariant manifold of codimension 1 in \mathcal{H} . There is a bijection W from $P_\perp \mathcal{H} \times \mathbb{R}$ to the set of solutions whose orbits are on $\mathcal{S}_+(Q)$, such that $\vec{u} := W(\gamma, a)$ satisfies $u - Q \in L_t^2 L_x^6$ for large t and

$$\lim_{t \rightarrow \infty} \|\vec{u}(t) - \vec{Q} - \vec{v}(t)\|_{\mathcal{H}} = 0, \quad (2.1)$$

where $\vec{Q} := (Q, 0)$ and \vec{v} is the free solution with $\vec{v}(0) = \gamma$.

The parameter a essentially determines the stable mode, which decays as $t \rightarrow \infty$. See (2.2) and Theorem 6 below for more precise meaning of this parameter. The above theorem extends [3] concerning the existence of the center-stable manifold of Q , by removing the restriction $E < E(Q) + \varepsilon^2$. We omit the similar statements for the other signs of $\mathcal{S}_\pm(\pm Q)$, obtained by the change of variables $u \mapsto -u$ and/or $t \mapsto -t$.

The next theorem describes the forward global dynamics around the manifold.

Théorème 4. For each $\varphi \in \mathcal{S}_+(Q)$, there is a neighborhood U of φ and a function $\sigma \in C^1(U; \mathbb{R})$ such that the following holds. For any $\psi \in U$, let u be the solution starting from ψ , then

1. u scatters to 0 as $t \rightarrow \infty$ if and only if $\sigma(\psi) > 0$.
2. u scatters to Q as $t \rightarrow \infty$ if and only if $\sigma(\psi) = 0$.
3. u blows up in $t > 0$ if and only if $\sigma(\psi) < 0$.

This extends [3] concerning the separation of forward dynamics by the center-stable manifold, removing the restriction $E < E(Q) + \varepsilon^2$. Again, the corresponding statements for $-Q$ and for $t \rightarrow -\infty$ are obvious.

The next theorem describes the entire dynamics around any homoclinic or heteroclinic orbits, either in the weak topology or in the scattering sense, connecting $\pm Q$ as $t \rightarrow \pm\infty$.

Théorème 5. *For each $\varphi \in \mathcal{S}_-(Q) \cap \mathcal{S}_+(\mathfrak{s}Q)$ with $\mathfrak{s} \in \{\pm\}$, there are a neighborhood U of φ and functions $\sigma_{\pm} \in C^1(U; \mathbb{R})$ such that the following holds. For any $\psi \in U$, let u be the solution starting from ψ , then*

1. u scatters to Q as $t \rightarrow -\infty$ if and only if $\sigma_-(\psi) = 0$.
2. u scatters to $\mathfrak{s}Q$ as $t \rightarrow \infty$ if and only if $\sigma_+(\psi) = 0$.
3. u scatters to 0 as $t \rightarrow \pm\infty$ if and only if $\sigma_{\pm}(\psi) > 0$.
4. u blows up in $\pm t > 0$ if and only if $\sigma_{\pm}(\psi) < 0$.

In other words, the signs of the local functionals σ_{\pm} classify the global dynamics into 9 sets, locally around any point of intersection of the center-unstable and the center-stable manifolds. Such a classification was given in [3] for the region $E < E(Q) + \varepsilon^2$, where we also showed that for each $\gamma_+ \in P_{\perp}\mathcal{H}$, the family of solutions $W(\varphi_+, a)$ (given in Theorem 3) is decomposed into 3 sets according to the backward evolution:

$$\begin{aligned} a > 0 &\implies W(\varphi_+, a) \text{ scatters to } 0 \text{ as } t \rightarrow -\infty, \\ a = 0 &\implies W(\varphi_+, a) \text{ scatters to } Q \text{ as } t \rightarrow -\infty, \\ a < 0 &\implies W(\varphi_+, a) \text{ blows up in the negative time direction.} \end{aligned} \tag{2.2}$$

This is not true for larger energy as we will show later. However, it remains valid for $|a| \gg 1$, at least if $E < 2E(Q)$. Let E_L denote the linearized energy

$$E_L((\gamma_1, \gamma_2)) := \frac{1}{2}[\|\gamma_2\|_2^2 + \langle L_+\gamma_1 | \gamma_1 \rangle_{L^2}]. \tag{2.3}$$

Théorème 6. *For any $\gamma_+ \in P_{\perp}\mathcal{H}$, let $\{W(\gamma_+, a)\}_{a \in \mathbb{R}}$ be the family of solutions given in Theorem 3. Then, $W(\gamma_+, a)$ blows up in the negative time direction for $a \ll -1$. If $E_L(\gamma_+) < E(Q)$ then $W(\gamma_+, a)$ scatters to 0 as $t \rightarrow -\infty$ for $a \gg 1$.*

It seems very difficult to determine the backward dynamics for intermediate values of $a \in \mathbb{R}$ without energy restriction. However, we can prove that there are plenty of weakly heteroclinic orbits between Q and $-Q$. We expect that this is the minimal energy event violating (2.2) or the global 9-set decomposition.

Théorème 7. *There are two invariant manifolds $\mathcal{M}_{\pm} \subset \mathcal{H}$ of codimension 2 such that any solution starting there scatters to $\pm Q$ as $t \rightarrow -\infty$, scatters to Q as $t \rightarrow \infty$, and $\inf_t \|u(t)\|_{L_x^6} \ll \|Q\|_{L_x^6}$. Moreover, $\inf E(\mathcal{M}_{\pm}) \leq 2E(Q)$.*

Numerical experiments suggest that the above infimum energy is actually quite below $2E(Q)$, such as $< 1.4E(Q)$, but for now we have no proof, even for the strict inequality $\inf E(\mathcal{M}_{\pm}) < 2E(Q)$.

References

- [1] K. Nakanishi J. Krieger and W. Schlag. Global dynamics above the ground state energy for the one-dimensional nlkg equation. *to appear in Math. Z.* arXiv:1011.1776.
- [2] K. Nakanishi J. Krieger and W. Schlag. Global dynamics away from the ground state for the energy-critical nonlinear wave equation. *to appear in Amer. J. Math.* arXiv:1010.3799.
- [3] K. Nakanishi and W. Schlag. Global dynamics above the ground state energy for the focusing nonlinear klein-gordon equation. *J. Differential Equations*, 250:2299–2333, 2011.
- [4] K. Nakanishi and W. Schlag. *Invariant manifolds and dispersive Hamiltonian evolution equations*. European Mathematical Society, Zürich, 2011.
- [5] K. Nakanishi and W. Schlag. Global dynamics above the ground state energy for the cubic nls equation in 3d. *Calc. Var. Partial Differential Equations*, 44:1–45, 2012.
- [6] K. Nakanishi and W. Schlag. Global dynamics above the ground state for the nonlinear klein-gordon equation without a radial assumption. *Arch. Ration. Mech. Anal.*, 203:809–851, 2012.
- [7] K. Nakanishi and W. Schlag. Invariant manifolds around soliton manifolds for the nonlinear klein-gordon equation. *SIAM J. Math. Anal.*, 44:1175–1210, 2012.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
KYOTO 606-8502
JAPAN
n-kenji@math.kyoto-u.ac.jp