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Unique Continuation, Runge Approximation and the Fractional Calderón Problem

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Abstract

In these notes for the proceedings of the "Journée Équations aux Dérivées Partielles", we survey some of the recent progress in and the interplay of unique continuation, approximation and some related nonlocal inverse problems. In particular, we discuss the qualitative and quantitative global unique continuation properties of the fractional Laplacian and its Runge approximation properties. We explain how this leads to surprising results on the inverse problems for the associated operators.

1. Introduction

In this survey we discuss the interplay of unique continuation and approximation for the fractional Laplacian and related nonlocal equations. As a main application we discuss the fractional Calderón problem, which is a nonlocal inverse problem which had been introduced in [23]. The fractional Laplacian is here regarded as a prototypical nonlocal operator which appears naturally in many applications including thin free boundary value problems, fluid mechanics, relativistic quantum mechanics and the description of dislocations. There is a vast literature on it and on equations involving it; we refer to [15] and the references therein. In our situation of uniqueness and approximation problems the nonlocality of the fractional Laplacian gives rise to striking rigidity and flexibility properties which we will discuss in the sequel.

2. Global unique continuation and Runge approximation

Analytic functions enjoy very strong rigidity properties which are for instance reflected in the maximum principle or the identity theorem. It is natural to ask for which type of other function classes some of these properties persist. One property which can be meaningfully extended to the setting of more general equations is the *unique continuation property*. It can be phrased in different versions. For instance, the *weak unique continuation property* addresses the following question:

> Let u be a solution to some equation $P(x, D)u = 0$ in a domain Ω for a differential or pseudodifferential operator $P(x, D)$. Assume that u vanishes in an open subset $\Omega' \subset \Omega$. Does this already imply the vanishing of u in the whole of Ω?

A prototypical class of equations for which this holds are for instance second order elliptic equations with sufficiently regular metrics, potentials and gradient potentials [31]. Similarly, it is also interesting to study the *strong unique continuation property*:

> Let u be a solution to some equation $P(x, D)u = 0$ in a domain Ω for a differential or pseudodifferential operator $P(x, D)$. Assume that u vanishes of infinite order (in a weak sense) in $x_0 \in \Omega$. Does this already imply the vanishing of u in the whole of Ω?

Again, typical examples of equations for which this is true are sufficiently regular, second order elliptic equations [31].

In addition to its intrinsic interest as capturing the rigidity properties of an equation, the unique continuation property has a number of ramifications in other areas: The question of weak unique continuation is for instance closely linked to uniqueness and stability properties of Cauchy problems (see [48, 49] for reviews on this and the unique continuation property for various equations). Moreover, it plays an important role in control theory [53], for instance, in the context of approximate controllability. Both the strong and the weak unique continuation property also encode relevant information on the nodal domain of a function and are thus of interest in the study of PDE and free boundary value problems [30, 46, 47].

In the sequel, we are concerned with investigating *global* versions of these questions and some of their implications for a class of nonlocal operators, in particular for solutions to fractional Schrödinger equations

$$(-\Delta)^s u + qu = 0 \text{ in } \Omega \subset \mathbb{R}^n, \; n \geq 1, \; s \in (0,1),$$
$$u = f \text{ in } \Omega_e := \mathbb{R}^n \setminus \overline{\Omega}, \tag{2.1}$$

where $u \in H^s(\mathbb{R}^n)$, $f \in H^s(\mathbb{R}^n \setminus \overline{\Omega})$ and $q : \Omega \to \mathbb{R}$ is in a suitable function space. We recall that $\mathcal{F}((-\Delta)^s u)(\xi) := |\xi|^{2s} \mathcal{F} u(\xi)$, where $\mathcal{F}$ denotes the Fourier transform.

Before however turning to solutions of (2.1), we state a striking *global* (weak) unique continuation result for arbitrary functions in $H^s(\mathbb{R}^n)$:

Theorem 2.1 ([23, Theorem 1.1]). *Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^n)$. Assume that for some open set $\Omega \subset \mathbb{R}^n$ it holds*

$$u(x) = 0, \; (-\Delta)^s u(x) = 0 \text{ for almost every } x \in \Omega. \tag{2.2}$$

Then, $u \equiv 0$ in $\mathbb{R}^n$.

Let us comment on this:

Firstly, we note that this result is a very nonlocal property in that it clearly fails for the case $s = 1$. For the case $s \in (0,1)$ it states that *global* information on u can be retrieved from *local* information, provided the two pieces of information from (2.2) are given. Heuristically, this is due to the nonlocality of the operator; however a word of caution is needed: Nonlocality is not sufficient for such a result, there are many nonlocal operators which do not enjoy such a strong uniqueness property. In this sense, the result can be read as a strong rigidity property of the fractional Laplacian. This will play an important role in the analysis of the fractional Calderón problem, an associated inverse problem.

Secondly, it is worth mentioning that this result (for a more restricted class of settings) had already been observed by Riesz [37], but was recently rediscovered and extended in [23]. The argument in [23] relies on Carleman estimates which had been derived in [38] in the analysis of the (strong) unique continuation properties of solutions to (2.1), see also [19, 20, 22, 39, 45, 52] for similar (weak and strong) unique continuation properties.

Thirdly, it is interesting to observe that the strong global rigidity properties of the fractional Laplacian had already been well-known in the physics community under the name of the Reeh-Schlieder theorem. We refer to [51] and the references therein for the physical significance of this property.

Let us explain how the rigidity result of Theorem 2.1 can be obtained from a corresponding (weak) unique continuation result.

Sketch of the proof of Theorem 2.1. We first recall that due to the seminal work of Caffarelli–Silvestre [6], the fractional Laplacian can be written as a weighted Dirichlet-to-Neumann map of a degenerate elliptic equation: Consider $u \in H^s(\mathbb{R}^n)$ and a solution

$$\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s}) := \left\{ v : \mathbb{R}^{n+1}_+ \to \mathbb{R} : \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2s} |\nabla v|^2 \mathrm{d}x < \infty \right\}$$

to the equation

$$\nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{u} = 0 \text{ in } \mathbb{R}_+^{n+1},$$
$$\tilde{u} = u \text{ in } \mathbb{R}^n \times \{0\},$$

(2.3)

where $\mathbb{R}_+^{n+1}$ denotes the upper half-space. Then,

$$(-\Delta)^s u(x) = c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x, x_{n+1}),$$

where this limit has to be understood in an H^{-s} sense and where $c_{n,s} \neq 0$ is an only dimension and s dependent constant.

Now, with the Caffarelli–Silvestre extension at hand, the statement of Theorem 2.1 turns into a *boundary unique continuation property* (in the spirit of [1]) for the Caffarelli–Silvestre extension, i.e., it can be phrased as follows:

If $u = 0$ and $\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} = 0$ on a (relatively) open set $\Omega \subset \mathbb{R}^n \times \{0\} = \partial \mathbb{R}_+^{n+1}$, then $u \equiv 0$ in $\mathbb{R}_+^{n+1}$.

This however follows from the arguments which are used in the derivation of most of the known unique continuation results for fractional Schrödinger equations (as in most of the articles dealing with these nonlocal equations the authors approach the unique continuation property by means of the Caffarelli–Silvestre extension). $\qquad\square$

We remark that in the setting of Theorem 2.1, it is possible to deal with the situation of variable coefficients, i.e. it is possible to replace $(-\Delta)^s$ by $(-\nabla \cdot a^{ij} \nabla)^s$, where a^{ij} is a sufficiently regular, uniformly elliptic metric. Similar results also hold in the context of the strong and the weak unique continuation properties for fractional Schrödinger equations, see [38, Section 7] and [52]. In both settings the Caffarelli–Silvestre extension [6] which allows to "localize" the problem plays a key role.

Furthermore, the global weak unique continuation properties from Theorem 2.1 persist for a larger class of nonlocal operators, see [14, 40]. Here a striking feature is that the elliptic/parabolic/hyperbolic nature of the problem plays a much less prominent role than in the context of local equations.

Moreover, the weak and strong unique continuation properties for the Caffarelli–Silvestre extension and thus for fractional Schrödinger equations can also be extended to the *unique continuation property from measurable sets* in which one assumes that a solution to (2.1) vanishes on a set of positive measure and seeks to deduce that this already implies the vanishing of u in Ω, see [19, 22]. Quantifications of these results are possible [40, 41, 42].

Let us now turn to solutions to the fractional Schrödinger equation (2.1). For the purpose of these notes, let us always assume that zero is not a Dirichlet eigenvalue of the Schrödinger operator (2.1). Using energy estimates and the Fredholm alternative this in particular implies that the problem (2.1) is well-posed. It is thus possible to define the Poisson operator

$$P_q : H^s(W) \mapsto H^s(\mathbb{R}^n), \ f \mapsto u,$$

which maps a boundary datum f to the associated solution u to (2.1). For solutions to (2.1) the rigidity properties of Theorem 2.1 have striking dual flexibility properties as a consequence.

Theorem 2.2 ([13] and [23, Theorem 1.2]). *Let $s \in (0,1)$, let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $W \subset \mathbb{R}^n \setminus \overline{\Omega}$ be an open, non-empty set. Let $q \in L^\infty(\Omega)$ and let P_q denote the Poisson operator associated with the fractional Schrödinger equation (2.1). Let $v \in \widetilde{H}^s(\Omega)$ be given. Then, for any $\epsilon > 0$, there exists $f \in H^s(W)$ such that*

$$\|v - P_q(f)\|_{L^2(\Omega)} \leq \epsilon.$$

We remark that this flexibility of nonlocal equations had first been observed in the work [13], but had first been interpreted as a *dual property* to Theorem 2.1 in [23]. Viewing Theorems 2.1 and 2.2 as dual statements allows one to study very flexible (geometric) situations involving quite rough domains and equations.

Again the property of Theorem 2.2 is a very nonlocal feature: For instance, in the *local* case $s = 1$ in general maximum principles make such a result impossible. While the nonlocal equations of

the form (2.1) under suitable conditions also enjoy maximum principles and Harnack inequalities, these however only hold on *global* scales, locally solutions to the described nonlocal equations can be much more flexible [26, 27].

Historically, results of the flavour of Theorem 2.2 had first been studied by Runge [44] in the context of approximation results for analytic functions and are thus known as *Runge approximation*. Later Runge type approximation results were studied by Lax [34], Malgrange [35] and Browder [4, 5] in the context of much more general equations. These properties rely on a duality between unique continuation and approximation. This duality is an important and frequently used tool in the context of control theory problems [53], but has also been employed in the context of inverse problems [2, 32] or the analysis of qualitative properties of PDE [17, 16, 18].

Just as in the case of the global unique continuation property it is possible to quantify the approximation results from [41] and to study different operators [9, 10, 14]. This has been done in [39, 40, 41, 42] for different equations and operators.

3. The fractional Calderón problem

As an application of the previous rigidity and flexibility properties of the fractional Laplacian we discuss a nonlocal inverse problem, the *fractional Calderón problem*. The fractional Calderón problem should be viewed as the nonlocal analogue of the Calderón problem, in which one seeks to recover the conductivity of a material from boundary voltage and current measurements which are encoded in the Dirichlet-to-Neumann map. In the case of an isotropic material, after a Liouville transform this reduces to an inverse problem for a Schrödinger equation. Let us explain the set-up for the inverse problem for the local Schrödinger equation in more detail: In the classical Calderón problem for the Schrödinger equation, one considers the problem

$$(-\Delta)u + qu = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega. \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^n$ is open, bounded Lipschitz, $q : \Omega \to \mathbb{R}$ is a potential in a suitable function space and $f : \partial\Omega \to \mathbb{R}$ is sufficiently regular. Assuming that zero is not a Dirichlet eigenvalue of the operator (3.1), it is possible to define the Dirichlet-to-Neumann map

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \ f \mapsto \nu \cdot \nabla u|_{\partial\Omega},$$

where ν denotes the outer unit normal field. The associated inverse problem deals with the question of whether it is possible to recover information on q given the knowledge of the Dirichlet-to-Neumann map Λ_q. This is an intensively studied problem, we refer to the survey [50] for references on this.

In the *fractional Calderón problem* which had been introduced in [23] one studies a nonlocal version of this inverse problem. Instead of investigating (3.1) one considers the equation (2.1). Assuming that zero is not a Dirichlet eigenvalue of this problem, it is possible to define a "Dirichlet-to-Neumann" (or rather Dirichlet-to-adjoint-Dirichlet) map. With slight abuse of notation, this can formally be viewed as the mapping

$$\Lambda_q : H^s(\Omega_e) \ni f \mapsto (-\Delta)^s u|_{\Omega_e} \in H^{-s}(\Omega_e). \tag{3.2}$$

A rigorous definition requires an appropriate weak formulation of this. In the very recent work [12] it was established that as in the classical Calderón problem also in the fractional Calderón problem it is possible to relate a fractional conductivity equation by means of a Liouville transform to a fractional Schrödinger equation.

In analogy to the classical Calderón problem for the Schrödinger equation, in the fractional Calderón problem, one seeks to recover information on q from the knowledge of Λ_q. Since the introduction of this problem in [23] there has been substantial progress in understanding its properties. In particular, the following issues have been addressed:

Uniqueness. This is the question whether

$$\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2?$$

For the fractional Calderon problem, this was first addressed by [23] dealing with L^∞ potentials (see Theorem 3.1 below). This was extended to the class of scaling critical potentials in [40]. The

article [21] dealt with the case of the variable coefficient fractional Laplacian. From dimensional arguments one might hope that even a *single measurement* suffices to recover the potential. This turns out to be correct [22] and will be discussed in Section 3.3 below.

Stability. Here one asks whether the following continuity type property holds

$$\Lambda_{q_1} \sim \Lambda_{q_2} \Rightarrow q_1 \sim q_2?$$

The closeness has to be phrased in suitable norms. Similarly as the classical Calderón problem, also the fractional Calderón problem is notoriously ill-posed and only conditional stability can be expected, i.e. stability can only be expected under suitable a priori conditions ensuring that q is in some compact subset. In this context an important question deals with the impact of the nonlocality of the fractional problem: Does it (as in some other nonlocal problems [25]) improve the stability properties of the inverse problem compared to its local analogue? This question has been studied in [40] and [41], where it was shown that as in the classical obstacle problem *only a logarithmic modulus of continuity* holds and that this is indeed optimal, we refer to the discussion below for the precise statement.

Recovery. In addition to the "abstract" uniqueness and stability properties, it is desirable to be able to explicitly, algorithmically reconstruct the unknown potential. This has been addressed in [24] for the infinite data problem by means of monotonicity methods and in [22] for the fractional Calderón problem with a single measurement by exploiting the strong rigidity properties of the fractional Laplacian (see Section 3.3 below).

The methods in the study of the fractional Calderón problem are rather robust. They for instance also allow for the treatment of further important and interesting problems such as semilinear problems of the fractional Laplacian [33], inclusion detection by means of single measurements [7] or Helmholtz type inverse problems [8]. Also lower order perturbations (local and nonlocal) have been considered in [3, 11].

In the sequel, we however focus on the three aspects introduced above for the model equation (2.1) and explain the main results and ideas in the treatment of the associated inverse problem. Here we emphasize the connection to the rigidity and flexibility properties from Section 2.

3.1. Partial data uniqueness

We begin by discussing the uniqueness properties for the fractional Calderón problem. These were first studied in [23], where the following result was proved:

Theorem 3.1 ([23])**.** *Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Let $W_j \subset \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{W}_j \cap \overline{\Omega} = \emptyset$ for $j \in 1,2$ be open, non-empty sets. Suppose that $q_1, q_2 \in L^\infty(\overline{\Omega})$. Then, if for all $f_1 \in \widetilde{H}^s(W_1)$, $f_2 \in \widetilde{H}^s(W_2)$ it holds*

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{L^2(\mathbb{R}^n)} = 0,$$

then $q_1 = q_2$ in Ω.

Here

$$\widetilde{H}^s(W_1) := \text{ completion of } C_c^\infty(W_1) \text{ in } H^s(\mathbb{R}^n),$$

where $H^s(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} : \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u\|_{L^2(\mathbb{R}^n)} < \infty\}$ denotes the usual fractional L^2-based Sobolev space, see [15, 36].

Particularly important features of this result in which the nonlocality of the problem enters are the fact that it is a *partial data* uniqueness problem. Instead of requiring the knowledge of the full Dirichlet-to-Neumann map (3.2), it suffices to have information on potentially very small subsets $W_j \in \mathbb{R}^n \setminus \overline{\Omega}$ in order to prove uniqueness. This also extends to the anisotropic problem if one considers sufficiently regular (known) variable coefficients in the principal symbol [21]. This is in strong contrast to the classical Calderón problem, where a result in such a generality is not known [28, 29].

Further, different from the classical Calderón problem in which one can only hope to recover the potential in dimensions larger or equal to two, the fractional Calderón problem is always

overdetermined: Indeed, counting dimensions, one observes that the Dirichlet-to-Neumann map for the fractional Calderón problem contains $2n$ degrees of determination (through the Schwartz kernel of the Dirichlet-to-Neumann map) while one seeks to recover a function of n variables.

It is possible to generalize this result to rougher classes of potentials, see [40], where q is taken in an almost sharp class of multipliers. In order to avoid technicalities, we do not discuss this in the present review.

We outline the argument for Theorem 3.1:

Sketch of the proof of Theorem 3.1. The argument for Theorem relies on an integration by parts identity (an "Alessandrini type identity"):

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{L^2(\mathbb{R}^n)} = ((q_1 - q_2)u_1, u_2)_{L^2(\Omega)}, \tag{3.3}$$

where u_1, u_2 are solutions to the fractional Schrödinger equations with data f_1, f_2 and potentials q_1, q_2 respectively. Since by the assumption of the theorem,

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{L^2(\mathbb{R}^n)} = 0,$$

by (3.3) it suffices to prove that the set of products of solutions $u_1 u_2$ with data $f_1 \in \widetilde{H}^s(W_1)$, $f_2 \in \widetilde{H}^s(W_2)$ forms a dense set in $L^1(\Omega)$. In the classical Calderón problem this follows from the construction of *complex geometric optics* solutions. Lacking these in the fractional setting, the key idea of [23] was to use the strong approximation property from Theorem 2.2 to prove this density result. □

3.2. Stability

An interesting question is concerned with the influence of the nonlocality of the operator on the stability properties of the inverse problem at hand. Does nonlocality improve these properties? In [40] it is shown that as in the classical Calderón problem also the fractional Calderón problem enjoys logarithmic stability estimates.

Theorem 3.2 ([40]). *Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open, smooth, bounded set. Let $W_j \subset \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{W}_j \cap \overline{\Omega} = \emptyset$ for $j \in 1,2$ be open, non-empty sets. Suppose that $q_1, q_2 \in L^\infty(\overline{\Omega})$ with $\|q_j\|_{L^\infty(\Omega)} \leq M < \infty$ for $j \in \{1,2\}$. Then, there exist constants $C, \mu > 0$ depending on n, s, Ω, W_j, M such that*

$$\|q_1 - q_2\|_{H^{-s}(\Omega)} \leq \frac{C}{\log(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*)^\mu},$$

where

$$\|A\|_* := \sup_{\|f_1\|_{H^s(\mathbb{R}^n)} = \|f_2\|_{H^s(\mathbb{R}^n)} = 1} \{(Af_1, f_2)_{L^2(\mathbb{R}^n)} : f_1 \in \widetilde{H}^s(W_1), \ f_2 \in \widetilde{H}^s(W_2)\}.$$

We emphasize that in analogy to the uniqueness result from the previous section, also the stability estimate of Theorem 3.2 is a partial data result in the sense that $\|\cdot\|_*$ only takes partial data into account.

One might wonder whether this result is optimal, or whether it is possible to improve on the logarithmic modulus of continuity. In [42] it was proved that the logarithmic modulus is indeed necessary.

Analogously to the classical case, if a priori information is available, e.g. if the potential is a priori known to be in a suitably regular finite dimensional subspace, it is possible to improve on this and prove Lipschitz estimates [43].

We give a very rough outline of the proof of Theorem 3.2.

Outline of the proof of Theorem 3.2. The proof starts with Alessandrini's identity (3.3). Now however the left hand side no longer vanishes. In this situation a *quantitative version* of the Runge approximation from Theorem 2.2 is used. Here the *cost* of approximating an arbitrary function v by a solution u to (2.1) with a prescribed error threshold $\epsilon > 0$ is measured by the size of its boundary data $f \in \widetilde{H}^s(W)$ in terms of the admissible error $\epsilon > 0$. This quantitative version of Theorem 2.2 is obtained by reducing it to a quantitative unique continuation result which is obtained by localizing the problem by means of the Caffarelli–Silvestre extension. □

The argument provides an interesting connection between inverse problems and questions and techniques from control theory.

3.3. Single measurement recovery

Last but not least we discuss the single measurement recovery and uniqueness result for the fractional Calderón problem.

Theorem 3.3 ([22]). *Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Let $W \subset \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{W} \cap \overline{\Omega} = \emptyset$ be an open, non-empty set. Then q can be (algorithmically) reconstructed from a single $f \in H^s(W) \setminus \{0\}$ and the measurement $\Lambda_q(f)$.*

An alternative recovery scheme (which however is slightly more restrictive in the class of admissible potentials q and which is based on infinitely many measurements) is obtained by monotonicity methods [24].

Let us emphasize that the reconstruction and uniqueness result in Theorem 3.3 is completely algorithmic. As the previous two results, it is of very nonlocal flavour in the sense that it strongly exploits the rigidity properties from Theorem 2.1. Also, it relies on the overdetermination of the inverse problem. It is further possible to extend this result to potentials $q \in L^\infty(\Omega)$ if $s \in [\frac{1}{4}, 1)$.

We us outline the argument for Theorem 3.3.

Sketch of proof of Theorem 3.3. The proof relies on two main steps:

Step 1: Constructive recovery of u from $(f, \Lambda_q(f))$ for a single function $f \in \widetilde{H}^s(W) \setminus \{0\}$. This can for instance be achieved by a form of Tikhonov regularization. It essentially encodes that a general function $u \in H^s(\mathbb{R}^n)$ is already determined by the two pieces of information given by $u|_W$ and $(-\Delta)^s u|_W$, where W is an open set. This hence essentially reduces to a rigidity estimate of the type given in Theorem 2.1.

Step 2: Solving the equation for q and unique continuation. Once the function u is (constructively) obtained on the whole of $\mathbb{R}^n$, it is possible to solve the equation (2.1) for the potential:

$$q(x) = \frac{(-\Delta)^s u(x)}{u(x)}.$$

Using the unique continuation either from open sets (if $q \in C^0$) or from measurable sets (if q is less regular) we obtain that the quotient is well-defined on sufficiently many points $x \in \Omega$, which allows to reconstruct $q(x)$ on these. $\qquad\square$

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