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Applications of a metaplectic calculus to Schrödinger evolutions with non-self-adjoint generators

Joe Viola

Abstract

We review the calculus of metaplectic operators and shifts in phase space applied to Gaussian wave packets. Using holomorphic extensions of this calculus, one can reduce the L^2 theory of evolution equations with non-selfadjoint quadratic generators to symplectic linear algebra. We illustrate these methods through an application to the quantum harmonic oscillator with complex perturbation ix .

1. Introduction

The goal of this note is to discuss methods of complexifying the metaplectic group and shifts on phase space to analyze the Schrödinger evolution of certain non-self-adjoint generators. The author has studied these methods recently (with coauthors) in works like [1], [2], [16], [17]. The methods are classical, particularly in the case of self-adjoint generators, and the study of non-self-adjoint operators has seen renewed interest in recent years [15]. The present discussion draws particularly on classical works like [6], [8], [13], though this is far from a complete list.

We will present these ideas through an application to the evolution corresponding to

$$P = \frac{1}{2}((D_x - iv_\xi)^2 + (x - iv_x)^2) \quad (1.1)$$

where $D_x = -i\frac{d}{dx}$ and $\mathbf{v} = (v_x, v_\xi) \in \mathbb{R}^2$ is fixed. The generator P is a complex perturbation of the quantum harmonic oscillator

$$Q_0 = \frac{1}{2}(D_x^2 + x^2). \quad (1.2)$$

The operator P is one of the simplest examples of a non-self-adjoint perturbation of a self-adjoint operator. It has many interesting spectral and pseudo-spectral properties [9], [11] and appears in the study of hypoelliptic operators as a simple and powerful model [4], [5], [12].

In [2, Prop. 2.23], it is shown that the solution operator

$$\exp(-itP) : u(x) \mapsto U(t, x)$$

solving

$$\begin{cases} (-i\partial_t + P)U(t, x) = 0, & t \in \mathbb{C}, \\ U(0, x) = u(x) \end{cases}$$

has a maximal weak extension for $t \in \mathbb{C}$, as recalled in Example 4.3. When $\exp(-itQ_0)$ is compact on $L^2(\mathbb{R})$ (that is, when $\text{Im } t < 0$), the operator $\exp(-itP)$ is also compact, and when $\exp(-itQ_0)$ is unbounded on $L^2(\mathbb{R})$ (meaning $\text{Im } t > 0$), the operator $\exp(-itP)$ is also unbounded. When $\text{Im } t < 0$, the $L^2(\mathbb{R})$ theory of $\exp(-itP)$ (its norm, singular value decomposition, etc.) is known [17]. Our goal here is to explore the intermediate case $t \in \mathbb{R}$ and the effect of $\exp(-itP)$ on wave packets, and through this exploration to discuss some fundamental ideas used in works like [6], [8] and [13].

To state some results, we introduce a wave packet in dimension one as a Gaussian shifted in phase space: when $\mathbf{w} = (w_x, w_\xi) \in \mathbb{C}^2$ and $\tau \in \mathbb{C}$ with $\text{Im } \tau > 0$, let

$$g_{\mathbf{w},\tau}(x) = \exp\left(-\frac{i}{2}w_x w_\xi + i w_\xi x + \frac{i}{2}\tau(x - w_x)^2\right). \quad (1.3)$$

When $\mathbf{w} \in \mathbb{R}^2$,

$$\|g_{\mathbf{w},\tau}\|_{L^2(\mathbb{R})} = \|g_{\mathbf{0},\tau}\|_{L^2(\mathbb{R})} = (\pi \text{Im } \tau)^{-1/4}.$$

We also recall the symplectic form

$$\sigma((x, \xi), (y, \eta)) = \xi y - \eta x, \quad (x, \xi), (y, \eta) \in \mathbb{C}^2$$

which for $(x, \xi), (y, \eta) \in \mathbb{R}^2$ coincides with $-(x, \xi) \times (y, \eta)$, the opposite of the cross product. Finally, we recall that $\exp(-itQ_0)$ is associated with its Hamilton flow

$$\mathbf{K}_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad (1.4)$$

which is simply rotation by the angle $-t$.

Theorem 1.1. Fix $\mathbf{v} = (v_x, v_\xi) \in \mathbb{R}^2$ and let the shifted harmonic oscillator P be as in (1.1). Also fix $\mathbf{w} \in \mathbb{R}^2$ and $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ with $\text{Im } \tau > 0$; we recall the wave packet $g_{\mathbf{w},\tau}$ defined in (1.3). With \mathbf{K}_t the Hamilton flow associated with the harmonic oscillator (1.4), we introduce

$$\mathbf{u}(t) = (u_x(t), u_\xi(t)) = (\mathbf{K}_{-t} - 1)\mathbf{v} = ((\cos t - 1)v_x - (\sin t)v_\xi, (\sin t)v_x - (\cos t - 1)v_\xi).$$

Then, with norms in $L^2(\mathbb{R})$,

$$\|\exp(-itP)g_{\mathbf{w},\tau}\| = e^{-\sigma(\mathbf{u}(t), \mathbf{w})} \|g_{\mathbf{u}(t),\tau}\| \quad (1.5)$$

and

$$\frac{\|g_{\mathbf{u}(t),\tau}\|}{\|g_{\mathbf{0},\tau}\|} = \exp\left(\frac{1}{2}\left(\frac{1}{\tau_2}(u_\xi(t) - \tau_1 u_x(t))^2 + \tau_2 u_x(t)^2\right)\right). \quad (1.6)$$

1.1. Discussion

The wave packet analysis of $\exp(-itP)$ both reinforces and significantly sharpens our natural intuitions concerning this non-self-adjoint operator. To fix ideas, let $\mathbf{v} = (1, 0)$ so the operator being investigated is

$$\begin{aligned} P &= \frac{1}{2}(D_x^2 + (x - i)^2) \\ &= Q_0 - ix - \frac{1}{2}. \end{aligned}$$

For small times, one could guess that $\|\exp(-itP)f\| \approx \|\exp(-tx)f\|$ in $L^2(\mathbb{R})$ because evolution by what remains of the operator, $\exp(-it(Q_0 - \frac{1}{2}))$, is unitary. One could correctly guess that the wave packet $g_{(w_x, 0), \tau}$ for τ fixed and $w_x \in \mathbb{R}$ should have norm similar to $\exp(-tw_x)$ for t small and w_x large. A certain number of tricks (such as periodicity and time inversion) are available, but these are not necessarily applicable to other degree-2 generators.

But we have much more precise information: in this case,

$$\mathbf{u}(t) = (\cos t - 1, \sin t)$$

and so

$$-\sigma(\mathbf{u}(t), \mathbf{w}) = w_\xi(\cos t - 1) - w_x \sin t$$

which has the geometric interpretation of the magnitude of the cross product between $\mathbf{u}(t)$, which traces a circle counterclockwise around $(-1, 0)$, and \mathbf{w} . To illustrate this, we draw $\|\exp(-itP)f\|/\|f\|$ for f a variety of wave packets. In Figure 1.1, we take $g_{\mathbf{w},i}$ for $\mathbf{w} \in \{(0, 0), (3, 0), (0, 3), (-3, 0), (0, -3)\}$. In Figure 1.2, various centered Gaussians $g_{\mathbf{0},\tau}$ with different ‘‘shapes’’ τ are considered.

Many of the precise features of the curves in Figures 1.1 and 1.2 can be understood with elementary geometry. For instance, the norm corresponding to $\mathbf{w} = (3, 0)$ decreases because $\mathbf{u}(t)$ starts off in the positive vertical direction which is counter-clockwise from $(3, 0)$. The norm then increases beyond the norm of the centered Gaussian as t passes π because the vertical component of $\mathbf{u}(t)$ becomes negative, and $\mathbf{u}(t)$ is then in the clockwise direction from \mathbf{w} .

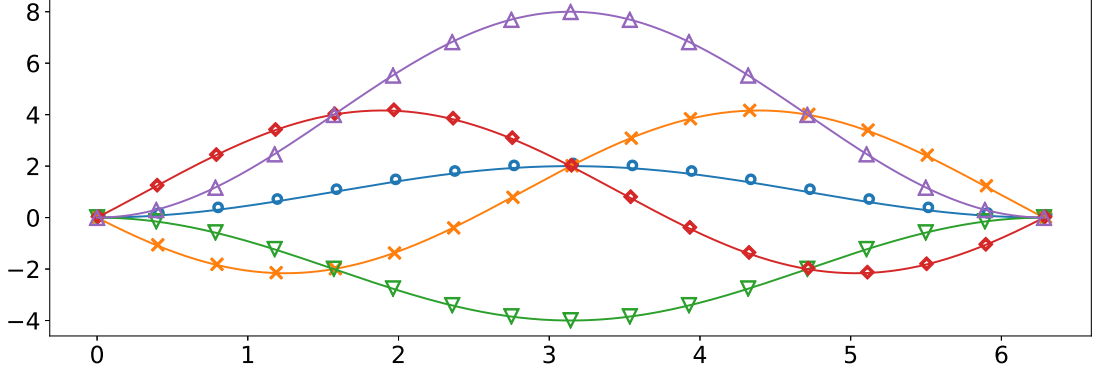


Figure 1.1: Logarithmic norm change $\log(\|\exp(-itP)g_{\mathbf{w},i}\|/\|g_{\mathbf{w},i}\|)$ for $t \in [0, 2\pi]$ and phase-space centers $\mathbf{w} = (0, 0), (3, 0), (0, 3), (-3, 0), (0, -3)$, with markers $\circ, \times, \nabla, \diamond, \triangle$, respectively.

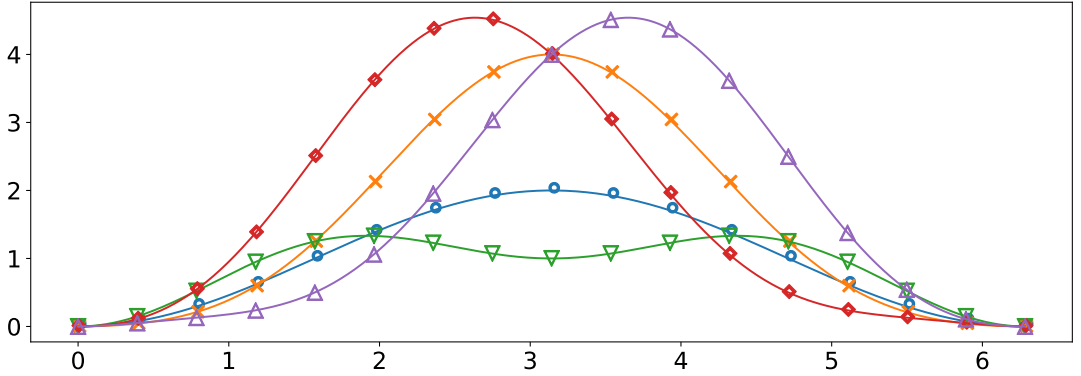


Figure 1.2: Logarithmic norm change $\log(\|\exp(-itP)g_{\mathbf{0},\tau}\|/\|g_{\mathbf{0},\tau}\|)$ for $t \in [0, 2\pi]$ and centered Gaussians with “shapes” $\tau = i, 2i, i/2, 1 + i, -1 + i$, with markers $\circ, \times, \nabla, \diamond, \triangle$, respectively.

A long list of questions can then be easily answered. If we scale $\mathbf{w} \mapsto \lambda\mathbf{w}$, does the maximum norm

$$\sup_{t \in \mathbb{R}} \|\exp(-itP)g_{\lambda\mathbf{w},\tau}\|$$

blow up as $\lambda \rightarrow \infty$? (Almost always, *except* if $\mathbf{w} = (0, w_\xi)$ and $w_\xi < 0$.) How quickly? (Like $e^{c_0\lambda + \mathcal{O}(1)}$ for some $c_0 \in \mathbb{R}$.) What is the growth rate c_0 ? (If $(\cos t, \sin t) = \frac{1}{\|\mathbf{w}\|}(w_\xi, -w_x)$, then

$$e^{-\sigma(\mathbf{u}(t), \mathbf{w})} = e^{\|\mathbf{w}\| - w_\xi}$$

which is of leading order as $\|\mathbf{w}\| \rightarrow \infty$.) Is $\|\exp(-itP)g_{\lambda\mathbf{w},\tau}\|$ ever exponentially *small*? (Yes, if $(\cos t, \sin t) = -\frac{1}{\|\mathbf{w}\|}(w_\xi, -w_x)$, then $\|\exp(-itP)g_{\lambda\mathbf{w},\tau}\|$ becomes exponentially small as $\lambda \rightarrow +\infty$ so long as \mathbf{w} is not $(0, w_\xi)$ with $w_\xi > 0$.) Are there centered wave packets, meaning $\mathbf{w} = \mathbf{0}$, for which $\|\exp(-itP)g_{\mathbf{0},\tau}\|$ grows arbitrarily large? (Yes, if $t \notin 2\pi\mathbb{Z}$ we can take any $\tau_1 \neq 0$ and $\tau_2 \rightarrow 0^+$.) Can we choose τ such that the evolution of a centered wave packet becomes small? (No, the exponent in (1.6) is positive.)

1.2. Plan of paper

In Section 2 we recall the calculus of metaplectic operators and shifts in phase space, as well as the elementary but somewhat involved computation giving the norm of a complex shift of a wave packet. With these tools in place, we prove Theorem 1.1 in several lines in Section 3. In Section 4 we recall following [1], [2] the general theory of a Bargmann reduction, based on the works of J. Sjöstrand, which allows us to make a weak definition of $\exp(-itP)$ for a wide variety of degree-2

generators P . Finally, in Section 5, we recall the implications of this theory for the L^2 norm of the Schrödinger evolution, following [17].

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2. Shifts in phase space and metaplectic operators

We present shifts in phase space and metaplectic operators on $L^2(\mathbb{R}^n)$ or subspaces thereof. Most of the material is classical and can be found in, for instance, [7] or [10]. We view these operators as Schrödinger evolutions

$$\exp(-itA)$$

for A the Weyl quantization of a homogeneous polynomial $a(x, \xi)$ of degree 1 (shifts) or 2 (metaplectic operators). When the symbol a has real coefficients, the operator A is self-adjoint and the evolution is unitary, but if the symbol is complex-valued, significant complications arise (see, for instance, the many references in [15]). Throughout one uses symplectic linear algebra, based on the usual symplectic inner product on \mathbb{C}^{2n} ,

$$\sigma((x, \xi), (y, \eta)) = \xi \cdot y - \eta \cdot x. \quad (2.1)$$

2.1. Shifts

A shift in phase space corresponding to $\mathbf{v} = (v_x, v_\xi) \in \mathbb{R}^{2n}$ is the Schrödinger evolution of the quantization of

$$a(x, \xi) = \sigma((x, \xi), (v_x, v_\xi)) = v_x \xi - v_\xi x.$$

For a degree-one polynomial, the Weyl quantization simply takes ξ to $D_x = -i\nabla_x$. It is straightforward to check that, if

$$\mathcal{S}_{\mathbf{v}} f(x) = e^{-\frac{i}{2}v_x \cdot v_\xi + i v_\xi \cdot x} f(x - v_x),$$

then

$$\partial_t \mathcal{S}_{t\mathbf{v}} f(x) = -i(v_x D_x - v_\xi x) \mathcal{S}_{t\mathbf{v}} f(x),$$

confirming that

$$\mathcal{S}_{\mathbf{v}} = \exp(-i\sigma((x, D_x), \mathbf{v})).$$

If $\mathbf{v} \in \mathbb{R}^{2n}$, then the shift $\mathcal{S}_{\mathbf{v}}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and therefore its dual $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, these shifts are unitary on $L^2(\mathbb{R}^n)$.

If $\mathbf{v} \in \mathbb{C}^{2n}$ is not real, then $\mathcal{S}_{\mathbf{v}}$ is unbounded on $\mathcal{S}(\mathbb{R}^n)$. The definition is still well-defined on any Gaussian, so one may take a maximal definition by having a complex shift act on the wave packet decomposition, as we describe in Section 4.

2.2. Wave packets

For $T \in \mathbb{M}_{n \times n}(\mathbb{C})$ a symmetric n -by- n matrix, we introduce the centered Gaussian

$$g_{\mathbf{0}, T}(x) = e^{\frac{i}{2}x \cdot T x}.$$

(We are, of course, most interested in the case $\text{Im } T > 0$ in the sense of positive definite matrices.) To change the center of a wave packet in phase space, we simply take the shift by $\mathbf{w} = (w_x, w_\xi) \in \mathbb{C}^{2n}$,

$$g_{\mathbf{w}, T}(x) = \mathcal{S}_{\mathbf{w}} g_{\mathbf{0}, T}(x) = e^{-\frac{i}{2}w_x \cdot w_\xi + i w_\xi \cdot x + \frac{i}{2}(x - w_x) \cdot T (x - w_x)}.$$

The complex shift of a wave packet does not necessarily conserve the norm, and in Section 2.4 we discuss how to find the norm and corresponding real shift by projecting $\mathbf{w} \in \mathbb{C}^{2n}$ to \mathbb{R}^{2n} along the Lagrangian plane associated to $g_{\mathbf{0}, T}$.

2.3. Calculus for the evolution of shifts

The way the Schrödinger evolution of a wave packet depends on the shape T of the wave packet is somewhat complicated; for this reason, it is enormously convenient to have a stable wave packet on hand. Identifying this choice of wave packet is due to [13], which we recall in Section 4.2. The evolution of the phase-space center, on the other hand, is very simple, and can be largely understood using two rules.

Commuting shifts contributes a factor depending on the symplectic inner product:

$$\mathcal{S}_{\mathbf{v}}\mathcal{S}_{\mathbf{w}} = e^{\frac{i}{2}\sigma(\mathbf{v},\mathbf{w})}\mathcal{S}_{\mathbf{v}+\mathbf{w}} = e^{i\sigma(\mathbf{v},\mathbf{w})}\mathcal{S}_{\mathbf{w}}\mathcal{S}_{\mathbf{v}}.$$

Commuting shifts past a metaplectic operator composes the shift with the associated canonical transformation: if the operator \mathcal{K} is associated with the canonical transformation \mathbf{K} , then

$$\mathcal{K}\mathcal{S}_{\mathbf{v}} = \mathcal{S}_{\mathbf{K}\mathbf{v}}\mathcal{K}. \quad (2.2)$$

Example 2.1. When the generator is the harmonic oscillator, the so-called Egorov relation (2.2) may be formally verified by hand, with some difficulty. Suppose that $U(0, x) = u(x)$ and that

$$(-i\partial_t + \frac{1}{2}(D_x^2 + x^2))U(t, x) = 0.$$

To show that $\exp(-itQ_0)\mathcal{S}_{\mathbf{w}} = \mathcal{S}_{\mathbf{K}_t\mathbf{w}}\exp(-itQ_0)$ for $\mathbf{w} = (w_x, w_\xi) \in \mathbb{R}^2$ and \mathbf{K}_t in (1.4), one can verify that

$$F(t, x) = e^{-\frac{i}{2}(w_x \cos t + w_\xi \sin t)(-w_x \sin t + w_\xi \cos t) + i(-w_x \sin t + w_\xi \cos t)x} U(t, x - (w_x \cos t + w_\xi \sin t))$$

also satisfies $(-i\partial_t + Q_0)F(t, x) = 0$, but the author does not particularly recommend carrying out the computation even in this simplest of cases.

2.4. Interaction between shifts and the shape of a wave packet

Equivalent shifts for a wave packet $g_{\mathbf{v},T}$ (which are useful when seeking the norm, because a real shift is unitary) depend on the Lagrangian plane

$$\Lambda_T = \{(\zeta, T\zeta)\}_{\zeta \in \mathbb{C}^n}$$

associated with the wave packet. (See [8, Sect. 5.5].) To begin, for any $\zeta \in \mathbb{C}^n$,

$$g_{(\zeta, T\zeta), T}(x) = e^{\frac{i}{2}(x-\zeta) \cdot T(x-\zeta) + iT\zeta \cdot x - \frac{i}{2}\zeta \cdot T\zeta} = g_{\mathbf{0}, T}(x) \quad (2.3)$$

because T is symmetric. Therefore if $\mathbf{v} - \mathbf{w} \in \Lambda_T$, then

$$g_{\mathbf{v}, T} = \mathcal{S}_{\mathbf{w}}\mathcal{S}_{-\mathbf{w}}\mathcal{S}_{\mathbf{v}}g_{\mathbf{0}, T} = e^{\frac{i}{2}\sigma(\mathbf{v}, \mathbf{w})}\mathcal{S}_{\mathbf{w}}\mathcal{S}_{\mathbf{v}-\mathbf{w}}g_{\mathbf{0}, T} = e^{\frac{i}{2}\sigma(\mathbf{v}, \mathbf{w})}g_{\mathbf{w}, T}.$$

In particular, we can determine the effect of a complex shift $\mathcal{S}_{\mathbf{v}}$ for $\mathbf{v} = (v_x, v_\xi) \in \mathbb{C}^{2n}$ on the L^2 norm of a wave packet. We assume that

$$\mathbf{v} = i\mathbf{u}, \quad \mathbf{u} = (u_x, u_\xi) \in \mathbb{R}^{2n},$$

because

$$\|g_{\mathbf{v}, T}\| = \|e^{-i\sigma(\operatorname{Re} \mathbf{v}, \operatorname{Im} \mathbf{v})}\mathcal{S}_{\operatorname{Re} \mathbf{v}}\mathcal{S}_{i\operatorname{Im} \mathbf{v}}g_{\mathbf{0}, T}\| = \|g_{i\operatorname{Im} \mathbf{v}, T}\|.$$

Proposition 2.2. *Let $T = T_1 + iT_2 \in \mathbb{M}_{n \times n}(\mathbb{C})$ be a complex symmetric matrix with T_1, T_2 real and T_2 positive definite. Let $\mathbf{u} = (u_x, u_\xi) \in \mathbb{R}^{2n}$. Define*

$$p = u_\xi - T_1 u_x \in \mathbb{R}^{2n}.$$

Then

$$\frac{\|g_{i\mathbf{u}, T}\|}{\|g_{\mathbf{0}, T}\|} = \exp\left(\frac{1}{2}(p \cdot T_2^{-1}p + u_x \cdot T_2 u_x)\right) \geq 1$$

with equality only when $\mathbf{u} = \mathbf{0}$.

Proof. We search for a $\zeta \in \mathbb{C}^n$ such that

$$i\mathbf{u} - (\zeta, T\zeta) \in \mathbb{R}^{2n},$$

since $g_{\mathbf{0}, T}$ is invariant under $\mathcal{S}_{(\zeta, T\zeta)}$ and the L^2 norm is invariant under a real shift. We write $\zeta = \zeta_1 + i\zeta_2$ for $\zeta_1, \zeta_2 \in \mathbb{R}^n$. Because $i u_x - \zeta$ is real, $\zeta_2 = u_x$. Because $i u_\xi - T\zeta$ is also real,

$$u_\xi = \operatorname{Im}(T\zeta) = T_2 \zeta_1 + T_1 u_x.$$

Therefore, recalling that $T = T_1 + iT_2$ is symmetric and so $T^* = \bar{T} = T_1 - iT_2$,

$$\begin{aligned}\zeta &= T_2^{-1}(u_\xi - T_1 u_x) + iu_x \\ &= T_2^{-1}p + iu_x.\end{aligned}$$

We also have

$$\begin{aligned}T\zeta &= (T_1 + iT_2)(T_2^{-1}p + iu_x) \\ &= T_1 T_2^{-1}p + iT_1 u_x + i(u_\xi - T_1 u_x) - T_2 u_x \\ &= iu_\xi + T_1 T_2^{-1}p - T_2 u_x.\end{aligned}$$

Since $\mathcal{S}_{(\zeta, T\zeta)}g_{\mathbf{0}, T} = g_{\mathbf{0}, T}$ by (2.3),

$$g_{i\mathbf{u}, T} = \mathcal{S}_{i\mathbf{u}}\mathcal{S}_{-(\zeta, T\zeta)}g_{\mathbf{0}, T} = e^{\frac{1}{2}\sigma(i\mathbf{u}, -(\zeta, T\zeta))}\mathcal{S}_{i\mathbf{u}-(\zeta, T\zeta)}g_{\mathbf{0}, T} = e^{\frac{1}{2}\sigma(\mathbf{u}, (\zeta, T\zeta))}\mathcal{S}_{i\mathbf{u}-(\zeta, T\zeta)}g_{\mathbf{0}, T}.$$

Because $i\mathbf{u} - (\zeta, T\zeta) \in \mathbb{R}^{2n}$, the corresponding shift is unitary and

$$\|g_{i\mathbf{u}, T}\| = |e^{\frac{1}{2}\sigma(\mathbf{u}, (\zeta, T\zeta))}|\|g_{\mathbf{0}, T}\|.$$

We simplify the exponential,

$$\begin{aligned}\sigma(\mathbf{u}, (\zeta, T\zeta)) &= \sigma(\mathbf{u}, (\zeta, T\zeta) - i\mathbf{u}) = \sigma(\mathbf{u}, (T_2^{-1}p, T_1 T_2^{-1}p - T_2 u_x)) \\ &= u_\xi \cdot T_2^{-1}p - (T_1 T_2^{-1}p - T_2 u_x) \cdot u_x = p \cdot T_2^{-1}p + u_x \cdot T_2 u_x.\end{aligned}\quad \square$$

3. Proofs for the shifted harmonic oscillator

Having set up the necessary elements of the metaplectic calculus, the proof of Theorem 1.1 is a few lines.

Proof of Theorem 1.1. Let $\text{Im } \tau > 0$ and $\mathbf{w} \in \mathbb{C}^2$. Since $P = \mathcal{S}_{i\mathbf{v}}Q_0\mathcal{S}_{-i\mathbf{v}}$,

$$\begin{aligned}\exp(-itP)g_{\mathbf{w}, \tau} &= \mathcal{S}_{i\mathbf{v}}\exp(-itQ_0)\mathcal{S}_{i\mathbf{v}}^{-1}\mathcal{S}_{\mathbf{w}}g_{\mathbf{0}, \tau} \\ &= \exp(-itQ_0)\mathcal{S}_{i\mathbf{K}_{-t}\mathbf{v}}\mathcal{S}_{-i\mathbf{v}}\mathcal{S}_{\mathbf{w}}g_{\mathbf{0}, \tau} \\ &= e^{-\frac{i}{2}\sigma(\mathbf{K}_{-t}\mathbf{v}, -\mathbf{v}) - \sigma((\mathbf{K}_{-t}-1)\mathbf{v}, \mathbf{w})}\exp(-itQ_0)\mathcal{S}_{\mathbf{w}}\mathcal{S}_{i(\mathbf{K}_{-t}-1)\mathbf{v}}g_{\mathbf{0}, \tau}.\end{aligned}$$

When t and \mathbf{w} are real, $\exp(-itQ_0)\mathcal{S}_{\mathbf{w}}$ is unitary. Having also assumed that \mathbf{v} is real,

$$\|\exp(-itP)g_{\mathbf{w}, \tau}\| = e^{-\sigma((\mathbf{K}_{-t}-1)\mathbf{v}, \mathbf{w})}\|\mathcal{S}_{i(\mathbf{K}_{-t}-1)\mathbf{v}}g_{\mathbf{0}, \tau}\|.$$

Defining $\mathbf{u}(t) = (\mathbf{K}_{-t} - 1)\mathbf{v}$ and applying Proposition 2.2 proves Theorem 1.1. \square

4. A weak definition of the Schrödinger evolution for supersymmetric degree-2 polynomials

Let P be a degree-two polynomial in (x, D_x) with complex coefficients. If we set out to solve the Schrödinger evolution problem

$$\begin{cases} -i\partial_t U(t, x) + PU(t, x) = 0, \\ U(0, x) = u(x), \end{cases}$$

on $L^2(\mathbb{R}^n)$, we are confronted with several challenges. In the elliptic case we can describe the spectrum thanks to [13], but the spectrum gives limited information about the evolution equation in the non-self-adjoint case. We recall the spectral theory in Section 4.1 below.

Another natural approach is to use wave packets as a definition, but the effect on the norm of decomposition into wave packets, transformation of wave packets, and recomposition is somewhat unclear. This is greatly simplified by the supersymmetric structure of many quadratic operators, which allows us to identify a wave packet whose shape (the matrix T) is stable under the action of the operator. We recall this structure in Section 4.2, and we recall associated Bargmann transformation which allows us to make a weak definition of a Schrödinger evolution in Section 4.3.

4.1. Spectral theory in the elliptic quadratic case

Let P be an operator given by the Weyl quantization of $q(x, \xi)$, a quadratic form in $(x, D_x) = (x, -i\partial_x)$:

$$Q = \sum_{|\alpha+\beta|=2} \frac{1}{2} q_{\alpha\beta} (x^\alpha D_x^\beta + D_x^\beta x^\alpha).$$

(Here, $\alpha, \beta \in \mathbb{N}^n$ are multiindices and $q_{\alpha\beta}$ are complex numbers.) For example, the Weyl quantization in dimension one takes

$$ax^2 + bx\xi + c\xi^2$$

to the operator

$$ax^2 + \frac{b}{2i} \left(x \frac{d}{dx} + \frac{d}{dx} x \right) - c \frac{d^2}{dx^2}.$$

It has long been known [13] that, under certain ellipticity hypotheses such as

$$\lim_{|(x,\xi)| \rightarrow \infty} \operatorname{Re} q(x, \xi) = +\infty,$$

the spectrum of a non-self-adjoint operator which is a degree-2 polynomial in (x, D_x) can be explicitly computed. In the case where q is elliptic and quadratic, there is a complete set of eigenfunctions of the form

$$f_\alpha(x) e^{\frac{i}{2} x \cdot A_+ x},$$

and the (generalized) eigenvalues are a lattice

$$\left\{ \sum_{j=1}^n \frac{1}{i} \lambda_j \left(\alpha_j + \frac{1}{2} \right) : \alpha \in \mathbb{N}^n \right\}.$$

The values λ_j are the eigenvalues of the Hamilton map H_q in the upper half-plane $\{\operatorname{Im} \lambda > 0\}$. The Hamilton map may be defined using the symplectic inner product as the unique (complex-)linear operator such that

$$q(X) = \frac{1}{2} \sigma(X, H_q X), \quad \forall X \in \mathbb{R}^{2n}$$

and

$$\frac{1}{2} \sigma(X, H_q Y) = -\frac{1}{2} \sigma(H_q X, Y), \quad \forall X, Y \in \mathbb{R}^{2n}.$$

Recalling the linear map $\mathbf{J}(x, \xi) = (-\xi, x)$ which defines the symplectic inner product via $\sigma(X, Y) = X \cdot \mathbf{J}Y$, the Hamilton map is simply

$$H_q = -\mathbf{J} \operatorname{Hess} q$$

where $\operatorname{Hess} q$ is the usual Hessian.

Example 4.1. The harmonic oscillator has symbol

$$q_0(x, \xi) = \frac{1}{2} (x, \xi) \cdot (x, \xi)$$

for which the Hessian is the identity matrix. Therefore the Hamilton map is $H_{q_0} = -\mathbf{J}$, or in other words rotation by $-\pi/2$.

4.2. Supersymmetry

At the heart of J. Sjöstrand's proof of the spectrum of non-self-adjoint elliptic quadratic operators is the existence of stable centered Gaussians for the operator and its adjoint. These come from a decomposition of a quadratic symbol $q(x, \xi)$ as

$$q(x, \xi) = (\xi - T_-^* x) \cdot B(\xi - T_+ x) \tag{4.1}$$

for $B, T_+, T_- \in \mathbb{M}_{n \times n}(\mathbb{C})$ matrices with T_+, T_- symmetric and $\operatorname{Im} T_+, \operatorname{Im} T_-$ positive definite. The Weyl quantization gives

$$Q = q^w(x, \xi) = (D_x - T_-^* x) \cdot B(D_x - T_+ x) + \mu_0 \tag{4.2}$$

for a constant $\mu_0 \in \mathbb{C}$. Recalling that $D_x = -i\partial_x$ and $g_T(x) = \exp(\frac{i}{2} x \cdot T x)$, it is immediate that

$$(Q - \mu_0)g_{T_+} = (Q^* - \bar{\mu}_0)g_{T_-} = 0.$$

Since formally

$$D_x - T_+ x = g_{T_+} D_x g_{T_+}^{-1},$$

we may (with minimal abuse of terminology) say that an operator in the form (4.2) is *supersymmetric*.

4.3. Adapted Bargmann transforms

If $q(x, \xi)$ can be expressed as in (4.1), there exists (see e.g. [2, Prop. 3.3] which follows [13]) a complex symplectic linear transformation \mathbf{B} and a matrix M such that, with H_q the Hamilton map of Q ,

$$\mathbf{B}H_q\mathbf{B}^{-1} = \begin{pmatrix} iM & 0 \\ 0 & -iM^\top \end{pmatrix}.$$

Consequently, for $X = (x, \xi) \in \mathbb{C}^{2n}$,

$$q(\mathbf{B}^{-1}X) = \frac{1}{2}\sigma(\mathbf{B}^{-1}X, H_q\mathbf{B}^{-1}X) = \frac{1}{2}\sigma(X, \mathbf{B}H_q\mathbf{B}^{-1}X) = Mx \cdot i\xi.$$

Much as there are metaplectic operators \mathcal{K} corresponding to real canonical transformations \mathbf{K} for which

$$\mathcal{K}p^w\mathcal{K}^{-1} = (p \circ \mathbf{K}^{-1})^w$$

for the Weyl quantization of symbols $p = p(x, \xi)$, one can find a Bargmann transform \mathcal{B} such that, when $Q = q^w$ is the Weyl quantization of $q(x, \xi)$,

$$\mathcal{B}Q\mathcal{B}^{-1} = Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M,$$

which is the Weyl quantization of $Mx \cdot i\xi$. The operator \mathcal{B} is unitary from $L^2(\mathbb{R}^n)$, but the image is $H_\Phi(\mathbb{C}^n)$, the space of holomorphic functions on \mathbb{C}^n for which

$$\|f\|_{H_\Phi}^2 = \int_{\mathbb{C}^n} |f(x)|^2 e^{-2\Phi(x)} \operatorname{d} \operatorname{Re} x \operatorname{d} \operatorname{Im} x$$

is finite. The function $\Phi(x)$ is quadratic and strictly convex. For details in the general case, we refer the reader to references like [18, Ch. 13], [14, Ch. 12.2] or the computations carried out in [2, Sect. 3.2].

Example 4.2. For the harmonic oscillator in dimension one, this reduction is accomplished using the Bargmann transform [3]:

$$\mathcal{B}f(x) = \pi^{-3/4} \int e^{-\frac{1}{2}(x^2+y^2)+\sqrt{2}xy} f(y) \operatorname{d}y.$$

The weight is given by $\Phi(x) = \frac{1}{2}|x|^2$ for $x \in \mathbb{C}$. For the quantum harmonic oscillator Q_0 in (1.2),

$$\mathcal{B}Q_0\mathcal{B}^{-1} = x \cdot \frac{\operatorname{d}}{\operatorname{d}x} + \frac{1}{2},$$

and if $\tilde{f} = \mathcal{B}f$, we have the equivalent definition for the evolution

$$\mathcal{B} \exp(-itQ_0) f(x) = e^{-\frac{it}{2}} \tilde{f}(e^{-it}x).$$

In particular, with a change of variables, the fact that $\exp(-itQ_0)$ is unitary for $t \in \mathbb{R}$ corresponds to the fact that $\Phi(x) = \frac{1}{2}|x|^2$ is invariant under multiplying x by e^{-it} .

The Bargmann reduction greatly simplifies the spectral theory [13]: the eigenfunctions of $Mx \cdot \partial_x$ are monomials, up to a change of variables putting M in Jordan normal form. The Bargmann reduction also simplifies the Schrödinger evolution, since for a holomorphic function $f(x)$,

$$F(t, x) = e^{-\frac{it}{2} \operatorname{tr} M} f(e^{-itM}x)$$

is the unique holomorphic solution to

$$\begin{cases} (-i\partial_t + Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M)F = 0, \\ F(0, x) = f(x). \end{cases}$$

The maximal domain D of $\exp(-itQ)$ can be easily identified on the Bargmann side:

$$\mathcal{B}(D) = \{f \in H_{\Phi} : f(e^{-itM}x) \in H_{\Phi}\}$$

By a change of variables, $f(e^{-itM}x) \in H_{\Phi}$ if and only if $f(x) \in H_{\Phi(e^{itM}x)}$. This domain coincides with the graph closure of $\exp(-itQ)$ restricted to the generalized functions of Q as described in Section 4.1.

The Bargmann reduction also reduces the Egorov relation (2.2) to an easily verified change of variables,

$$\begin{aligned} \exp\left(-it\left(Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M\right)\right) \mathcal{S}_{(x_0, \xi_0)} f &= \mathcal{S}_{(e^{itM}x_0, e^{-itM^\top} \xi_0)} \exp\left(-it\left(Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M\right)\right) f \\ \iff e^{-\frac{i}{2}x_0 \cdot \xi_0 + i\xi_0 \cdot e^{-itM}x} f(e^{-itM}x - x_0) &= e^{-\frac{i}{2}e^{itM}x_0 \cdot e^{-itM^\top} \xi_0 + ie^{-itM^\top} \xi_0 \cdot x} f(e^{-itM}(x - e^{itM}x_0)). \end{aligned}$$

If one perturbs $Q = q^w$ via a shift corresponding to $\mathbf{v} \in \mathbb{C}^{2n}$, the operator

$$P = \mathcal{S}_{\mathbf{v}} Q \mathcal{S}_{\mathbf{v}}^{-1}$$

corresponds to

$$\mathcal{B}P\mathcal{B}^{-1} = \mathcal{S}_{\mathbf{B}\mathbf{v}} \left(Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M \right) \mathcal{S}_{-\mathbf{B}\mathbf{v}}.$$

For the moment, let $\tilde{\mathbf{v}} = (\tilde{v}_x, \tilde{v}_\xi) = \mathbf{B}\mathbf{v}$. The principal advantage when describing the Schrödinger evolution is that, since functions on the Bargmann side are holomorphic, there is no doubt about the (complicated) definition

$$\begin{aligned} \exp(-itP)f(x) &= \mathcal{S}_{\tilde{\mathbf{v}}} \exp\left(-it\left(Mx \cdot \partial_x + \frac{1}{2} \operatorname{tr} M\right)\right) \mathcal{S}_{-\tilde{\mathbf{v}}} f(x) \\ &= e^{-\frac{it}{2} \operatorname{tr} M - i\tilde{v}_x \tilde{v}_\xi - i\tilde{v}_\xi e^{-itM}(x - \tilde{v}_x) + i\tilde{v}_\xi x} f(e^{-itM}(x - \tilde{v}_x) + \tilde{v}_x), \end{aligned}$$

only whether the resulting function is in H_{Φ} (that is, square-integrable against $e^{-2\Phi(x)} d\operatorname{Re} x d\operatorname{Im} x$).

Example 4.3. For the classical Bargmann transform in Example 4.2 the corresponding canonical transformation gives

$$\tilde{\mathbf{iv}} = (i\tilde{v}_x, \tilde{v}_\xi) = \mathbf{iB}\mathbf{v} = \frac{\mathbf{i}}{\sqrt{2}}(v_x - iv_\xi, -iv_x + v_\xi).$$

With P as in (1.1), we can say that f is in the domain of $\exp(-itP)$ if and only if, when $\tilde{f} = \mathcal{B}f$,

$$e^{-\tilde{v}_\xi(1-e^{-it})x} \tilde{f}(e^{-it}(x - (1-e^{it})i\tilde{v}_x)) \in H_{\Phi},$$

meaning that it is square-integrable against $e^{-|x|^2} d\operatorname{Re} x d\operatorname{Im} x$.

5. Geometric formulas for the L^2 operator norm

We are now in a position to summarize the results of [17], describing the norm of the Schrödinger evolution of a degree-2 polynomial in (x, D_x) under a weak ellipticity hypothesis.

Definition 5.1. We say that $\mathbf{K} : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ is a strictly positive linear canonical transformation in the sense of Melin-Sjöstrand if it is canonical, meaning that it preserves the symplectic inner product (2.1), and if

$$i\sigma(\overline{\mathbf{K}\mathbf{z}}, \mathbf{K}\mathbf{z}) > i\sigma(\bar{\mathbf{z}}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{C}^{2n} \setminus \{0\}.$$

If $q(x, \xi) : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ is quadratic and supersymmetric, then the weak definition of $\exp(-iQ)$, where $Q = q^w$ is the Weyl quantization of q , is compact when acting on $L^2(\mathbb{R}^n)$ if and only if the Hamilton flow $\exp H_q$ is strictly positive as a linear canonical transformation. Conversely, if q is quadratic and if the Hamilton flow $\exp H_q$ is strictly positive, then q is supersymmetric.

In the case where $\mathbf{K} = \exp H_q$ is strictly positive, $\exp(-iQ)$ is associated with \mathbf{K} and the adjoint $\exp(-iQ)^*$ is associated with $\overline{\mathbf{K}}^{-1}$. As shown in [6], the operator $\exp(-iQ)^* \exp(-iQ)$ is associated with $\overline{\mathbf{K}}^{-1} \mathbf{K}$ which can be shown to correspond to a Schrödinger evolution of quantum harmonic

oscillator type. The spectrum, and therefore the norm, of such an evolution can be read off from the eigenvalues of $\overline{\mathbf{K}}^{-1}\mathbf{K}$, and one obtains

$$\|\exp(-iQ)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \prod \{\mu^{1/4} : \mu \in \text{Spec } \overline{\mathbf{K}}^{-1}\mathbf{K} \cap (0, 1)\} \quad (5.1)$$

(where of course the eigenvalues are counted for multiplicity).

A similar analysis can be performed, when $Q = q^w$ is the Weyl quantization of a quadratic form with strictly positive Hamilton flow and $\mathbf{v} \in \mathbb{C}^{2n}$, for the conjugated operator

$$P = \mathcal{S}_{\mathbf{v}}Q\mathcal{S}_{\mathbf{v}}^{-1}.$$

Because strict positivity of the Hamilton flow implies that q is non-degenerate, every operator of the form $P = Q + w_x \cdot x + w_\xi \cdot D_x + p_0$, for $(w_x, w_\xi) \in \mathbb{C}^{2n}$ and $p_0 \in \mathbb{C}$, can be written as $P = \mathcal{S}_{\mathbf{v}}Q\mathcal{S}_{\mathbf{v}}^{-1} + c_0$ for some choice of $\mathbf{v} \in \mathbb{C}^{2n}$ and $c_0 \in \mathbb{C}$.

In a relatively straightforward imitation of the proof of the singular value decomposition for matrices, one can show that there exist real vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^{2n}$ where the canonical transformations corresponding to

$$\exp(-iP) = \mathcal{S}_{\mathbf{v}} \exp(-iQ)\mathcal{S}_{\mathbf{v}}^{-1}$$

and

$$\mathcal{S}_{\mathbf{a}_2} \exp(-iQ)\mathcal{S}_{\mathbf{a}_1}^{-1}$$

coincide, meaning that

$$\mathbf{K}(\mathbf{z} - \mathbf{v}) + \mathbf{v} = \mathbf{K}(\mathbf{z} - \mathbf{a}_1) + \mathbf{a}_2, \quad \forall \mathbf{z} \in \mathbb{C}^{2n}.$$

Specifically,

$$\mathbf{a}_1 = \text{Re } \mathbf{v} + (\text{Im } \mathbf{K})^{-1}(\text{Re } \mathbf{K} - 1) \text{Im } \mathbf{v},$$

and \mathbf{a}_2 is given by the same formula with $\overline{\mathbf{K}}^{-1}$ replacing \mathbf{K} and $\overline{\mathbf{v}}$ replacing \mathbf{v} .

It is here that the proof for the non-compact case differs from the compact case, and why Theorem 1.1 is not covered by the analysis in [17]: for real t , the evolution $\exp(-itQ_0)$ corresponds to a real canonical transformation \mathbf{K}_t in (1.4). One cannot invert $\text{Im } \mathbf{K}_t = 0$, of course, which corresponds to the fact that there is no phase-space center \mathbf{a}_1 which witnesses the largest norm of $\exp(-itP)$. Instead, the norm of the evolution of a wave packet goes exponentially quickly to infinity as the center goes to infinity in certain directions.

In contrast with the case of quadratic generators, equality of canonical transformations is not equivalent to equality of Schrödinger evolutions: if this were so, a complex perturbation would have no effect on the L^2 norm. But, using the Mehler formula in a computation not dissimilar to the proof of Proposition 2.2, one can show that

$$\mathcal{S}_{\mathbf{v}} \exp(-iQ)\mathcal{S}_{\mathbf{v}}^{-1} = e^{\frac{1}{2}\sigma(\mathbf{u}, \mathbf{v})} \exp(-iQ)\mathcal{S}_{\mathbf{u}}^{-1} = e^{\frac{1}{2}\sigma(\mathbf{v}, \mathbf{w})} \mathcal{S}_{\mathbf{w}} \exp(-iQ)$$

whenever $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^{2n}$ are such that the canonical transformations agree,

$$\mathbf{K}(\mathbf{z} - \mathbf{v}) + \mathbf{v} = \mathbf{K}(\mathbf{z} - \mathbf{u}) + \mathbf{u}, \quad \forall \mathbf{z} \in \mathbb{C}^{2n},$$

and as usual Q is the Weyl quantization of a quadratic form with strictly positive Hamilton flow \mathbf{K} .

Using this rule and taking advantage of a remarkable amount of available simplification, one obtains

$$\exp(-iP) = e^{\frac{1}{2}\sigma(\mathbf{v}, \mathbf{a}_2 - \mathbf{a}_1)} \mathcal{S}_{\mathbf{a}_2} \exp(-iQ)\mathcal{S}_{\mathbf{a}_1}^{-1}.$$

Since real shifts are unitary,

$$\|\exp(-iP)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = e^{-\frac{1}{2}\sigma(\text{Im } \mathbf{v}, \mathbf{a}_2 - \mathbf{a}_1)} \|\exp(-iQ)\|_{\mathcal{L}(L^2(\mathbb{R}^n))},$$

where the norm $\|\exp(-iQ)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ can be computed as in (5.1).

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