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# Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity: some results and open problems

Mihai Mariş

## Abstract

This text is a survey of recent results on traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity. We present the existence, nonexistence and stability results and we describe the main ideas used in proofs.

## 1. Introduction

We consider the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial\Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \quad \text{in } \mathbf{R}^N \times \mathbf{R}, \quad (1.1)$$

where  $\Phi$  is a complex-valued function that satisfies the "boundary condition"  $|\Phi| \rightarrow r_0$  as  $|x| \rightarrow \infty$ ,  $r_0$  is a positive constant and  $F$  is a real-valued function on  $\mathbf{R}_+$  satisfying  $F(r_0^2) = 0$ .

The analysis of Eq. (1.1) is an extremely active research field at the moment. This equation, with the considered non-zero conditions at infinity, is relevant in a large variety of physical problems such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensates ([BGMP89], [BaMa88], [Be08], [Co98], [GR74], [Gro63], [IoSm78], [JR82], [JPR86]). In nonlinear optics, it appears in the context of dark solitons, that is, localized waves which exist on a stable continuous background ([KiLD98], [KPS95]). Two important particular cases of (1.1) have been extensively studied in the literature: the Gross-Pitaevskii equation (where  $F(s) = 1 - s$ ) and the so-called "cubic-quintic" Schrödinger equation (where  $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$ ,  $\alpha_1, \alpha_3, \alpha_5$  are positive and  $F$  has two positive roots).

In contrast to the case of zero boundary conditions at infinity (when the dynamics associated to (1.1) is essentially governed by dispersion and scattering), the non-zero boundary condition allows a much richer dynamics and gives rise to a

remarkable variety of special solutions, such as traveling-waves, standing waves or vortex solutions.

Using the Madelung transformation  $\Phi(x, t) = \sqrt{\rho(x, t)}e^{i\theta(x, t)}$  (which is well-defined whenever  $\Phi \neq 0$ ), equation (1.1) is equivalent to a system of Euler's equations for a compressible inviscid fluid of density  $\rho$  and velocity  $2\nabla\theta$ . In this context it has been shown that, if  $F$  is  $C^1$  near  $r_0^2$  and  $F'(r_0^2) < 0$ , the sound velocity at infinity associated to (1.1) is  $v_s = r_0\sqrt{-2F'(r_0^2)}$  (see, e.g., the introduction of [M08]).

Eq. (1.1) is Hamiltonian, the "energy"

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2 dx + \int_{\mathbf{R}^N} V(|\Phi|^2) dx \quad (1.2)$$

is (formally) conserved by the flow, where  $V(s) = \int_s^{r_0^2} F(\tau) d\tau$ .

Let  $a = \sqrt{-\frac{1}{2}F'(r_0^2)} > 0$ . Then the sound velocity at infinity is  $v_s = 2ar_0$  and it is easy to see that we have the Taylor expansion

$$V(s) = a^2(s - r_0^2)^2 + o((s - r_0^2)^2) \quad \text{as} \quad s \rightarrow r_0^2. \quad (1.3)$$

Thus  $V(|\Phi|^2)$  is well approximated by  $a^2(|\Phi|^2 - r_0^2)^2$  if  $|\Phi|$  is close to  $r_0$ . Take a nondecreasing cut-off function  $\varphi \in C^\infty([0, \infty))$  such that  $\varphi(s) = s$  for  $s \in [0, 2r_0]$ , and  $\varphi$  is constant near infinity. The following *Ginzburg-Landau energy* is relevant in the study of (1.1):

$$\tilde{E}_{GL}(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2 dx + a^2 \int_{\mathbf{R}^N} (\varphi(|\Phi|)^2 - r_0^2)^2 dx.$$

The function space naturally associated to (1.1) is

$$\tilde{\mathcal{E}} = \{\psi \in H_{loc}^1(\mathbf{R}^N) \mid \nabla\psi \in L^2(\mathbf{R}^N), \varphi(|\psi|)^2 - r_0^2 \in L^2(\mathbf{R}^N)\}.$$

Another important quantity for (1.1) is the *momentum*  $P(\Phi) = (P_1(\Phi), \dots, P_N(\Phi))$ . Notice that the momentum is also (formally) conserved by the flow. A rigorous definition of the momentum will be given in Sect. 3; at the present stage we only give its formal definition,

$$P_k(\Phi) = \int_{\mathbf{R}^N} \langle i \frac{\partial\Phi}{\partial x_k}, \Phi \rangle dx.$$

The Cauchy problem associated to (1.1) has been addressed in several papers. In [BS99], global well-posedness was proved in the case of the Gross-Pitaevskii nonlinearity (that is,  $F(s) = 1 - s$ ) for initial data in the affine space  $1 + H^1(\mathbf{R}^N)$ ,  $N = 1, 2, 3$ . Later, C. Gallo studied the Schrödinger group on Zhidkov spaces  $X^k(\mathbf{R}^N) = \{u \in L^\infty(\mathbf{R}^N) \mid \nabla u \in H^{k-1}(\mathbf{R}^N)\}$  and proved well-posedness results for (1.1) in  $X^k(\mathbf{R}^N)$ , in the case of general nonlinearities, provided that  $k$  is large enough (see [Ga04], [Ga06], [Ga08]). Finally, P. Gérard ([G06]; see also the survey paper [G08]) studied the Cauchy problem for the Gross-Pitaevskii equation in the energy space  $\mathcal{E} = \{\psi \in H_{loc}^1(\mathbf{R}^N) \mid \nabla\psi \in L^2(\mathbf{R}^N), |\psi|^2 - 1 \in L^2(\mathbf{R}^N)\}$ . He proved that the solutions exist and are global in time for any initial data in  $\mathcal{E}$  if  $N = 1, 2, 3$ . He also proved a global existence result for small initial data in  $\mathcal{E}$  if  $N = 4$  (notice that the equation becomes critical in space dimension four, so one could not expect global existence for any initial data in that case). The same arguments should presumably work in any space dimension and for general nonlinearities provided that

one imposes appropriate conditions on the growth of the nonlinearity  $F$  at infinity and works with the space  $\tilde{\mathcal{E}}$  introduced above instead of  $\mathcal{E}$ .

On the other hand, Gustafson, Nakanishi and Tsai established in [GNT06], [GNT09] the scattering theory of small solutions to the Gross-Pitaevskii equation in space dimension three and four. In dimension two, where a scattering theory is excluded due to the existence of small energy traveling-waves, they constructed dispersive solutions with some prescribed data at infinity (see [GNT07]).

In despite of these results, the long time behavior of the solutions of Eq. (1.1) remains largely unknown. The first step in understanding the long-time dynamics associated to (1.1) would be to understand the special solutions (such as traveling waves, standing waves, or vortex solutions) and the behavior of those solutions which are close to the special solutions.

In a series of papers (see, e.g., [GR74], [JR82], [JPR86], [BaMa88], [BGMP89], [PR91], [PN93], [KR95a], [KR95b]), particular attention has been paid to a special class of solutions of (1.1), namely the traveling waves. These are solutions of the form  $\Phi(x, t) = \psi(x - cty)$ , where  $y \in S^{N-1}$  is the direction of propagation and  $c \in \mathbf{R}^*$  is the speed of the traveling wave. Without loss of generality we may assume that  $y = (1, 0, \dots, 0)$ . The equation satisfied by the traveling wave profile  $\psi$  is then

$$-ic \frac{\partial \psi}{\partial x_1} + \Delta \psi + F(|\psi|^2) \psi = 0 \quad \text{in } \mathbf{R}^N. \quad (1.4)$$

We say that  $\psi$  has finite energy if  $\nabla \psi \in L^2(\mathbf{R}^N)$  and  $V(|\psi|^2) \in L^1(\mathbf{R}^N)$ . Under general assumptions it can be proved that this is equivalent to  $\tilde{E}_{GL}(\psi) < \infty$ .

Traveling waves are supposed to play an important role in the dynamics of (1.1) (although this role is still not sufficiently understood). Travelling waves also model various phenomena observed in Helium II such as vortices and sound waves. In view of formal computations and numerical experiments, a list of conjectures, often referred to as *the Roberts programme*, has been formulated about the existence, the stability and the qualitative properties of traveling waves. We will try to describe below, to the best of our knowledge, the main lines of the Roberts programme and the parts of these conjectures that have already been proved.

*1. Existence and nonexistence.* It has been conjectured that finite energy traveling waves of speed  $c$  exist if and only if  $|c| < v_s$ , where  $v_s$  is the sound velocity at infinity.

In space dimension one, Eq. (1.4) can be integrated (almost) explicitly and in many interesting applications one can check directly that the above conjecture holds true.

In higher dimensions, the nonexistence of traveling waves with supersonic speeds ( $c > v_s$ ) has been proved in [Gr03] in the case of the Gross-Pitaevskii equation, respectively in [M08] for more general nonlinearities.

Despite of many attempts, a rigorous proof of the existence of subsonic, finite energy traveling waves in space dimension  $N \geq 2$  has been a long lasting problem. In the particular case of the Gross-Pitaevskii (GP) equation, this problem was considered in a series of papers. In space dimension two, the existence of traveling waves for any sufficiently small speed has been proved in [BS99]. In space dimension  $N \geq 3$ , the existence has been proved in [BOS04] for a sequence of speeds  $c_n \rightarrow 0$  by using constrained minimization; a similar result has been established in [Ch04]

for all sufficiently small speeds by using a mountain-pass argument. In a recent paper [BGS09], the existence of traveling waves for (GP) has been proved in space dimension  $N = 2$  and  $N = 3$  for any speed in a set  $A \subset (0, v_s)$ . If  $N = 2$ ,  $A$  contains points arbitrarily close to 0 and to  $v_s$  (although it is not clear that  $A = (0, v_s)$ ), while in dimension three we have  $A \subset (0, v_0)$ , where  $v_0 < v_s$  and  $0, v_0$  are limit points of  $A$ . The traveling waves are obtained in [BGS09] by minimizing the energy at fixed momentum (see section 3 for the definition of the momentum) and the propagation speed is the Lagrange multiplier associated to minimizers. In the case of cubic-quintic type nonlinearities, it has been proved in [M02] that traveling waves exist for any sufficiently small speed if  $N \geq 4$ . Notice that even for very specific nonlinearities, none of the previous results covers the whole range  $(0, v_s)$  of possible speeds. In dimension  $N \geq 3$ , the existence of traveling waves for any subsonic speed and under general assumptions on the nonlinearity  $F$  has been obtained in [M09b] (the proof is presented in section 3 below). In dimension two the problem is still open, although some progress has been made ([CM10]).

*2. Uniqueness.* In dimension  $N \geq 2$ , traveling wave solutions to (1.1) are not expected to be unique (up to the invariances of the equation). However, it has been conjectured that those traveling waves that minimize the energy at fixed momentum are unique up to the invariances of the equation. No rigorous result has been obtained in that direction.

*3. Qualitative properties.*

The regularity of traveling waves has been proved in [Fa03] and [M08].

All traveling waves found up to now are axially symmetric.

The asymptotic behavior of traveling waves as  $|x| \rightarrow \infty$  has been accurately described by the physicists in the seventies by using formal computations. More recently, this behavior has been rigorously established in a series of works by P. Gravejat (see [Gr04a], [Gr04b], [Gr05], [Gr06]) in the case of the Gross-Pitaevskii equation. Probably his proofs can be adapted to more general nonlinearities.

The fact that traveling waves do or do not present vortices should depend both on their speed and on the nonlinearity in the equation. For Gross-Pitaevskii-type nonlinearities, numerical experiments suggest that there is a critical speed  $v_c < v_s$  such that travelling-waves of speed less than  $v_c$  should present vortices, while traveling waves moving faster than  $v_c$  should not have vortices. For cubic-quintic nonlinearities, all traveling-waves should not have vortices. As far as we know, no rigorous result has been obtained in this direction. It is very likely that traveling waves moving with small speeds have vortices if (1.1) does not admit finite energy stationary solutions.

*4. Transsonic limit.* Let  $(\psi_{c_n})$  be a sequence of finite energy traveling waves for (1.1), where  $c_n \rightarrow v_s$ . It has been conjectured that, after a suitable anisotropic rescaling and up to a subsequence, the functions  $r_0^2 - |\psi_{c_n}|^2$  and the phases of  $\psi_{c_n}$  should tend to solitary waves of the Korteweg-de Vries equation in one dimension, respectively to the solitary waves of the Kadomtsev-Petviashvili I equation in dimensions two and three. In the case of the Gross-Pitaevskii nonlinearity, this has been proved in [BGSS09] in dimension one, respectively in [BGS08] in dimension two for those solutions that minimize the energy at fixed momentum. All other cases

are still unknown (a major difficulty in proving this convergence seems to be the obtention of good estimates on the energy of traveling waves).

5. *Stability.* In one dimension, the orbital stability (or instability) of traveling waves has been proved in [Lin02] for cubic-quintic nonlinearities, respectively in [BGSS08] and [GZ08] for the Gross-Pitaevskii nonlinearity. Even in one dimension, no asymptotic stability result seems to be available.

The orbital stability of the set of traveling waves that minimize the energy at fixed momentum is proved in [CM10] in any dimension  $N \geq 2$  and for general nonlinearities, under the assumption that the potential  $V$  is nonnegative.

In the next section we present the proof of the nonexistence of supersonic traveling waves. Section 3 is devoted to the proof of existence of such solutions for any subsonic speed in the case  $N \geq 3$ . Other recent existence results are described in the last section.

## 2. Nonexistence of supersonic traveling waves

Using a new and quite unexpected integral identity, P. Gravejat proved in [Gr03] the nonexistence of supersonic traveling waves with finite energy in the particular case of the Gross-Pitaevskii equation. He also proved in [Gr04b] the nonexistence of sonic traveling waves in space dimension 2. Simplifying his arguments, in [M08] we were able to generalize his integral identity and to prove that the nonexistence of finite energy supersonic traveling waves is a general phenomenon, which holds true for a large class of equations of the form (1.1) (including the Gross-Pitaevskii and cubic-quintic cases) as well as for systems of equations of this type. The nonexistence of finite energy sonic traveling waves in dimension two, as well as in dimension  $N \geq 3$  under the additional assumption that their phase is integrable on the exterior of a large ball, is also proved in [M08].

In the sequel we sketch the nonexistence proof. Throughout this section we suppose that the following assumptions are satisfied:

**C1.** The function  $F$  is continuous on  $[0, \infty)$ ,  $C^1$  in a neighborhood of  $r_0^2$ ,  $F(r_0^2) = 0$  and  $F'(r_0^2) < 0$ .

**C2.** There exist  $C, \alpha > 0$  such that for  $s$  sufficiently large we have  $F(s) \leq -Cs^\alpha$ .

The next result gives the regularity properties of finite energy traveling waves.

**Proposition 2.1.** *Assume that conditions C1 and C2 are satisfied. Let  $\psi$  be a finite energy solution of (1.4). Then:*

- i) *We have  $\psi \in L^\infty \cap W_{loc}^{2,p}(\mathbf{R}^N)$  for any  $p \in [1, \infty)$ .*
- ii) *We have  $\nabla \psi \in W^{1,p}(\mathbf{R}^N)$  for any  $p \in [2, \infty)$  and there exists  $R_* > 0$  such that  $\psi$  admits a lifting  $\psi = \rho e^{i\theta}$  on  $\mathbf{R}^N \setminus B(0, R_*)$ , where  $\rho, \theta \in W_{loc}^{2,p}(\mathbf{R}^N)$ ,  $p \in [1, \infty)$ .*
- iii) *Moreover, if  $F \in C^k([0, \infty))$ , then  $\psi \in W_{loc}^{k+2,p}(\mathbf{R}^N)$  for any  $p \in [1, \infty)$ .*

Notice that the classical bootstrap argument (which consists in using the equation, standard elliptic estimates and Sobolev embeddings in order to successively improve the regularity of the solution) could not work because in most applications the nonlinearity  $F$  is critical or supercritical. We use a method developed by A. Farina in [Fa98, Fa03] for Ginzburg-Landau systems (and which is based on Kato's inequality,

see [Ka72]) to prove that finite energy travelling waves are in  $L^\infty(\mathbf{R}^N)$ . Then the standard elliptic regularity theory can be used to complete the proof of Proposition 2.1.

At least formally, the solutions of (1.4) are critical points of the functional  $\tilde{E}_c(\psi) = E(\psi) + c\tilde{Q}(\psi)$ , where  $E$  is given by (1.2) and  $\tilde{Q}$  is the "momentum" with respect to the  $x_1$  direction (a precise definition will be given later; for the moment,  $\tilde{Q}$  is just a functional whose derivative  $\tilde{Q}'$  satisfies  $\tilde{Q}'(\phi) = 2i\phi_{x_1}$  for any  $\phi$ .) This variational characterization enables us to prove Pohozaev identities:

**Proposition 2.2.** *Let  $\psi$  be a finite energy solution of (1.4). Then  $\psi$  satisfies*

$$-\int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_1} \right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=2}^N \left| \frac{\partial \psi}{\partial x_j} \right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2) dx = 0 \quad \text{and} \quad (2.1)$$

$$\int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_1} \right|^2 + \frac{N-3}{N-1} \sum_{k=2}^N \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2) dx + c\tilde{Q}(\psi) = 0. \quad (2.2)$$

The Pohozaev identities are a consequence of the behavior of  $\tilde{E}_c$  with respect to dilations in  $\mathbf{R}^N$ . Given a function  $v$  defined on  $\mathbf{R}^N$ , we will use the notation

$$v_{\lambda, \sigma}(x) = v\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right), \quad \text{where } x' = (x_2, \dots, x_N). \quad (2.3)$$

If  $\psi$  is a critical point of  $\tilde{E}_c$ , we should have  $\left. \frac{d}{d\lambda} \right|_{\lambda=1} \tilde{E}_c(\psi_{\lambda,1}) = 0$  and  $\left. \frac{d}{d\sigma} \right|_{\sigma=1} \tilde{E}_c(\psi_{1,\sigma}) = 0$ ; these identities are precisely (2.1) and (2.2), respectively. Of course, this argument is purely formal because, in general,  $\left. \frac{d}{d\lambda} \right|_{\lambda=1} (\psi_{\lambda,1}) = -x_1 \frac{\partial \psi}{\partial x_1}$  et  $\left. \frac{d}{d\sigma} \right|_{\sigma=1} (\psi_{1,\sigma}) = -\sum_{j=2}^N x_j \frac{\partial \psi}{\partial x_j}$  do not belong to the function space on which  $\tilde{E}'_c(\psi)$  is defined.

In order to prove (2.1) and (2.2) rigorously, we multiply (1.4) by  $\chi\left(\frac{x}{n}\right) x_j \frac{\partial \psi}{\partial x_j}$ , where  $\chi \in C_c^\infty(\mathbf{R}^N)$  is a cut-off function that equals 1 in a neighborhood of the origin, we perform some integrations by parts, then we pass to the limit as  $n \rightarrow \infty$ . To justify the integration by parts, one needs some smoothness for  $\psi$ . The regularity provided by Proposition 2.1 ( $\psi \in L^\infty \cap W_{loc}^{2,p}(\mathbf{R}^N)$  and  $\nabla \psi \in W^{1,p}(\mathbf{R}^N)$  for  $p \in [2, \infty)$ ) is enough to get Pohozaev identities.

**Theorem 2.3.** *Suppose that  $c^2 > v_s^2$  and  $\psi$  is a finite energy solution of (1.4). Then  $\psi$  satisfies the identity*

$$\int_{\mathbf{R}^N} |\nabla \psi|^2 - F(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) dx + c \left(1 - \frac{v_s^2}{c^2}\right) \tilde{Q}(\psi) = 0. \quad (2.4)$$

*Proof.* Let  $\psi_1 = \text{Re}(\psi)$ ,  $\psi_2 = \text{Im}(\psi)$ . Eq. (1.4) is equivalent to the system

$$c \frac{\partial \psi_2}{\partial x_1} + \Delta \psi_1 + F(|\psi|^2)\psi_1 = 0 \quad \text{and} \quad (2.5)$$

$$-c \frac{\partial \psi_1}{\partial x_1} + \Delta \psi_2 + F(|\psi|^2)\psi_2 = 0. \quad (2.6)$$

We multiply (2.5) by  $\psi_2$  and (2.6) by  $-\psi_1$ , then we add the resulting equalities to obtain

$$\frac{c}{2} \frac{\partial}{\partial x_1} (|\psi|^2 - r_0^2) = \text{div}(\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1). \quad (2.7)$$

Multiplying (2.5) by  $\psi_1$  and (2.6) by  $\psi_2$ , then summing up the resulting equalities we get

$$|\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(|\psi|^2)|\psi|^2 - c\left(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1}\right) = \frac{1}{2}\Delta(|\psi|^2 - r_0^2). \quad (2.8)$$

Let  $R_*$  be as in Proposition 2.1 (ii) so that  $\psi$  has a lifting  $\psi = \rho e^{i\theta}$  on  $\mathbf{R}^N \setminus B(0, R_*)$ . Let  $\chi \in C^\infty(\mathbf{R}^N)$  be a cut-off function such that  $\chi = 0$  on  $B(0, 2R_*)$  and  $\chi = 1$  on  $\mathbf{R}^N \setminus B(0, 3R_*)$ . Denote  $G_j = \psi_1\frac{\partial\psi_2}{\partial x_j} - \psi_2\frac{\partial\psi_1}{\partial x_j} - r_0^2\frac{\partial}{\partial x_j}(\chi\theta)$ ,  $j = 1, \dots, N$ . One can prove that  $G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$  and the momentum  $\tilde{Q}(\psi)$  with respect to the de  $x_1$  direction is given by

$$\tilde{Q}(\psi) = - \int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta) dx = - \int_{\mathbf{R}^N} G_1 dx.$$

From (2.7) and (2.8) we infer that

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2 - r_0^2) = \operatorname{div}(\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1 - r_0^2\nabla(\chi\theta)) + r_0^2\Delta(\chi\theta), \quad (2.9)$$

respectively

$$\begin{aligned} & \frac{1}{2}\Delta(|\psi|^2 - r_0^2) - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \\ &= |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \\ & \quad - c\left(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)\right) - cr_0^2\frac{\partial}{\partial x_1}(\chi\theta). \end{aligned} \quad (2.10)$$

Let

$$H = |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) - c\left(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)\right).$$

Take the derivative with respect to  $x_1$  of (2.9) and multiply it by  $c$ , then take the Laplacian of (2.10). Summing up the resulting equalities we find

$$\frac{1}{2}\left(\Delta^2 - v_s^2\Delta + c^2\frac{\partial^2}{\partial x_1^2}\right)(|\psi|^2 - r_0^2) = \Delta H + c\frac{\partial}{\partial x_1}(\operatorname{div}(G)). \quad (2.11)$$

Taking the Fourier transform of (2.11) we obtain

$$\frac{1}{2}\left(|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2\right)\mathcal{F}(|\psi|^2 - r_0^2) = -|\xi|^2\widehat{H} - c\sum_{k=1}^N \xi_1\xi_k\widehat{G}_k. \quad (2.12)$$

Let  $\Gamma = \{\xi \in \mathbf{R}^N \mid |\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2 = 0\}$ . If  $c^2 \leq v_s^2$ , then  $\Gamma = \{0\}$ . In the case  $c^2 > v_s^2$ ,  $\Gamma$  is a nontrivial submanifold of  $\mathbf{R}^N$  and using (2.12) we get

$$|\xi|^2\widehat{H}(\xi) + c\sum_{k=1}^N \xi_1\xi_k\widehat{G}_k(\xi) = 0 \quad \text{for any } \xi \in \Gamma. \quad (2.13)$$

It is obvious that  $\Gamma = \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1} \mid |\xi'|^2 = \frac{1}{2}(-v_s^2 - 2\xi_1^2 + \sqrt{v_s^4 + 4c^2\xi_1^2})\}$ . Let  $f(t) = \sqrt{\frac{1}{2}(-v_s^2 - 2t^2 + \sqrt{v_s^4 + 4c^2t^2})}$ . The function  $f$  is well defined on  $[-\sqrt{c^2 - v_s^2}, \sqrt{c^2 - v_s^2}]$ , we have  $f(0) = 0$  and  $\lim_{t \rightarrow 0} \frac{f^2(t)}{t^2} = -1 + \frac{c^2}{v_s^2}$ . For



$j \in \{2, \dots, N\}$  and  $t \in (0, \sqrt{c^2 - v_s^2}]$ , let  $\xi(t) = (t, 0, \dots, 0, f(t), 0, \dots, 0)$  and  $\tilde{\xi}(t) = (t, 0, \dots, 0, -f(t), 0, \dots, 0)$ , where  $f(t)$  and  $-f(t)$ , respectively, stand at the  $j^{\text{th}}$  place. It is clear that  $\xi(t), \tilde{\xi}(t) \in \Gamma$ . From (2.13) we obtain

$$(t^2 + f^2(t))\widehat{H}(\xi(t)) + ct^2\widehat{G}_1(\xi(t)) + ctf(t)\widehat{G}_j(\xi(t)) = 0, \quad \text{and} \quad (2.14)$$

$$(t^2 + f^2(t))\widehat{H}(\tilde{\xi}(t)) + ct^2\widehat{G}_1(\tilde{\xi}(t)) - ctf(t)\widehat{G}_j(\tilde{\xi}(t)) = 0. \quad (2.15)$$

We multiply (2.14) and (2.15) by  $\frac{1}{t^2}$ , then take the limit as  $t \downarrow 0$  to get

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G}_1(0) + c\sqrt{-1 + \frac{c^2}{v_s^2}\widehat{G}_j(0)} = 0, \quad \text{respectively} \quad (2.16)$$

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G}_1(0) - c\sqrt{-1 + \frac{c^2}{v_s^2}\widehat{G}_j(0)} = 0. \quad (2.17)$$

From (2.16) and (2.17) we find  $\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G}_1(0) = 0$ , and this equality is precisely (2.4).  $\square$

**Theorem 2.4.** *Let  $N \geq 2$ . Assume that the conditions (C1) and (C2) hold,  $c^2 > v_s^2$  and, moreover, there exists  $\alpha \in \left[-1 + \frac{N-3}{N-1}\left(1 - \frac{v_s^2}{c^2}\right), \frac{v_s^2}{c^2}\right]$  such that*

$$sF(s) + \frac{v_s^2}{2}(s - r_0^2) + \left(1 - \alpha - \frac{v_s^2}{c^2}\right)V(s) \leq 0 \quad \text{for } s \geq 0.$$

*Let  $\psi$  be a finite energy traveling wave of (1.1) moving with velocity  $c$ . Then  $\psi$  is constant.*

*Proof.* Multiply (2.2) by  $1 - \frac{v_s^2}{c^2}$  and subtract the resulting equality from (2.4) to get

$$\int_{\mathbf{R}^N} \frac{v_s^2}{c^2} \left| \frac{\partial \psi}{\partial x_1} \right|^2 + \left(1 - \left(1 - \frac{v_s^2}{c^2}\right) \frac{N-3}{N-1}\right) \sum_{k=2}^N \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx \quad (2.18)$$

$$- \int_{\mathbf{R}^N} F(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + \left(1 - \frac{v_s^2}{c^2}\right)V(|\psi|^2) dx = 0.$$

Let  $\alpha$  be as in Theorem 2.4. Multiply (2.1) by  $\alpha$  and take the sum of the resulting equality and of (2.18) to get

$$\int_{\mathbf{R}^N} \left(\frac{v_s^2}{c^2} - \alpha\right) \left| \frac{\partial \psi}{\partial x_1} \right|^2 + \left(\alpha + 1 - \left(1 - \frac{v_s^2}{c^2}\right) \frac{N-3}{N-1}\right) \sum_{k=2}^N \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx \quad (2.19)$$

$$= \int_{\mathbf{R}^N} F(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + \left(1 - \alpha - \frac{v_s^2}{c^2}\right)V(|\psi|^2) dx.$$

It is easily seen that the right hand side of (2.19) is nonpositive, while the coefficients on the left hand side are nonnegative and at least one is positive. Since  $\nabla \psi \in L^2(\mathbf{R}^N)$ , we infer that  $\psi$  is constant.  $\square$

Notice that the assumptions of Theorem 2.4 are satisfied in the case  $F(s) = 1 - s$  as well as in the case  $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$ , where  $\alpha_i > 0$  and  $F$  has two positive roots. Hence the conclusion of Theorem 2.4 holds as well for the Gross-Pitaevskii equation and for the cubic-quintic NLS.

### 3. Existence of traveling waves for any subsonic speed ( $N \geq 3$ )

In addition to conditions (C1) and (C2) in the previous section, we introduce the following one:

**C3.** There exists  $p_0 < \frac{2^*}{2} - 2 = \frac{2}{N-2}$  and  $C > 0$  such that  $|F(s)| \leq Cs^{p_0}$  for any  $s > M_1$ .

The main result of [M09b] is

**Theorem 3.1.** *Let  $N \geq 3$ . Assume that (C1) and one of the conditions (C2) or (C3) are satisfied. Then for any  $c \in (0, v_s)$ , (1.1) has finite energy traveling waves of speed  $c$ .*

*Sketch of the proof.* If the conditions (C1) and (C2) in Section 2 are satisfied, the proof of Proposition 2.1 gives an uniform estimate of the  $L^\infty$  norm of the traveling waves: there exists a positive constant  $M$  (depending only on  $F$ ) such that any solution  $\psi$  of (1.4) satisfies  $|\psi(x)| \leq M$  on  $\mathbf{R}^N$ . Thus we may replace the function  $F$  by a function  $\tilde{F}$  such that  $F = \tilde{F}$  on  $[0, M_1]$ , where  $M_1 > M$ ,  $\tilde{F}$  satisfies (C1) and (C2) (perhaps with another constant  $\beta \in (0, \alpha)$  instead of  $\alpha$ ) and, moreover,  $\tilde{F}$  has a subcritical growth at infinity. If  $\psi$  satisfies the equation (1.4) with  $\tilde{F}$  instead of  $F$ , Proposition 2.1 guarantees again that  $|\psi| \leq M$ , and consequently  $\psi$  is a solution of (1.4). Hence we may always suppose that (C1) and (C3) are satisfied.

It can be shown that any finite-energy traveling wave tends to a constant of modulus  $r_0$  at infinity. Since eq. (1.4) is invariant under multiplication by complex numbers of modulus one, it suffices to search for solutions that tend to  $r_0$  at infinity. Hence we look for traveling waves of the form  $\psi = r_0 - u$ , where  $u \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then  $u$  satisfies the equation

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0 \quad \text{in } \mathbf{R}^N. \quad (3.1)$$

Formally, the solutions of (3.1) are precisely the critical points of the functional

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) dx, \quad (3.2)$$

where  $Q$  is the momentum with respect to  $x_1$ .

Denote  $a = \sqrt{-\frac{1}{2}F'(r_0^2)}$ , so that  $v_s = 2ar_0$  and we have the Taylor expansion (1.3) in a neighborhood of  $r_0^2$ . Then for  $u$  close to zero we may approximate  $V(|r_0 - u|^2)$  by  $a^2(|r_0 - u|^2 - r_0^2)^2$ . Fix a cut-off function  $\varphi \in C^\infty([0, \infty), \mathbf{R})$  as in the Introduction, such that  $\varphi(s) = s$  for  $s \in [0, 2r_0]$ ,  $\varphi$  is nondecreasing and  $\varphi(s) = 3r_0$  for  $s \geq 4r_0$ . Given a function  $u$  and a domain  $\Omega \subset \mathbf{R}^N$ , we consider the Ginzburg-Landau energy

$$E_{GL}^\Omega(u) = \int_\Omega |\nabla u|^2 dx + a^2 \int_\Omega (\varphi^2(|r_0 - u|) - r_0^2)^2 dx.$$

We denote  $E_{GL}(u) = E_{GL}^{\mathbf{R}^N}(u)$ . Taking (1.3) into account, the function space naturally associated to  $E_c$  is

$$\mathcal{X} = \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid E_{GL}(u) < \infty\} = \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid r_0 - u \in \tilde{\mathcal{E}}\}.$$

By the Sobolev embedding we have  $\mathcal{X} \subset L^{2^*}(\mathbf{R}^N)$ . Let  $u \in \mathcal{X}$ . If  $u(x)$  is close to zero, we have the estimate  $|V(|r_0 - u(x)|^2)| \leq C(\varphi^2(|r_0 - u|) - r_0^2)^2$  thanks to (1.3).

If  $u(x)$  is far away from zero, using (C3) we get the estimate  $|V(|r_0 - u(x)|^2)| \leq C|u|^{2^*}(x)$ . Hence  $V(|r_0 - u|^2) \in L^1(\mathbf{R}^N)$  for any  $u \in \mathcal{X}$ .

The next step is to define the momentum with respect to  $x_1$  for any function in  $\mathcal{X}$ . It is clear that if  $u \in H^1(\mathbf{R}^N)$  the momentum should be  $Q(u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle dx$ . On the other hand, if  $u \in \mathcal{X}$  is a function such that  $r_0 - u$  has a lifting  $r_0 - u = \rho e^{i\theta}$ , then we have formally  $Q(u) = - \int_{\mathbf{R}^N} \rho^2 \theta_{x_1} dx = - \int_{\mathbf{R}^N} (\rho^2 - r_0^2) \theta_{x_1} dx$ , where  $\rho^2 - r_0^2, \theta_{x_1} \in L^2(\mathbf{R}^N)$ .

The key point is the observation that for any  $u \in \mathcal{X}$  we have  $\langle iu_{x_1}, u \rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$ , where  $\mathcal{Y} = \{\partial_{x_1} \phi \mid \phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)\}$ . For  $v \in L^1(\mathbf{R}^N)$  and  $w \in \mathcal{Y}$ , let  $L(v + w) = \int_{\mathbf{R}^N} v dx$ . It is standard to check that  $L$  is well-defined and is a continuous linear form of  $L^1(\mathbf{R}^N) + \mathcal{Y}$ . This enables us to define

$$Q(u) = L(\langle iu_{x_1}, u \rangle) \quad \text{for any } u \in \mathcal{X}.$$

Then it is not hard to prove that the functional  $Q$  has very convenient properties for our variational approach.

Because of different scaling properties of the terms appearing in (3.2), we also introduce the functionals

$$\begin{aligned} A(u) &= \int_{\mathbf{R}^N} \sum_{k=2}^N \left| \frac{\partial u}{\partial x_k} \right|^2 dx, \\ B_c(u) &= \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) dx, \\ P_c(u) &= \frac{N-3}{N-1} A(u) + B_c(u). \end{aligned} \tag{3.3}$$

It is clear that  $E_c(u) = A(u) + B_c(u) = \frac{2}{N-1} A(u) + P_c(u)$ . By Proposition 2.2, any solution of (3.1) satisfies the Pohozaev identity  $P_c(u) = 0$ , and consequently  $B_c(u) = -\frac{N-3}{N-1} A(u)$ , which gives  $B_c(u) < 0$  if  $N \geq 4$ , and  $B_c(u) = 0$  if  $N = 3$ , respectively. Using the notation (2.3), it is easy to see that

$$\begin{aligned} E_c(v_{1,\sigma}) &= \sigma^{N-3} A(v) + \sigma^{N-1} B_c(v) \quad \text{and} \\ \frac{d}{d\sigma} (E_c(v_{1,\sigma})) &= (N-3)\sigma^{N-4} A(v) + (N-1)\sigma^{N-2} B_c(v). \end{aligned} \tag{3.4}$$

Assume that  $N \geq 4$ . Let  $v$  be a function such that  $B_c(v) < 0$ . From (3.4) we see that there is a unique  $\sigma_v > 0$  such that  $P_c(v_{1,\sigma_v}) = 0$ . Moreover, the function  $\sigma \mapsto E_c(v_{1,\sigma_v})$  is increasing on  $(0, \sigma_v]$  and decreasing on  $[\sigma_v, \infty)$ . This simple observation suggests that the functional  $E_c$  has a mountain-pass geometry and the manifold  $\{v \in \mathcal{X} \mid v \neq 0, P_c(v) = 0\}$  separates two regions in  $\mathcal{X}$  where  $E_c$  takes lower values. Consequently, it would be interesting to minimize  $E_c$  under the constraint  $P_c = 0$ . This is precisely our approach to find critical points of  $E_c$ .

An essential tool in proving Theorem 3.1 is a "regularization procedure" which enables us to get rid of the small-scale topological defects of functions in  $\mathcal{X}$ . More

precisely, given  $u \in \mathcal{X}$ ,  $h > 0$  and a domain  $\Omega \subset \mathbf{R}^N$ , we consider the functional

$$G_{h,\Omega}^u(v) = E_{GL}^\Omega(v) + \frac{1}{h^2} \int_{\Omega} \varphi \left( \frac{|v-u|^2}{32r_0} \right) dx.$$

We prove that  $G_{h,\Omega}^u$  has minimizers in the set

$$\{v \in \mathcal{X} \mid v = u \text{ on } \mathbf{R}^N \setminus \Omega, v - u \in H_0^1(\Omega)\}.$$

Of course that we cannot expect for minimizers to be unique. However, for any choice of a minimizer  $v_h$  of the above functional, we have:

- $\|v_h - u\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$  as  $h \longrightarrow 0$ ,
- We may estimate  $\| |v_h - r_0| - r_0 \|_{L^\infty(\omega)}$  in terms of  $h$  and  $E_{GL}^\Omega(u)$  on any compact subset  $\omega \subset \Omega$  and we find that  $\| |v_h - r_0| - r_0 \|_{L^\infty(\omega)}$  is arbitrarily small if the Ginzburg-Landau energy  $E_{GL}^\Omega(u)$  is sufficiently small.

Using this regularization procedure, we prove:

**Lemma 3.2.** *Assume that  $0 \leq c < v_s$  and let  $\varepsilon \in (0, 1 - \frac{c}{v_s})$ . There exists  $K > 0$  such that for any  $u \in \mathcal{X}$  with  $E_{GL}(u) < K$  we have*

$$E_c(u) > \varepsilon E_{GL}(u).$$

*Idea of proof.* Let  $\delta > 0$  be sufficiently small, so that  $\delta < \frac{r_0}{2}$  and  $\frac{c}{2a(r_0-\delta)} < 1 - \varepsilon$  (such  $\delta$  exist because  $v_s = 2ar_0$  and  $\varepsilon < 1 - \frac{c}{v_s}$ ).

Assume first that a function  $v \in \mathcal{X}$  satisfies  $r_0 - \delta < |r_0 - v| \leq r_0 + \delta$  on  $\mathbf{R}^N$ . Then  $r_0 - v$  admits a lifting  $r_0 - v = \rho e^{i\theta}$  and we have  $|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2$  and  $Q(v) = - \int_{\mathbf{R}^N} (\rho^2 - r_0^2) \theta_{x_1} dx$ . Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \frac{c}{1-\varepsilon} |Q(v)| &\leq 2a(r_0 - \delta) |Q(v)| \leq 2a(r_0 - \delta) \|\theta_{x_1}\|_{L^2(\mathbf{R}^N)} \|\rho^2 - r_0^2\|_{L^2(\mathbf{R}^N)} \\ &\leq (r_0 - \delta)^2 \int_{\mathbf{R}^N} |\theta_{x_1}|^2 dx + a^2 \int_{\mathbf{R}^N} (\rho^2 - r_0^2)^2 dx \\ &\leq \int_{\mathbf{R}^N} \rho^2 |\nabla \theta|^2 + a^2 (\rho^2 - r_0^2)^2 dx \leq E_{GL}(v). \end{aligned}$$

Consequently  $E_{GL}(v) - c|Q(v)| > \varepsilon E_{GL}(v)$ .

One can prove that  $\int_{\mathbf{R}^N} V(|r_0 - v|^2) dx$  is arbitrarily close to  $a^2 \int_{\mathbf{R}^N} (\varphi^2(|r_0 - v|) - r_0^2)^2 dx$  if  $E_{GL}(v)$  is small enough and we infer that  $v$  satisfies the conclusion of Lemma 3.2.

In the general case, let  $u \in \mathcal{X}$  be a function with small Ginzburg-Landau energy. First we choose a small  $h > 0$ , then we choose a minimizer  $v_h$  of  $G_{h,\mathbf{R}^N}^u$ . If the Ginzburg-Landau energy  $E_{GL}(u)$  is sufficiently small, we have  $\| |v_h - r_0| - r_0 \|_{L^\infty(\mathbf{R}^N)} < \delta$ , therefore the conclusion of Lemma 3.2 holds for  $v_h$ . If  $h$  has been chosen sufficiently small,  $v_h$  is close to  $u$  and we can prove that the conclusion of Lemma 3.2 also holds for  $u$ .

Using Lemma 3.2, it is easy to see that for any  $k > 0$ , the functional  $E_c$  is bounded on  $\{u \in \mathcal{X} \mid E_{GL}(u) \leq k\}$ . Let

$$E_{c,min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X}, E_{GL}(u) = k\}.$$

**Lemma 3.3.** *Assume that  $0 < c < v_s$ . The function  $E_{c,min}$  has the following properties:*

- i) *There exists  $k_0 > 0$  such that  $E_{c,min}(k) > 0$  for any  $k \in (0, k_0)$ .*
- ii) *We have  $\lim_{k \rightarrow \infty} E_{c,min}(k) = -\infty$ .*
- iii) *For any  $k > 0$  we have  $E_{c,min}(k) < k$ .*

Part (i) follows directly from Lemma 3.2. Notice that in the case  $c > v_s$ , Lemmas 3.2 and 3.3 (i) do not hold. More precisely, one can prove that the function  $k \mapsto E_{c,min}(k)$  is decreasing (and negative) on  $(0, \infty)$ .

From Lemma 3.3 we infer that

$$S_c := \sup \{E_{c,min}(k) \mid k > 0\} > 0. \quad (3.5)$$

**Lemma 3.4.** *The set  $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}$  is not empty and we have*

$$T_c := \inf \{E_c(u) \mid u \in \mathcal{C}\} \geq S_c > 0.$$

*Proof.* Let  $w \in \mathcal{X}$  be such that  $E_c(w) < 0$  (such functions exist by Lemma 3.3 (ii)). Then  $P_c(w) = E_c(w) - \frac{2}{N-1}A(w) < 0$  and we have

$$P_c(w_{\sigma,1}) = \frac{1}{\sigma} \int_{\mathbf{R}^N} \left| \frac{\partial w}{\partial x_1} \right|^2 dx + \frac{N-3}{N-1} \sigma A(w) + cQ(w) + \sigma \int_{\mathbf{R}^N} V(|r_0 - w|^2) dx. \quad (3.6)$$

Since  $P_c(w_{1,1}) = P_c(w) < 0$  and  $\lim_{\sigma \rightarrow 0} P_c(w_{\sigma,1}) = \infty$ , there exists  $\sigma_0 \in (0, 1)$  such that  $P_c(w_{\sigma_0,1}) = 0$ , that is  $w_{\sigma_0,1} \in \mathcal{C}$ .

For the second part, assume first that  $N \geq 4$ . Let  $v \in \mathcal{C}$ . Then  $A(v) > 0$  and  $B_c(v) = -\frac{N-3}{N-1}A(v) < 0$ . Using (3.4) we infer that  $\sigma \mapsto E_c(v_{1,\sigma})$  is increasing on  $(0, 1]$  and decreasing on  $[1, \infty)$ , therefore it achieves its maximum at  $\sigma = 1$ . Fix  $k > 0$ . It is easy to see that there exists a unique  $\sigma(k, v) > 0$  such that  $E_{GL}(v_{1,\sigma(k,v)}) = k$ . Then

$$E_{c,min}(k) \leq E_c(v_{1,\sigma(k,v)}) \leq E_c(v_{1,1}) = E_c(v).$$

Take the sup for  $k \geq 0$  in the last inequality to get  $S_c \leq E_c(v)$ .

Now consider the case  $N = 3$ . Let  $v \in \mathcal{C}$ . Then  $E_c(v_{1,\sigma}) = E_c(v) = A(v) = \text{constant}$  for  $\sigma > 0$ . Let  $k > 0$ . We distinguish two cases:

- If  $A(v) \geq k$ , we have  $E_c(v) = A(v) \geq k > E_{c,min}(k)$  by Lemma 3.3 (iii).
- If  $A(v) < k$ , there exists a unique  $\sigma(k, v) > 0$  such that  $E_{GL}(v_{1,\sigma(k,v)}) = k$ . Then  $E_c(v) = E_c(v_{1,\sigma(k,v)}) \geq E_{c,min}(k)$ .

In both cases we get  $E_c(v) \geq E_{c,min}(k)$  for any  $k > 0$  and  $v \in \mathcal{C}$ . Lemma 3.4 is proved.  $\square$

**Lemma 3.5.** *Let  $T_c$  be as in Lemma 3.4. Then:*

- i) *For any  $w \in \mathcal{X}$  with  $P_c(w) < 0$  we have  $A(w) > \frac{N-1}{2}T_c$ .*
- ii) *Let  $(u_n)_{n \geq 1} \subset \mathcal{X}$  be a sequence such that  $(E_{GL}(u_n))_{n \geq 1}$  is bounded and  $\lim_{n \rightarrow \infty} P_c(u_n) = \mu < 0$ . Then  $\liminf_{n \rightarrow \infty} A(u_n) > \frac{N-1}{2}T_c$ .*

*Proof.* We only prove part (i). Recall that  $P_c(w_{\sigma,1})$  is given by (3.6). As in the proof of Lemma 3.3, there exists  $\sigma_0 \in (0, 1)$  such that  $P_c(w_{\sigma_0,1}) = 0$ , and consequently  $w_{\sigma_0,1} \in \mathcal{C}$ . Then we have  $E_c(w_{\sigma_0,1}) \geq T_c$  and we infer that  $A(w_{\sigma_0,1}) = \frac{N-1}{2}(E_c(w_{\sigma_0,1}) - P_c(w_{\sigma_0,1})) \geq \frac{N-1}{2}T_c$ . This implies  $A(w) = \frac{1}{\sigma_0}A(w_{\sigma_0,1}) \geq \frac{N-1}{2} \frac{1}{\sigma_0}T_c > \frac{N-1}{2}T_c$ .  $\square$

In order to prove Theorem 3.1, we show that the functional  $E_c$  has a minimizer in  $\mathcal{C}$ , then we prove that the minimizers satisfy (1.4). The proofs are quite different in the case  $N = 3$  and in the case  $N \geq 4$ . We begin with the simpler case  $N \geq 4$ .

**Theorem 3.6.** *Let  $N \geq 4$ . Let  $(u_n)_{n \geq 1} \subset \mathcal{X} \setminus \{0\}$  be a sequence satisfying the following properties:*

$$P_c(u_n) \longrightarrow 0 \quad \text{and} \quad E_c(u_n) \longrightarrow T_c \quad \text{as } n \longrightarrow \infty. \quad (3.7)$$

*Then there exists a subsequence  $(u_{n_k})_{k \geq 1}$ , a sequence  $(x_k)_{k \geq 1} \subset \mathbf{R}^N$  and  $u \in \mathcal{C}$  such that*

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad \varphi^2(|r_0 - u_{n_k}(\cdot + x_k)|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \quad \text{in } L^2(\mathbf{R}^N).$$

*Moreover, we have  $E_c(u) = T_c$ , that is  $u$  minimizes  $E_c$  in  $\mathcal{C}$ .*

*Sketch of proof.* Since  $A(u_n) = \frac{N-1}{2}(E_c(u_n) - P_c(u_n)) \longrightarrow \frac{N-1}{2}T_c$ , from (3.7) we infer that  $(A(u_n))_{n \geq 1}$  is bounded. Then we prove that  $(E_{GL}(u_n))_{n \geq 1}$  is bounded. We use the concentration-compactness method of P.-L. Lions [Lio84] to prove the convergence of a subsequence of  $(u_n)_{n \geq 1}$ .

Passing to a subsequence, we may assume that  $E_{GL}(u_n) \longrightarrow \alpha_0 > 0$  as  $n \longrightarrow \infty$ . Let  $q_n(t)$  be the concentration function of  $E_{GL}(u_n)$ , that is

$$q_n(t) = \sup_{x \in \mathbf{R}^N} E_{GL}^{B(x,t)}(u_n).$$

Each  $q_n$  is a nondecreasing function on  $[0, \infty)$  and tends to  $E_{GL}(u_n)$  as  $t \longrightarrow \infty$ . By a standard argument, there is a subsequence (still denoted  $(u_n)_{n \geq 1}$ ) and a non-decreasing function  $q : [0, \infty) \longrightarrow \mathbf{R}_+$  such that  $q_n(t) \longrightarrow q(t)$  as  $n \longrightarrow \infty$  for a.e.  $t \in [0, \infty)$ .

Let  $\alpha = \lim_{t \rightarrow \infty} q(t)$ . It is obvious that  $\alpha \in [0, \alpha_0]$ . Our aim is to prove that the energy of  $u_n$  "concentrates," that is  $\alpha = \alpha_0$ .

The fact that  $\alpha > 0$  follows from the next lemma.

**Lemma 3.7.** *Let  $(u_n)_{n \geq 1} \subset \mathcal{X}$  be a sequence satisfying:*

a)  $M_1 \leq E_{GL}(u_n) \leq M_2$  for some positive constants  $M_1, M_2$ .

b)  $\lim_{n \rightarrow \infty} P_c(u_n) = 0$ .

*There exists  $k > 0$  such that  $\sup_{y \in \mathbf{R}^N} E_{GL}^{B(y,1)}(u_n) \geq k$  for all sufficiently large  $n$ .*

The proof of Lemma 3.7 is delicate. It relies on the regularization procedure described above and on Lieb's lemma ([Li83]). The main ideas are as follows:

1. Assume, by contradiction, that  $\lim_{n \rightarrow \infty} \sup_{x \in \mathbf{R}^N} E_{GL}^{B(x,1)}(u_n) = 0$ .

Then we prove that there is a sequence  $h_n \longrightarrow 0$  and for each  $n$  there is a minimizer  $v_n$  of  $G_{h_n, \mathbf{R}^N}^{u_n}$  such that

$$\| |v_n - r_0| - r_0 \|_{L^\infty(\mathbf{R}^N)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (3.8)$$

2. Let  $\varepsilon \in (0, 1 - \frac{c}{v_s})$ . Using (3.8), we argue as in the proof of Lemma 3.2 to show that for  $n$  sufficiently large we have

$$\begin{aligned} & \int_{\mathbf{R}^N} \left| \frac{\partial v_n}{\partial x_1} \right|^2 dx + \frac{N-3}{N-1} A(v_n) + \int_{\mathbf{R}^N} V(|r_0 - v_n|^2) dx + cQ(u_n) \\ & \geq \varepsilon \left( \int_{\mathbf{R}^N} \left| \frac{\partial v_n}{\partial x_1} \right|^2 dx + \frac{N-3}{N-1} A(v_n) + a^2 \int_{\mathbf{R}^N} (\varphi^2(|r_0 - v_n|) - r_0^2)^2 dx \right) \end{aligned} \quad (3.9)$$

3. Since  $h_n \rightarrow 0$ , it is clear that  $v_n$  is close to  $u_n$  for large  $n$ , hence (3.9) holds as well for  $(u_n)$  instead of  $v_n$  and for some  $\varepsilon_1 \in (0, \varepsilon)$  instead of  $\varepsilon$ . This is in contradiction with  $P_c(u_n) \rightarrow 0$  and  $E_{GL}(u_n) \geq M_1 > 0$ .

The next step is to prove that  $\alpha \notin (0, \alpha_0)$ . We argue again by contradiction and we suppose that  $\alpha \in (0, \alpha_0)$ . Then a standard argument shows that there exists a sequence  $R_n \rightarrow \infty$  and a sequence  $(x_n)_{n \geq 1} \subset \mathbf{R}^N$  such that

$$E_{GL}^{B(x_n, R_n)}(u_n) \rightarrow \alpha \quad \text{and} \quad E_{GL}^{\mathbf{R}^N \setminus B(x_n, 2R_n)}(u_n) \rightarrow \alpha_0 - \alpha. \quad (3.10)$$

Obviously, (3.10) implies

$$E_{GL}^{B(x_n, 2R_n) \setminus B(x_n, R_n)}(u_n) \rightarrow 0.$$

Since the Ginzburg-Landau energy of  $u_n$  in the annulus  $B(x_n, 2R_n) \setminus B(x_n, R_n)$  is small, using again the regularization procedure we infer that for each sufficiently large  $n$  there exist two functions  $u_{n,1}$  and  $u_{n,2}$  such that  $r_0 - u_{n,1} = e^{i\theta_n}(r_0 - u_n)$  on  $B(x_n, R_n)$  (where  $\theta_n$  is a constant),  $\text{supp}(u_{n,1}) \subset B(x_n, 2R_n)$ ,  $u_{n,2} = u_n$  on  $\mathbf{R}^N \setminus B(x_n, 2R_n)$ ,  $u_{n,2}$  is constant on  $B(x_n, R_n)$  and

$$E_{GL}(u_{n,1}) \rightarrow \alpha \quad \text{and} \quad E_{GL}(u_{n,2}) \rightarrow \alpha_0 - \alpha, \quad (3.11)$$

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \rightarrow 0, \quad (3.12)$$

$$|P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

It is easy to see that the sequences  $(P_c(u_{n,i}))_{n \geq 1}$  are bounded,  $i = 1, 2$ . Passing again to a subsequence, we may assume that there exist two constants  $p_1, p_2$  such that

$$P_c(u_{n,1}) \rightarrow p_1 \quad \text{and} \quad P_c(u_{n,2}) \rightarrow p_2 \quad \text{as } n \rightarrow \infty.$$

Then (3.13) implies that  $p_1 + p_2 = 0$  and we distinguish two cases:

Case a: One of  $p_i$ 's is negative, for instance  $p_1 < 0$ . By Lemma 3.5 (ii) we get  $\liminf_{n \rightarrow \infty} A(u_{n,1}) > \frac{N-1}{2} T_c$ . Then (3.12) implies  $\liminf_{n \rightarrow \infty} A(u_n) > \frac{N-1}{2} T_c$  and using the fact that  $P_c(u_n) \rightarrow 0$  we find  $\liminf_{n \rightarrow \infty} E_c(u_n) > T_c$ , in contradiction with the assumption of Theorem 3.6.

Case b: We have  $p_1 = p_2 = 0$ . Then we use the next result.

**Lemma 3.8.** *Let  $(u_n)_{n \geq 1} \subset \mathcal{X}$  be a sequence satisfying the following properties:*

- a) *There are  $C_1, C_2 > 0$  such that  $C_1 \leq E_{GL}(u_n)$  and  $A(u_n) \leq C_2$  for any  $n \geq 1$ .*
- b)  *$P_c(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Then  $\liminf_{n \rightarrow \infty} E_c(u_n) \geq T_c$ , where  $T_c$  is as in Lemma 3.4.*

If  $p_1 = p_2 = 0$ , by Lemma 3.8 we get  $\liminf_{n \rightarrow \infty} E_c(u_{n,i}) \geq T_c$  for  $i = 1, 2$ . Using (3.12) and (3.13) we find  $\liminf_{n \rightarrow \infty} E_c(u_n) \geq 2T_c$ , which is a contradiction.

From the preceding arguments we infer that  $\lim_{t \rightarrow \infty} q(t) = \alpha_0$ . Then it is standard to prove that there is a sequence  $(x_n)_{n \geq 1} \subset \mathbf{R}^N$  such that, denoting  $\tilde{u}_n = u_n(\cdot + x_n)$ , we have:

$$\text{for any } \varepsilon > 0, \text{ there exist } R_\varepsilon > 0 \text{ and } n_\varepsilon \in \mathbf{N}^* \text{ such that} \quad (3.14)$$

$$E_{GL}^{\mathbf{R}^N \setminus B(0, R_\varepsilon)}(\tilde{u}_{n_k}) < \varepsilon \text{ for any } n \geq n_\varepsilon.$$

Since  $(E_{GL}(\tilde{u}_n))_{n \geq 1}$  is bounded, there is a subsequence  $(\tilde{u}_{n_k})_{k \geq 1}$  such that

$$\begin{aligned} \tilde{u}_{n_k} &\rightharpoonup u && \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N), \\ \tilde{u}_{n_k} &\longrightarrow u && \text{in } L_{loc}^p(\mathbf{R}^N) \text{ and a.e. on } \mathbf{R}^N. \end{aligned} \quad (3.15)$$

It follows that  $u \in \mathcal{X}$  and  $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2$  weakly in  $L^2(\mathbf{R}^N)$ .

Then we prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_{n_k}|^2) dx = \int_{\mathbf{R}^N} V(|r_0 - u|^2) dx \quad (3.16)$$

and

$$\lim_{k \rightarrow \infty} Q(\tilde{u}_{n_k}) = Q(u). \quad (3.17)$$

Using (3.15), (3.16), (3.17) and Lemma 3.5 (i) we prove that the subsequence  $(\tilde{u}_{n_k})_{k \geq 1}$  satisfies the conclusion of Theorem 3.6.  $\square$

**Proposition 3.9.** *Assume that  $N \geq 4$ ,  $0 \leq c < v_s$ , (C1) and one of the conditions (C2) or (C3) are satisfied. Let  $u$  be a minimizer of  $E_c$  in  $\mathcal{C}$ . Then  $u \in W_{loc}^{2,p}(\mathbf{R}^N)$ ,  $\nabla u \in W^{1,p}(\mathbf{R}^N)$  for  $p \in [2, \infty)$  and  $u$  is a solution of (3.1).*

*Proof.* It is obvious that  $u$  minimizes the functional  $A$  under the constraint  $P_c = 0$  and then it is not hard to see that  $u$  satisfies an Euler-Lagrange equation  $A'(u) = \alpha P'_c(u)$ .

We claim that  $\alpha < 0$ . Indeed, suppose that  $\alpha > 0$ . Let  $w$  be such that  $P'_c(u) \cdot w > 0$ . Then for  $t < 0$  and  $t$  close to zero we have  $P_c(u + tw) < 0$  and  $A(u + tw) < A(u) = \frac{N-1}{2}T_c$ , in contradiction with Lemma 3.5 (i). We cannot have  $\alpha = 0$  because this would imply  $A'(u) = 0$ , and thus  $u = 0$ . Consequently, we have  $\alpha < 0$  and the Euler-Lagrange equation above is equivalent to

$$-\frac{\partial^2 u}{\partial x_1^2} - \left( \frac{N-3}{N-1} - \frac{1}{\alpha} \right) \sum_{k=2}^N \frac{\partial^2 u}{\partial x_k^2} + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0. \quad (3.18)$$

As in Proposition 2.2 we prove that  $u$  satisfies a Pohozaev identity analogous to (2.2):

$$\frac{N-3}{N-1} \left( \frac{N-3}{N-1} - \frac{1}{\alpha} \right) A(u) + B_c(u) = 0. \quad (3.19)$$

From (3.19) and from the fact that  $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$  we infer that  $\frac{1}{\alpha} = -\frac{2}{N-1}$  and  $u$  satisfies (3.1). Finally, the regularity of  $u$  follows from Proposition 2.1.  $\square$

In the three dimensional case the proof follows the same lines, but is technically more difficult. This is mainly due to the invariance of the functionals  $A$  and  $B_c$  with



respect to dilations in the  $(x_2, x_3)$ -variables and to the fact that in three dimensions we have  $P_c = B_c$ , thus  $P_c$  does not contain the terms  $\int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_2} \right|^2 dx$  and  $\int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_3} \right|^2 dx$ .

For  $v \in \mathcal{X}$  we denote

$$D(v) = \int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_1} \right|^2 dx + a^2 \int_{\mathbf{R}^3} \left( \varphi^2(|r_0 - v|) - r_0^2 \right)^2 dx.$$

It is obvious that for any  $v \in \mathcal{X}$  and  $\sigma > 0$  we have

$$A(v_{1,\sigma}) = A(v), \quad B_c(v_{1,\sigma}) = \sigma^2 B_c(v) \quad \text{and} \quad D(v_{1,\sigma}) = \sigma^2 D(v). \quad (3.20)$$

Unlike in the case  $N \geq 4$ , (3.20) implies that there are sequences  $(u_n)_{n \geq 1} \subset \mathcal{C}$  such that  $E_c(u_n) \rightarrow T_c$  and  $D(u_n) \rightarrow \infty$ , and consequently  $E_{GL}(u_n) \rightarrow \infty$ . However, (3.20) also implies that there are sequences  $(u_n)_{n \geq 1} \subset \mathcal{C}$  such that  $E_c(u_n) \rightarrow T_c$  and  $D(u_n) = 1$  for each  $n$ .

Let

$$\Lambda_c = \left\{ \lambda \in \mathbf{R} \mid \text{there is a sequence } (u_n)_{n \geq 1} \subset \mathcal{X} \text{ such that } \right. \\ \left. D(u_n) \geq 1, B_c(u_n) \rightarrow 0 \text{ and } A(u_n) \rightarrow \lambda \text{ as } n \rightarrow \infty \right\}.$$

Let  $\lambda_c = \inf \Lambda_c$ . It is easy to see that  $T_c \in \Lambda_c$ , thus  $\lambda_c \leq T_c$ . One can prove that  $\lambda_c \geq S_c$ , where  $S_c$  is given by (3.5), but we do not know whether  $S_c = T_c$ .

Our main result is

**Theorem 3.10.** *Let  $N = 3$ . Let  $(u_n)_{n \geq 1} \subset \mathcal{X}$  be a sequence satisfying*

$$D(u_n) \rightarrow 1, \quad B_c(u_n) \rightarrow 0 \quad \text{and} \quad A(u_n) \rightarrow \lambda_c \quad \text{as } n \rightarrow \infty. \quad (3.21)$$

*There exists a subsequence  $(u_{n_k})_{k \geq 1}$ , a sequence  $(x_k)_{k \geq 1} \subset \mathbf{R}^3$  and a function  $u \in \mathcal{C}$  such that*

$$\nabla u_{n_k}(\cdot + x_k) \rightarrow \nabla u \quad \text{and} \quad |r_0 - u_{n_k}(\cdot + x_k)|^2 - r_0^2 \rightarrow |r_0 - u|^2 - r_0^2 \quad \text{in } L^2(\mathbf{R}^3).$$

*Moreover, we have  $E_c(u) = A(u) = T_c = \lambda_c$  and  $u$  is a minimizer of  $E_c$  in  $\mathcal{C}$ .*

If  $u$  minimizes  $E_c$  in  $\mathcal{C}$ , proceeding as in the proof of Proposition 3.9 one can show that there exists  $\alpha < 0$  such that  $A'(u) = \alpha B'_c(u)$ . Then it is not hard to see that there is  $\sigma > 0$  such that  $u_{1,\sigma}$  solves (3.1). The smoothness of solutions of (3.1) follows from Proposition 2.1.

Finally, Lemma 3.5 implies that any minimizer of  $E_c$  in  $\mathcal{C}$  is also a minimizer of the functional  $P_c$  under the constraint  $A = \frac{N-1}{2} T_c$ . Then it is an easy consequence of the general symmetry results in [M09a] that after a translation, any of these minimizers is axially symmetric with respect to  $Ox_1$ .

## 4. Further results and remarks

The geometry of the functional  $E_c$  is different in the two-dimensional case and the approach to find traveling waves presented in the previous section fails. It can be proved that Lemmas 3.2 and 3.3 still hold and this implies that  $E_c$  has a mountain-pass geometry for any  $c \in (0, v_s)$ . This suggests that finite energy traveling waves should exist for any subsonic speed.

However, if  $\psi = r_0 - u$  is a traveling wave, it satisfies the Pohozaev identities (2.1) and (2.2). Let  $v \in \mathcal{X}$ . It is easy to see that if  $\varphi = r_0 - v$  satisfies (2.1), the function  $\lambda \mapsto E_c(v_{\lambda,1})$  achieves its minimum at  $\lambda = 1$ , and if  $r_0 - v$  satisfies (2.2),

then  $\sigma \mapsto E_c(v_{1,\sigma})$  reaches its minimum at  $\sigma = 1$ . It can be proved that  $E_c$  has no minimizers in the set  $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}$ . Even if we consider the smaller set of functions that satisfy the two Pohozaev identities,  $\mathcal{C}_0 = \{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0 \text{ and } -\int_{\mathbf{R}^2} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + \int_{\mathbf{R}^2} \left| \frac{\partial u}{\partial x_2} \right|^2 dx + \int_{\mathbf{R}^2} V(|r_0 - u|^2) dx = 0\}$ , the infimum of  $E_c$  on  $\mathcal{C}_0$  is zero and it is never achieved.

A natural idea would be to use other constraints. For instance, it is possible to minimize  $E_c$  under the constraint  $E_{GL} = k$ . Then one would try to adjust the value of  $k$  in order to get traveling waves. Unfortunately it seems extremely difficult to control the Lagrange multipliers associated to minimizers. Notice that the advantage of minimizing a functional in the set of functions that satisfy a Pohozaev identity associated to this functional (as in the previous section) is that minimizers automatically have the "good" Lagrange multiplier. This observation is valid in a much more general context (see [M10]).

In the two-dimensional case, assumption (C3) becomes  $|F(s)| \leq Cs^{p_0}$  for large  $s$  and some  $p_0 \in \mathbf{R}$ . If it is satisfied, for any  $k > 0$  the functional  $E_c$  has minimizers in the set of functions  $u$  such that  $\int_{\mathbf{R}^2} |\nabla u|^2 dx = k$ . Then we use scaling properties in order to get rid of the Lagrange multipliers associated to minimizers. Following this approach, we prove:

**Theorem 4.1.** ([CM10]) *Let  $N = 2$ . Assume that  $F$  satisfies (C1) and one of the assumptions (C2) or (C3). There exists a set  $A \subset (0, v_s)$  such that 0 and  $v_s$  are limit points of  $A$  and Eq. (1.1) has finite energy traveling waves for any  $c \in A$ .*

Since both the energy  $E$  and the momentum  $Q$  are conserved by the flow associated to (1.1), it is very natural to minimize  $E$  at constant momentum. More precisely, consider the problem

$$(\mathcal{P}_p) \quad \text{minimize } E_c(u) \text{ in the set } \{u \in \mathcal{X} \mid Q(u) = p\}.$$

Clearly, any minimizer of  $(\mathcal{P}_p)$  satisfies an Euler-Lagrange equation  $E'(u) + c(u)Q'(u) = 0$ , thus any minimizer is a traveling wave and its speed is precisely its Lagrange multiplier.

In the case of the Gross-Pitaevskii equation, it has been proved in [BGS09] that  $(\mathcal{P}_p)$  admits solutions for any  $p > 0$  in dimension two, respectively for any  $p > p_0$  (where  $p_0 > 0$ ) in dimension three. The minimizers were obtained via a limit process. The authors considered the same problem on tori  $R\mathbf{T}^N$  ( $N = 2, 3$ ). They proved the existence of minimizers on  $R\mathbf{T}^N$  and obtained estimates on these minimizers uniformly in  $R$ , then passed to the limit as  $R \rightarrow \infty$  in order to get minimizers on  $\mathbf{R}^N$ . This approach clearly gives the existence of minimizers, but it does not imply the compactness of all minimizing sequences of  $(\mathcal{P}_p)$ , therefore it does not give the orbital stability of the set of minimizers.

Denote

$$E_{min}(p) = \inf\{E(u) \mid u \in \mathcal{X}, Q(u) = p\}. \quad (4.1)$$

If  $V \geq 0$ , it can be proved that the function  $E_{min}$  is continuous, increasing, concave and tends to infinity as  $p$  goes to infinity. If there exists  $s \geq 0$  such that  $V(s) < 0$ , it can be proved that  $E_{min}(p) = -\infty$  for any  $p > 0$ , hence  $(\mathcal{P}_p)$  does not admit solutions. We have the following result:

**Theorem 4.2.** ([CM10]) *Let  $N \geq 2$ . Assume that the assumptions (C1) and (C3) are satisfied and, moreover,  $V > 0$  on  $[0, r_0^2) \cup (r_0^2, \infty)$ . There exists  $p_0 \geq 0$  (with  $p_0 = 0$  if  $N = 2$ ) such that for any  $p > p_0$ , any minimizing sequence of  $(\mathcal{P}_p)$  has a convergent subsequence.*

*More precisely, for any sequence  $(u_n)_{n \geq 1} \subset \mathcal{X}$  such that  $Q(u_n) \rightarrow p$  and  $E(u_n) \rightarrow E_{\min}(p)$ , there exist a subsequence  $(u_{n_k})_{k \geq 1}$ , a sequence  $(x_k)_{k \geq 1} \subset \mathbf{R}^N$  and a function  $u \in \mathcal{X}$  such that*

$$\nabla u_{n_k}(\cdot + x_k) \rightarrow \nabla u \quad \text{and} \quad \varphi^2(|r_0 - u_{n_k}(\cdot + x_k)|) - r_0^2 \rightarrow \varphi^2(|r_0 - u|) - r_0^2 \quad \text{in } L^2(\mathbf{R}^N).$$

*Moreover,  $u$  is a solution of  $(\mathcal{P}_p)$ .*

In particular, Theorem 4.2 implies that for any  $p > p_0$ , Eq. (1.1) admits traveling waves  $u$  having momentum  $p$ , and moving with a speed  $c(u)$ . Let  $\mathcal{S}_p$  be the set of minimizers of  $(\mathcal{P}_p)$ . It follows from Theorem 4.2 and a classical result in [CaLi82] that  $\mathcal{S}_p$  is orbitally stable under the flow of (1.1).

As we can see, the main advantage of considering the problem  $(\mathcal{P}_p)$  is that the set of minimizers is orbitally stable. Its disadvantages are the difficulty to control Lagrange multipliers (which prevents us to prove that the speeds of traveling waves found in this way cover a whole interval) and the fact that  $(\mathcal{P}_p)$  does not have global minimizers if  $V$  takes negative values.

The following result establishes the relationship between the minimizers of  $(\mathcal{P}_p)$  and the traveling waves obtained in section 3.

**Proposition 4.3.** ([CM10]) *Let  $N \geq 3$ . Under the assumptions of Theorem 4.2, suppose that  $u$  is a minimizer of  $(\mathcal{P}_p)$  (with  $p > p_0$ ) and that  $u$  satisfies the Euler-Lagrange equation  $E'(u) + c(u)Q'(u) = 0$ . Then  $u$  minimizes  $E_{c(u)}$  under the Pohozaev constraint  $P_{c(u)} = 0$ .*

The converse of Proposition 4.3 is not true in general. For instance, in the case of the three dimensional Gross-Pitaevskii equation it has been proved in [BGS09] that there exists a critical speed  $v_0 < v_s$  such that any minimizer  $u$  of  $(\mathcal{P}_p)$  (for some  $p > 0$ ) has a speed  $c(u) \leq v_0$ . However, we have seen in section 3 that for any  $c \in (v_0, v_s)$  there are minimizers of  $E_c$  under the constraint  $P_c = 0$  and any of them is a traveling wave of speed  $c$ .

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