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# Emphasizing nonlinear behaviors for cubic coupled Schrödinger systems

Victor Vilaça Da Rocha

**Abstract**

The purpose of this note is to propose a study of various nonlinear behaviors for a system of two coupled cubic Schrödinger equations with small initial data. Depending on the choice of the spatial domain, we highlight different examples of nonlinear behaviors. The goal is to mix the approaches of the study on the torus (with a truly nonlinear behavior) and of the study on the real line (with an infinite behavior) in order to obtain on the product space $\mathbb{R} \times \mathbb{T}$ a truly nonlinear behavior in infinite time.

In this note, our aim is to highlight some nonlinear behaviors for the following system of two coupled cubic Schrödinger equations:

$$\begin{cases} i\partial_t u + \Delta_x u &= |v|^2 u, \\ i\partial_t v + \Delta_x v &= |u|^2 v. \end{cases} \tag{CNLS}$$

First, we see through a brief history of the research on the growth of Sobolev norms for the cubic Schrödinger equation, that the system (CNLS) is a perfect candidate for a study on the product space $\mathbb{R} \times \mathbb{T}$. Then, we expose, for three choices of spatial domains, the nonlinear behaviors we can highlight.

## 1. Motivation : about the growth of Sobolev norms

In this section, we present some results on the research about the growth of Sobolev norms for the cubic Schrödinger equation

$$i\partial_t u + \Delta_x u = |u|^2 u. \tag{NLS}$$

The goal is to link the research on the growth of Sobolev norms for the Schrödinger equation (NLS) to the introduction of the cubic coupled Schrödinger (CNLS).

### 1.1. The Bourgain's problem

In 1996, Bourgain highlights in [2] a polynomial bound in time for the Sobolev norms of the solutions of (NLS) on $\mathbb{T}^2$ et $\mathbb{T}^3$. This bound has been then several times improved, especially by Bourgain himself or by Staffilani (see [11]). Nevertheless, in all these works, nothing prove that these bounds are sharp, and no solution of the cubic NLS equation with a growth of Sobolev norms for an infinite time are constructed. However, if this kind of solution doesn't exist, we should be able to find better bounds that polynomial ones (which are growing in infinite time) to control the Sobolev norms of the solutions of the NLS equation.

In 2000, Bourgain states in [1] a list of problems which, according to him, "will not be by any means completed in the near future". Among these problems, convinced that the polynomial bounds obtained are sharp, Bourgain states the following problem for the solution of (NLS) on $\mathbb{T}^d$ (for $d = 2$ or 3):

$$\text{“ Is there a possible growth of} \quad \|u(t)\|_{H^s(\mathbb{T}^d)} \quad \text{for} \quad t \to +\infty \quad \text{? ”}$$

In order to construct such solutions, the strategy evoked is to construct transfers of energy from low Fourier modes to higher ones. The idea is to take profit of the higher weight of the high modes in the Sobolev norms. This is a so-called cascade of energy.

In this article, Bourgain also deals with the question of the growth on $\mathbb{R}^d$. The fact is that in these Euclidean spaces, the scattering results (due to the dispersion of the equation) are in opposition with the growth of the Sobolev norms. Indeed, if the solution $u(t)$ of (NLS) exhibits scattering in the $H^s$ norm to a function $W \in H^s(\mathbb{R}^d)$, then

$$\lim_{t\to+\infty} \|u(t)\|_{H^s(\mathbb{R}^d)} = \|W\|_{H^s(\mathbb{R}^d)},$$

and, in particular,

$$\sup_{t>0} \|u(t)\|_{H^s(\mathbb{R}^d)} < +\infty.$$

Although we are not looking for a growth of Sobolev norms for the coupled system (CNLS), let us see how these works on the growth of Sobolev norms lead us to the study on the product space $\mathbb{R} \times \mathbb{T}$.

### 1.2. Choice of the spatial domain

Let us present some results, through the study of (NLS) on different spatial domains, that give a (some time partial) answer to the question of the growth of Sobolev norms.

On the Euclidean spaces $\mathbb{R}^d$, we know that the dispersion of (NLS) may imply some scattering results and thus prevents any kind of growth of Sobolev norms. For example, for $d = 2$ or 3, the scattering of the solutions has been proved by Ginibre and Ozawa in 1993 in [4]. From this point of view, the critical case for the NLS equation is the one dimensional case: the study on $\mathbb{R}$. In this case, we know since 1991, thanks to Ozawa ([10], see also [8] and [9] for different proofs and generalizations), that we have a modified scattering result. This modification is made by a phase correction term, which doesn't change the Sobolev norms. Therefore, no growth of the Sobolev norms is possible in this critical case too.

The case of the periodic solutions, proposed by Bourgain, seems to be more conducive to the “growth” behaviors because there is no dispersion. In his paper, Bourgain dismisses the case of the one dimensional torus $\mathbb{T}$. Indeed, we know since 1972 and a work of Zakharov and Shabat ([12], see also [5] for a different proof), that the equation (NLS) is completely integrable. Therefore, there is a lot of constants of motion that prevents any growth of the Sobolev norms. On $\mathbb{T}^2$, a first kind of growth of the Sobolev norms, in finite time, has been highlighted by Colliander, Keel, Staffilani, Takaoka and Tao in 2010 in [3]. In this article, the authors show that, for all $s > 1$, for all $\varepsilon > 0$ as small as we want, and for all constant $M > 0$ as big as we want, there exists a time $T > 0$ such that

$$\|u(0)\|_{H^s} \leq \varepsilon \qquad \text{et} \qquad \|u(T)\|_{H^s} \geq M\,.$$

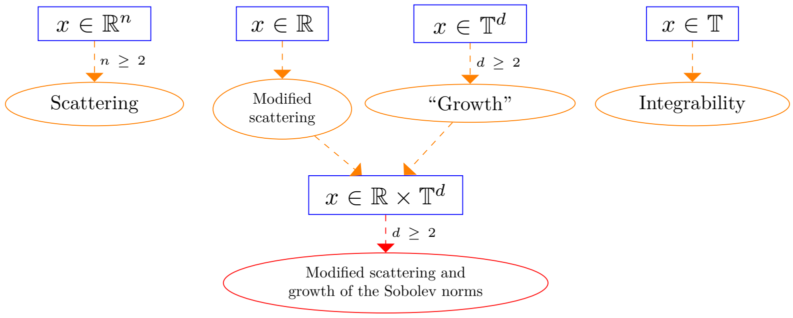
This is a first partial answer to the Bourgain's problem, because although this is a growth of the Sobolev norm, this is a finite time behavior. Nevertheless, this method doesn't allow to study the infinite time behavior of the solution.

The first truly positive answer to the Bourgain's problem is given by in 2013 by Hani, Pausader, Tzvetkov et Visciglia in [7]. In this article, the authors study the equation (NLS) on the product spaces $\mathbb{R} \times \mathbb{T}^d$, with $2 \leq d \leq 4$. The idea is to take advantage of the modified scattering associated to the space $\mathbb{R}$ and the energy cascades from $\mathbb{T}^d$ study, in order to prove a modified scattering result for the solutions of (NLS) to solutions of another equation that provide a growth of the Sobolev norms in infinite time.

These different results are summarized in the following sketch:

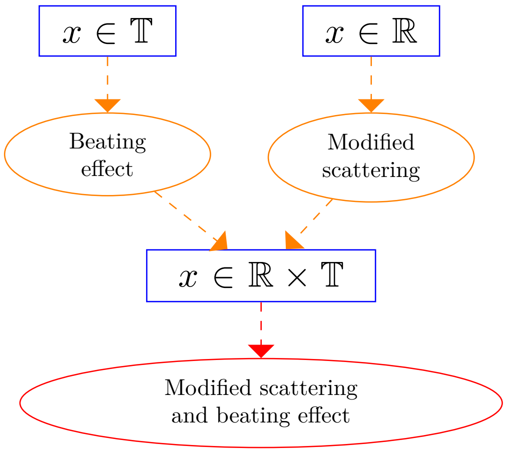
Although, from the point of view of the research of nonlinear behaviors, the study of the equation (NLS) is not relevant on the torus $\mathbb{T}$ due to its complete integrability, the case of the system (CNLS) is totally different. Indeed, it is possible on the torus $\mathbb{T}$ to highlight some exchange of energy for the solutions of (CNLS), this is the beating effect of Grébert, Paturel and Thomann ([6]). This

Cubic NLS and growth of the Sobolev norms: sketch of the known results




exchange of energy, which is a truly nonlinear behavior, is a finite time behavior. The idea of this note is thus to use the method employed by Hani, Pausader, Tzvetkov and Visciglia in [7]: on the product space $\mathbb{R} \times \mathbb{T}$, the goal is to exhibit solutions of (CNLS) which an exchange of energy in infinite time. This strategy is summarized in the following sketch:

Figure 1.1: Beating effect in infinite time: sketch of the strategy




## 2. Presentation of the results

We can now present the results obtained in order to highlight a beating effect in infinite time for the system (CNLS). According to the previous history, the important matter is the choice of the spatial domain. First, in Subsection 2.1, we choose the spatial domain $\mathbb{T}$ in order to present the beating effect in finite time of Grébert, Paturel and Thomann ([6]). Then, in Subsection 2.2, we highlight a modified scattering on $\mathbb{R}$. Finally, in Subsection 2.3, we show that the product space $\mathbb{R} \times \mathbb{T}$ allow us to construct solutions that exhibit a modified scattering to solutions of a resonant system that provide a beating effect in infinite time. For each result, we give the principal tools used and an idea of the proof.

### 2.1. Cubic Schrödinger systems on $\mathbb{T}$: the beating effect

We study the following system:

$$\begin{cases} i\partial_t u + \partial_{xx} u = |v|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{T}, \\ i\partial_t v + \partial_{xx} v = |u|^2 v, \end{cases} \tag{2.1}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus of dimension 1. The strategy of the study is to use:

- The Hamiltonian formalism of the equation to obtain a good framework.
- The study of a reduced model in order to highlight the desired nonlinear behavior.
- A Birkhoff Normal Form Theorem to study the large time behavior of the Hamiltonian.

### 2.1.1. The beating effect

The result we present is the following:

**Theorem A** (Grébert, Paturel, Thomann, 2013). *For all* $\gamma \in \left(0, \frac{1}{2}\right)$*, there exists:*

- $T_\gamma \in \left]0, \frac{|\ln \gamma|}{1-2\gamma}\right[$,
- $K_\gamma : \mathbb{R} \mapsto ]0,1[$ *a* $2T_\gamma$*-periodic function with*
$$K_\gamma(0) = \gamma, \qquad K_\gamma(T_\gamma) = 1-\gamma.$$
- $\varepsilon_0 < \gamma^2$,

*such as if* $p, q \in \mathbb{Z}$ *and* $0 < \varepsilon < \varepsilon_0$*, then there exists a solution of the system* (2.1) *which satisfies for all* $|t| \leq \varepsilon^{-\frac{5}{2}}$*:*

$$\begin{cases} u(t,x) = u_p(t)e^{ipx} + u_q(t)e^{iqx} + \varepsilon^{\frac{5}{2}} r_u(t,x), \\ v(t,x) = v_p(t)e^{ipx} + v_q(t)e^{iqx} + \varepsilon^{\frac{5}{2}} r_v(t,x), \end{cases}$$

*where*

$$\begin{cases} |u_q(t)|^2 = |v_p(t)|^2 = \varepsilon^2 K_\gamma(\varepsilon^2 t) \\ |u_p(t)|^2 = |v_q(t)|^2 = \varepsilon^2 (1 - K_\gamma(\varepsilon^2 t)). \end{cases}$$

*Moreover, the terms* $r_u$ *and* $r_v$ *satisfy :*

- $r_u$ *and* $r_v$ *are smooth in time and analytic in space on* $\left[-\varepsilon^{-\frac{5}{2}}, \varepsilon^{-\frac{5}{2}}\right] \times \mathbb{T}$,
- *for* $r = r_u$ *or* $r_v$*, and for some* $\rho > 0$*:*
$$\sup_{|t| \leq \varepsilon^{-\frac{5}{2}}} |\hat{r_j}(t)| \leq C e^{-\rho |j|},$$
*uniformly in* $\varepsilon > 0$ *and* $p, q \in \mathbb{Z}$.

As presented in the previous section, this is a truly nonlinear behavior, but in finite time. This nonlinear behavior, called the beating effect, is the exchange of energy between the modes $u_q$ and $v_p$ on the one side, and the modes $u_q$ and $v_p$ on the other side. This exchange is characterized by the function $K_\gamma$.

For example, denoting by $I_k = |a_k|^2$ and $J_k = |b_k|^2$ the actions, we obtain for $\gamma = 0.25$ and $\varepsilon = 0.1$:

### 2.1.2. Sketch of the proof

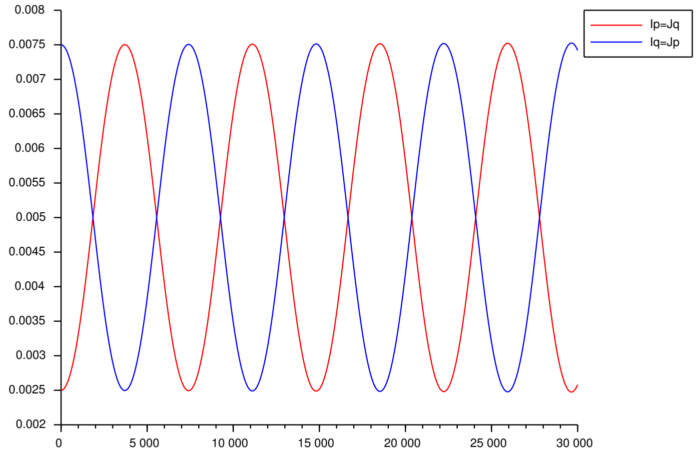
In order to study the system (2.1), the first step is to use its Hamiltonian structure. We set

$$H(u, \overline{u}, v, \overline{v}) := \int_{\mathbb{T}} \left(|\partial_x u|^2 + |\partial_x v|^2 + |u|^2|v|^2\right) \mathrm{d}x,$$

the total energy of the system. $H$ thus defined is the Hamiltonian of the system in the following sense:

$$\begin{cases} \dot{u} &= -i\dfrac{\partial H}{\partial \overline{u}}, \qquad \dot{\overline{u}} = i\dfrac{\partial H}{\partial u}, \\ \dot{v} &= -i\dfrac{\partial H}{\partial \overline{v}}, \qquad \dot{\overline{v}} = i\dfrac{\partial H}{\partial v}. \end{cases}$$

Figure 2.1: Example of beating effect




Therefore, we see that the study of the system is here reduced to the study of the Hamiltonian $H$. As we are on the torus $\mathbb{T}$, we write the Fourier expansions of the solutions $u$ and $v$:

$$u(t,x) = \sum_{j\in\mathbb{Z}} u_j(t)e^{ijx}, \qquad v(t,x) = \sum_{j\in\mathbb{Z}} v_j(t)e^{ijx}.$$

In these coordinates, we have

$$H = N_2 + P_4 := \sum_{j\in\mathbb{Z}} j^2(|u_j|^2 + |v_j|^2) + \sum_{\substack{p,q,r,s\in\mathbb{Z}\\ p-q+r-s=0}} u_p\overline{u_q}v_r\overline{v_s}.$$

The first term, $N_2$, is constant for the solutions of the linear equation: this is the conservation of the energy of the couple of solutions. The important fact here is that $N_2$ depends only on the actions $I_k = |a_k|^2$ and $J_k = |b_k|^2$ (for $k \in \mathbb{Z}$). Therefore, this term will not play a role in the dynamics of the solutions of the system. Thus, we have to focus on the second term $P_4$. The idea is to isolate the dominant terms in $P_4$. Thanks to the Hamiltonian structure of the system, we have a good tool for that purpose with the Birkhoff normal forms. We can write:

$$H^B := H \circ \tau = N_2 + Z_4 + R_6,$$

where :

- $\tau$ is a symplectic transformation (*i.e* which preserves the Hamiltonian structure) close to the identity. Therefore, the dynamics of the new Hamiltonian $H^B$ are close to the ones of the original Hamiltonian $H$.

- $Z_4$ is a $4^{th}$ order homogeneous polynomial which contains the effective terms of the polynomial $P_4$, given by

$$Z_4 = \sum_{\substack{p,q,r,s\in\mathbb{Z}\\ p-q+r-s=0\\ p^2-q^2+r^2-s^2=0}} u_p\overline{u}_q v_r\overline{v}_s. \tag{2.2}$$

  In particular, $Z_4$ commutes with $N_2$ in the sense of the Poisson brackets associated to the symplectic structure of the system.

- $R_6$ is the remainder of order 6, which can be neglected for large but finite times.

With this change of variables, the next idea is to study a reduced model. First, we forget the remainder term $R_6$ and we set up our reduced model by working on the space

$$\mathcal{J}(p,q) = \{(u,v) \ | \ u_j = \overline{u}_j = v_j = \overline{v}_j = 0 \quad \text{if} \quad j \neq p,q\}\,.$$

Since the term $N_2$ is preserved, we focus on the study of the resonant term $Z_4$. As on $\mathcal{J}(p,q)$, there are only 4 modes, we obtain an Hamiltonian system of 8 equations (for the four modes and their conjugates). Thanks to an adapted change of variables, we can prove the complete integrability (on $\mathcal{J}(p,q)$) of the reduced system associated to the Hamiltonian $Z_4$:

$$\begin{cases} i\partial_t u_p = \displaystyle\sum_{(p,q,r,s)\in\Gamma_0} v_q \overline{v}_r u_s, \\ i\partial_t v_p = \displaystyle\sum_{(p,q,r,s)\in\Gamma_0} u_q \overline{u}_r v_s, \end{cases} \tag{2.3}$$

where the resonant set $\Gamma_0$ is defined by

$$\Gamma_0 := \left\{(p,q,r,s) \in \mathbb{Z}^4 \ | \ p-q+r-s=0,\ p^2-q^2+r^2-s^2=0\right\}.$$

In particular, we obtain for the reduced system the existence of solutions that exchange energy in infinite time.

The last part of the proof consists on linking the behaviors observed for the reduced model (the infinite time beating effect) with the behavior of the solutions of the Hamiltonian $H^B$ (where the remainder term $R_6$ is no more neglected). Precisely, the control of the remainder term $R_6$ is the reason why we are forced to make a finite time analysis for the behaviors of the solutions of the Hamiltonian system associated to $H^B$ (in the construction of the Birkhoff normal form, $R_6$ can be neglected for large but finite times).

### 2.2. Cubic Schrödinger systems on $\mathbb{R}$: the modified scattering

We study the following system:

$$\begin{cases} i\partial_t u + \partial_{xx} u = |v|^2 u, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ i\partial_t v + \partial_{xx} v = |u|^2 v. \end{cases} \tag{2.4}$$

We use a classic strategy in order to prove the existence of the solutions to the system (2.4):

- The Duhamel's principle to obtain an expression of the solutions.
- A priori estimates to control the nonlinear effects.
- A fixed-point Theorem to conclude on the existence of the solutions.

The goal is to obtain a result of modified scattering, which is an infinite time behavior. For that purpose, we adapt the method of Kato and Pusateri in [9]. First, let us introduce some notations.

Working with small initial data, we can expect the nonlinearity to stay small, and thus the linear dynamics to be dominant. This is the reason why it is interesting to consider the profile of a solution, which is the backwards linear evolution of a solution of the nonlinear equation. We define $f$ (respectively $g$) the profile of the solution $u$ (respectively $v$) by

$$f(t,x) := e^{-it\partial_{xx}} u(t,x) \qquad\qquad g(t,x) := e^{-it\partial_{xx}} v(t,x). \tag{2.5}$$

We see in the sketch of the proof that after considering the Duhamel's formula of the solutions, the equations in $f$ and $g$ are easier to treat than the ones with $u$ and $v$. This is the main reason why the profiles are introduced here. Moreover, the profiles are the adapted variables in order to write a scattering result. Indeed, we say that the solution $u$ exhibits a scattering to a function $W$ for the norm $N$ if

$$\|u(t) - e^{it\partial_{xx}} W\|_N \underset{t\to+\infty}{\longrightarrow} 0.$$

If the Schrödinger operator $e^{it\partial_{xx}}$ is unitary for the norm $N$ (which is the case for the Sobolev norms), then $u$ exhibits a scattering to a function $W$ for the norm $N$ if

$$\|u(t) - e^{it\partial_{xx}} W\|_N = \|f(t) - W\|_N \underset{t\to+\infty}{\longrightarrow} 0. \tag{2.6}$$

Therefore, in order to prove a scattering result, we have to prove that the profiles tend to be constant.

We also introduce $I = [1, +\infty)$ and the initial data

$$u_1(x) = u(1, x), \qquad v_1(x) = v(1, x).$$

Due to the behavior of the solutions of the linear Schrödinger equation near 0, we can expect some integration problems in 0 for the construction of the solutions through the Duhamel's principle. Our purpose here is to highlight an asymptotic behavior, this is the reason why we just avoid this problem by taking a strictly positive time for the initial data.

Finally, for the control of the initial data, we introduce the space

$$H^{1,1} := \left\{\varphi \in L^2_x,\ \|\varphi\|_{H^1_x} + \|x\varphi\|_{L^2_x} < +\infty\right\}.$$

#### 2.2.1. The modified scattering

We can now state the result:

**Theorem B** (VVDR, 2015)**.** *Fix $0 < \nu < \frac{1}{4}$. Assume that for $\varepsilon$ small enough we have*

$$\|u_1\|_{H^{1,1}_x} + \|v_1\|_{H^{1,1}_x} \leq \varepsilon.$$

*Then the system (2.4) admits a unique pair of solutions $(u, v) \in \mathcal{C}\left(I; H^{1,1}(I)\right) \times \mathcal{C}\left(I; H^{1,1}(I)\right)$. The pair of solutions satisfies the following decay estimates:*

$$\|u(t)\|_{L^\infty_x} \lesssim \frac{1}{t^{\frac{1}{2}}}, \qquad \|v(t)\|_{L^\infty_x} \lesssim \frac{1}{t^{\frac{1}{2}}}.$$

*Furthermore, we have a modified scattering result in the following sense: there exists a unique pair of functions $(W_f, W_g) \in (L^\infty_\xi \cap L^2_\xi) \times (L^\infty_\xi \cap L^2_\xi)$ such that for $t \geq 1$,*

$$\begin{cases} \|\hat{f}(t,.) \exp\left(\dfrac{i}{2\sqrt{2\pi}} \displaystyle\int_1^t \frac{1}{s} |\hat{v}(s,.)|^2 \,\mathrm{d}s\right) - W_f\|_{L^\infty_\xi} & \lesssim t^{-\frac{1}{4}+\nu}, \\ \|\hat{g}(t,.) \exp\left(\dfrac{i}{2\sqrt{2\pi}} \displaystyle\int_1^t \frac{1}{s} |\hat{u}(s,.)|^2 \,\mathrm{d}s\right) - W_g\|_{L^\infty_\xi} & \lesssim t^{-\frac{1}{4}+\nu}, \end{cases}$$

*and*

$$\begin{cases} \|\hat{f}(t,.) \exp\left(\dfrac{i}{2\sqrt{2\pi}} \displaystyle\int_1^t \frac{1}{s} |\hat{v}(s,.)|^2 \,\mathrm{d}s\right) - W_f\|_{L^2_\xi} & \lesssim t^{-\frac{1}{8}+\frac{\nu}{2}}, \\ \|\hat{g}(t,.) \exp\left(\dfrac{i}{2\sqrt{2\pi}} \displaystyle\int_1^t \frac{1}{s} |\hat{u}(s,.)|^2 \,\mathrm{d}s\right) - W_g\|_{L^2_\xi} & \lesssim t^{-\frac{1}{8}+\frac{\nu}{2}}, \end{cases}$$

*where $f$ and $g$ are defined in (2.5).*

*Finally, we have an asymptotic formula. There exists an unique pair of functions $\Gamma_f, \Gamma_g \in L^\infty_\xi$ such that for large time $t$,*

$$\begin{cases} u(t,x) = \dfrac{1}{(2it)^{\frac{1}{2}}} W_f(\dfrac{x}{2t}) \exp\left(i\dfrac{x^2}{4t} - \dfrac{i}{2\sqrt{2\pi}}(|W_g|^2(\dfrac{x}{2t})\ln(t) + \Gamma_g(\dfrac{x}{2t}))\right) + \mathcal{O}(t^{-\frac{3}{4}+\nu}), \\ v(t,x) = \dfrac{1}{(2it)^{\frac{1}{2}}} W_g(\dfrac{x}{2t}) \exp\left(i\dfrac{x^2}{4t} - \dfrac{i}{2\sqrt{2\pi}}(|W_f|^2(\dfrac{x}{2t})\ln(t) + \Gamma_f(\dfrac{x}{2t}))\right) + \mathcal{O}(t^{-\frac{3}{4}+\nu}). \end{cases}$$

The advantage here is the infinite time for the nonlinear behavior. But the nonlinear behavior is just a phase correction term in the modified scattering result. We also remark that we see in this phase correction term the role of the coupling.

#### 2.2.2. Sketch of the proof

Writing the Duhamel formula, for example for $u$, we have:

$$u(t,x) = e^{i(t-1)\partial_{xx}} u_1(x) - i \int_1^t e^{i(t-s)\partial_{xx}} |v(s,x)|^2 u(s,x) \mathrm{d}s.$$

In order to simplify this expression, due to the presence of the terms $e^{-it\partial_{xx}}$, we introduce the profiles $f$ and $g$. The previous equation becomes

$$f(t,x) = e^{-i\partial_{xx}}u_1(x) - i\int_1^t e^{-is\partial_{xx}}\left(e^{is\partial_{xx}}g(s,x)e^{-is\partial_{xx}}\overline{g}(s,x)e^{is\partial_{xx}}f(s,x)\right)\mathrm{d}s.$$

In order to deal with the $e^{\pm is\partial_{xx}}$ terms, we perform a spatial Fourier transform. The product in the right-hand side thus becomes a convolution product. Applying a derivative in time, we obtain ($c>0$ is a constant depending on the convention chosen for the Fourier transform):

$$i\partial_t\hat{f}(t,\xi) = c\int_{\mathbb{R}^2} e^{2it\eta\kappa}\hat{g}(t,\xi-\kappa)\overline{\hat{g}}(t,\xi-\eta-\kappa)\hat{f}(t,\xi-\eta)\mathrm{d}\kappa\mathrm{d}\eta.$$

We have to study this integral, for that purpose, according to a stationary phase intuition, we split the integral to obtain:

$$i\partial_t\hat{f}(t,\xi) = \frac{\tilde{c}}{t}\left|\hat{g}(t,\xi)\right|^2\hat{f}(t,\xi) + R(t,\xi).$$

The remainder term $R$ has a good behavior for large time: it is integrable. To take profit of this good decay, we perform a new change of variables by introducing $\hat{w}_f$ and $\hat{w}_g$ defined by

$$\hat{w}_f(t,\xi) = \hat{f}(t,\xi)\exp\left(i\tilde{c}\int_1^t\frac{1}{s}\left|\hat{g}(s,\xi)\right|^2 ds\right) = \hat{f}(t,\xi)\exp\left(i\tilde{c}\int_1^t\frac{1}{s}\left|\hat{v}(s,\xi)\right|^2\mathrm{d}s\right).$$

We notice here that $\hat{w}$ is the first term of the scattering result of Theorem B. Therefore, the aim now is to study the convergence of this term. For that purpose, we use the good decay of $R$ since we have

$$\partial_t\hat{w}_f(t,\xi) = -i\exp\left(i\tilde{c}\int_1^t\frac{1}{s}\left|\hat{g}(s,\xi)\right|^2\mathrm{d}s\right)R(t,\xi).$$

The global existence of the solution, the dispersive inequalities and the asymptotic behavior are obtained by intermediate estimates of the proof of this modified scattering result.

### 2.3. Cubic Schrödinger systems on $\mathbb{R}\times\mathbb{T}$: modified scattering and beating effects

In order to take profit of the truly nonlinear behavior observed on the torus (the beating effect from Theorem A) and the infinite time behavior on the real line (the modified scattering from Theorem B), we consider the product space $\mathbb{R}\times\mathbb{T}$ and thus the following system:

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{R}\times\mathbb{T}}u &= |v|^2 u, \quad (t,x,y)\in[0,+\infty)\times\mathbb{R}\times\mathbb{T}, \\ i\partial_t v + \Delta_{\mathbb{R}\times\mathbb{T}}v &= |u|^2 v. \end{cases} \tag{2.7}$$

We adapt the method of Hani, Pausader, Tzvetkov and Visciglia ([7]) for the coupled system case. The goal is to obtain a correspondence between the behaviors of the solutions of the initial system (2.7) and those of the solutions of a simpler system, called the resonant system, defined by:

$$\begin{cases} i\partial_\tau W_u(\tau) = \mathcal{R}\left[W_v(\tau),W_v(\tau),W_u(\tau)\right], \\ i\partial_\tau W_v(\tau) = \mathcal{R}\left[W_u(\tau),W_u(\tau),W_v(\tau)\right], \end{cases} \tag{2.8}$$

where the nonlinearity $\mathcal{R}$ is defined in the spatial Fourier space by:

$$\mathcal{F}\mathcal{R}\left[f,g,h\right](\xi,p) = \sum_{\substack{p-q+r-s=0\\ p^2-q^2+r^2-s^2=0}}\hat{f}(\xi,q)\overline{\hat{g}(\xi,r)}\hat{h}(\xi,s).$$

We remark here that the resonant system (2.8) associated to the nonlinearity $\mathcal{R}$ is really close to the Hamiltonian system associated to the Hamiltonian $Z_4$ defined in (2.2), where we have a beating effect in infinite time. This aspect was the key point of the start of this work on the product space.

We use the same strategy as in the previous case in order to prove the existence of the solutions to the system (2.7):

- The Duhamel's principle to obtain an expression of the solutions.

- A priori estimates to control the nonlinear effects. We need some tools of Harmonic Analysis such as the Littlewood-Paley decomposition.

- A fixed-point Theorem to conclude on the existence of the solutions.

As in the previous study, we need to introduce the norm used for the control of the initial data. Let $N \geq 12$ be an integer, we define the norms $S$ and $S^+$ by

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}}, \quad \|F\|_{S^+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S.$$

We remark that they are $L^2$-based norms, controlling a large number of derivatives in both spatial variables and some localization in the Euclidean variable $x$. The only thing we have to remember from these definitions is that the spaces defined by these norms are Banach spaces (we want to perform a fixed-point Theorem) that contain the Schwartz space of functions with rapidly decreasing derivatives.

We can now state the results of this section, which are based on two main ideas:

- The set up of a modified scattering result and its dynamical consequences. In particular, we show that the modified scattering allows to control the Sobolev norms of the solutions of the system (2.7).

- The set up of a modified wave operator result and its dynamical consequences. The goal here is to use the modified wave operator in order to construct solutions of the system (2.7) which provide a beating effect in infinite time thanks to the study of the system (2.1) on the torus.

### 2.3.1. The modified scattering

First, we obtain for the system (2.7) a modified scattering result, analogue of the result for the NLS equation of Hani, Pausader, Tzvetkov and Visciglia in [7]:

**Theorem C** (Modified scattering, VVDR 2016)**.** *There exists $\varepsilon > 0$ such that if $u_0, v_0 \in S^+$ satisfies*

$$\|u_0\|_{S^+} + \|v_0\|_{S^+} \leq \varepsilon,$$

*and if $(u(t), v(t))$ solves the system* (2.7) *with initial data $(U_0, V_0)$, then the couple of solutions $(u, v) \in \mathcal{C}([0, +\infty) : H^N) \times \mathcal{C}([0, +\infty) : H^N)$ exists globally and exhibits modified scattering to its resonant dynamics* (2.8) *in the following sense: there exists $(W_{u,0}, W_{v,0})$ such that if $(W_u(t), W_v(t))$ is the solution of* (2.8) *with initial data $(W_{u,0}, W_{v,0})$, then*

$$\begin{cases} \|u(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_U(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad as \quad t \to +\infty, \\ \|v(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_V(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad as \quad t \to +\infty. \end{cases}$$

*Furthermore, we have the following decay estimate*

$$\begin{cases} \|u(t)\|_{L^\infty_x H^1_y} & \lesssim (1+|t|)^{-\frac{1}{2}}, \\ \|v(t)\|_{L^\infty_x H^1_y} & \lesssim (1+|t|)^{-\frac{1}{2}}. \end{cases}$$

We first remark here that we have the same dispersion as in the case of system (2.4) on $\mathbb{R}$. The fact is we have in both case a $L^\infty$ control on the Euclidean variable. Thus, the periodic variable doesn't seem to play a role in the dispersion estimate, this is due to the choice of the $H^1$ for this variable, a norm which is conserved in the linear system for the periodic variable.

Then, this theorem show that for small enough initial data, each couple of solution of the initial system (2.7) behaves asymptotically as a solution of the resonant system (2.8). In particular, a control of the Sobolev norms of the solutions of the resonant system (2.8) allows to obtain a control of the Sobolev norms of the initial system (2.7). This is the purpose of the following theorem:

**Theorem D** (Control of the Sobolev norms, VVDR 2016)**.** *There exists $\varepsilon > 0$ such that if $U_0, V_0 \in S^+$ satisfy*

$$\|U_0\|_{S^+} + \|V_0\|_{S^+} \leq \varepsilon,$$

*and if $(U(t), V(t))$ solves the system (2.7) with initial data $(U_0, V_0)$, then the couple of solutions $(U, V) \in \mathcal{C}([0, +\infty) : H^N) \times \mathcal{C}([0, +\infty) : H^N)$ exists globally and, for all $s \in \mathbb{R}$,*

$$\|u(t)\|_{H^s_{x,y}} + \|v(t)\|_{H^s_{x,y}} \lesssim \varepsilon.$$

*Moreover, there exists a constant $c \geq 0$ such that*

$$\lim_{t \to +\infty} \left( \|U(t)\|_{H^s_{x,y}} + \|V(t)\|_{H^s_{x,y}} \right) = c.$$

Thanks to this result, we know that no growth of the Sobolev norms in infinite time is possible for solutions of (2.7) with small initial data. On the one hand, this result is close to the behavior of the Sobolev norms of the solutions of system (2.4) posed on $\mathbb{R}$. On the other hand, this is a truly new result for this kind of system with periodic variables. Indeed, at the best of our knowledge, the same property is unknown for the same system (2.1) posed on $\mathbb{T}$, where the question of the complete integrability is unsolved.

### 2.3.2. The modified wave operator

After the existence of a scattering result, the natural question is to look for a wave operator. Indeed, if we know that each couple of solutions of the system (2.7) scatters to a couple of solutions of the resonant system (2.8), we want to know if all the couples of solutions of the resonant system (2.8) are limits of couples of solutions of the initial system. An answer is given by the following theorem:

**Theorem E** (Modified wave operator, VVDR 2016)**.** *There exists $\varepsilon > 0$ such that if $W_{u,0}, W_{v,0} \in S^+$ satisfy*

$$\|W_{u,0}\|_{S^+} + \|W_{v,0}\|_{S^+} \leq \varepsilon,$$

*and if $(W_u(t), W_v(t))$ solves the resonant system (2.8) with initial data $(W_{u,0}, W_{v,0})$, then there exists a couple $(u, v) \in \mathcal{C}([0, +\infty) : H^N) \times \mathcal{C}([0, +\infty) : H^N)$ solution of (2.7) such that*

$$\begin{cases} \|u(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_u(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad \text{as} \quad t \to +\infty, \\ \|v(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_v(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad \text{as} \quad t \to +\infty. \end{cases}$$

Thanks to the construction of the modified wave operator in Theorem E, the idea is to find some interesting nonlinear behavior of the resonant system (2.8) in order to transfer this behavior to the initial system.

In particular, we can use here the link between the resonant system (2.8) and the system associated to $Z_4$ in (2.3) from the study on the torus. As a consequence, we follow the strategy of [6] to construct couples of solutions of the initial system (2.7) which scatter to beating effect solutions of the resonant system (2.8):

**Theorem F** (Modified scattering and beating effect, VVDR 2016)**.** *Let $I \subset \mathbb{R}$ be a bounded open interval, $(p, q)$ a couple of different integers and $0 < \gamma < \frac{1}{2}$. For $\varepsilon = \varepsilon(p, q, I) > 0$ small enough, there exists:*

- *a constant $0 < T_\gamma \lesssim |\ln(\gamma)|$ and a $2\,T_\gamma$−periodic function $K_\gamma : \mathbb{R} \to (0, 1)$ such that*

$$K_\gamma(0) = \gamma, \qquad K_\gamma(T_\gamma) = 1 - \gamma;$$

- *a couple of solutions $(W_u, W_v)$ of the resonant system (2.8) which exhibits a beating effect in the following sense:*

$$\begin{cases} \hat{W}_u(t, \xi, y) = \mathcal{F}(W_u)(t, \xi, p)e^{ipy} + \mathcal{F}(W_u)(t, \xi, q)e^{iqy}, \\ \hat{W}_v(t, \xi, y) = \mathcal{F}(W_v)(t, \xi, p)e^{ipy} + \mathcal{F}(W_v)(t, \xi, q)e^{iqy}, \end{cases}$$

*and $\forall \xi \in I$,*

$$\begin{cases} |\hat{W}_v(t, \xi, p)|^2 = |\hat{W}_u(t, \xi, q)|^2 = \varepsilon^2 K_\gamma(\varepsilon^2 t), \\ |\hat{W}_u(t, \xi, p)|^2 = |\hat{W}_v(t, \xi, q)|^2 = \varepsilon^2(1 - K_\gamma(\varepsilon^2 t)); \end{cases}$$

- *a couple of solutions* $(u,v)$ *of the initial system* (2.7) *which exhibits modified scattering to this couple* $(W_u, W_v)$ *in the following sense:*

$$\begin{cases} \|u(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_u(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad \text{as} \quad t \to +\infty, \\ \|v(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} W_v(\pi \ln(t))\|_{H^N(\mathbb{R}\times\mathbb{T})} & \to 0 \quad \text{as} \quad t \to +\infty. \end{cases}$$

With this result, we have a truly nonlinear behavior (the beating effect, associated to the periodic variable) in infinite time (thanks to the modified scattering associated to the Euclidean variable). That was the goal of this work, and it shows the role of each variable of the chosen spatial domain $\mathbb{R}\times\mathbb{T}$.

#### 2.3.3. Sketch of the proofs

First, we study the structure of the nonlinearity defined by the system (2.7) to show that this system is close to the resonant one associated to the nonlinearity $\mathcal{R}$. This is the key of the modified scattering and modified wave operator theorems. Then, we show how to obtain the dynamical consequences (the control of the Sobolev norms and the beating effect in infinite time).

As in the study of the system (2.4) on $\mathbb{R}$, we start here by writing the Duhamel's formula for the solutions for the profiles. For example, for $f(t) = e^{-it\Delta_{\mathbb{R}\times\mathbb{T}}}u(t)$ the profile of $u$, we obtain

$$i\partial_t f(t) = e^{-it\Delta_{\mathbb{R}\times\mathbb{T}}} \left( e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} g(t) e^{-it\Delta_{\mathbb{R}\times\mathbb{T}}} \overline{g(t)} e^{it\Delta_{\mathbb{R}\times\mathbb{T}}} f(t) \right) =: \mathcal{N}^t[g(t), g(t), f(t)].$$

In the same way that the study of the Hamiltonian made it possible to know perfectly the system (2.1) in the study on the torus, the aim here is to study the nonlinearity $\mathcal{N}^t$, and in particular to exhibit the dominant terms. As in the previous parts, via the Fourier transform in the spatial variables, we obtain here in the frequency space

$$\mathcal{F}\mathcal{N}^t[f,g,h](\xi,p) = \sum_{p-q+r-s=0} e^{it(p^2-q^2+r^2-s^2)} \int_{\mathbb{R}^2} e^{2it\eta\kappa} \hat{f}_q(\xi-\eta)\overline{\hat{g}_r}(\xi-\eta-\kappa)\hat{h}_s(\xi-\kappa)\mathrm{d}\eta\mathrm{d}\kappa.$$

The integral part is close to the one obtain in the study on $\mathbb{R}$. Therefore, the same stationary phase intuition allows us to write

$$\int_{\mathbb{R}^2} e^{2it\eta\kappa} \hat{f}_q(\xi-\eta)\overline{\hat{g}_r}(\xi-\eta-\kappa)\hat{h}_s(\xi-\kappa)\mathrm{d}\eta\mathrm{d}\kappa \approx \frac{\pi}{t}\hat{f}_q(\xi)\overline{\hat{g}_r}(\xi)\hat{h}_s(\xi).$$

For the sum indexed by $p$, as in the study on $\mathbb{T}$ with the Birkhoff normal form, we can think that the dominant term is given by the resonance relation

$$p-q+r-s=0, \quad p^2-q^2+r^2-s^2=0.$$

Therefore, we see here the resonant nonlinearity $\mathcal{R}$ appear, and we decompose the nonlinearity $\mathcal{N}^t$ as

$$\mathcal{N}^t[f,g,h] = \frac{\pi}{t}\mathcal{R}[f,g,h] + \mathcal{E}^t[f,g,h].$$

Thus, the aim now is to show that the remainder term $\mathcal{E}^t$ decreases fast enough. This good decreasing behavior leads to the modified scattering and modified wave operator results. We thus obtain a correspondence between the solutions of the initial system (associated with the nonlinearity $\mathcal{N}^t$) and the solutions of the resonant system (associated with the non-linearity $\mathcal{R}$). We remark are that the $\frac{\pi}{t}$ factor in front of the nonlinearity $\mathcal{R}$ is responsible to the change of timescale in $\pi\ln(t)$ in both Theorems C and E.

For the dynamical consequences, we first remark that the control of the Sobolev norms of Theorem D is a direct consequence of the control of the Sobolev norms of the resonant system, which is made easy by the restrictive resonance relation (due to the dimension 1):

$$p-q+r-s=0 \quad \text{and} \quad p^2-q^2+r^2-s^2=0 \Rightarrow \{p,r\}=\{q,s\}.$$

In order to highlight the beating effect of Theorem F, we recall that by definition

$$\mathcal{F}\mathcal{R}\left[f,g,h\right](\xi,p) = \sum_{(p,q,r,s)\in\Gamma_0} \hat{f}(\xi,q)\overline{\hat{g}(\xi,r)}\hat{h}(\xi,s).$$

In this equation, the variable $\xi$ is just a parameter. Thus, we want to make a link between the nonlinearity $\mathcal{R}$ and the nonlinearity $R$ obtained by deleting this variable:

$$R\,[f,g,h]_p = \sum_{\substack{p,q,r,s\in\mathbb{Z}\\ p-q+r-s=0\\ p^2-q^2+r^2-s^2=0}} f_q\overline{g_r}h_s,$$

The nonlinearity $R$ define a new system called the reduced resonant system. We remark that this reduced resonant system is exactly the system (2.3) associated to $Z_4$ in the first section on the torus, where we have a beating effect for infinite time. Because of the "brutality" of the employed method to obtain this nonlinearity (we have deleted a variable), we can't expect a bilateral correspondence between the solutions of the reduced resonant system (associated to $R$) and the solutions of the resonant system (associated to $\mathcal{R}$). Nevertheless, one can hope to build a solution of the resonant system from a reduced resonant system solution, and thus enjoy all the dynamics of it. This transfer of solutions is given by the following method:

Let $(a,b)$ be a solution of the reduced resonant system of which we want to mimic the behavior (for example the beating effect). We have :

$$i\partial_t a_p = R[b,b,a]_p, \qquad i\partial_t b_p = R[a,a,b]_p.$$

We chose adapted initial data:

$$W_{u,0}(x,y) = \mathcal{F}^{-1}(\varphi)(x)\alpha(y), \qquad W_{v,0}(x,y) = \mathcal{F}^{-1}(\varphi)(x)\beta(y),$$

where $\varphi$ is a function, and $\alpha$,$\beta$ are given by $\alpha_p = a_p(0)$, $\beta_p = b_p(0)$. Thanks to the separating the variables for the initial data, a solution $(W_u, W_v)$ of the resonant system is then constructed by setting

$$\mathcal{F}(W_U)(t,\xi,p) = \varphi(\xi)a_p\left(\varphi(\xi)^2 t\right), \qquad \mathcal{F}(W_V)(t,\xi,p) = \varphi(\xi)b_p\left(\varphi(\xi)^2 t\right).$$

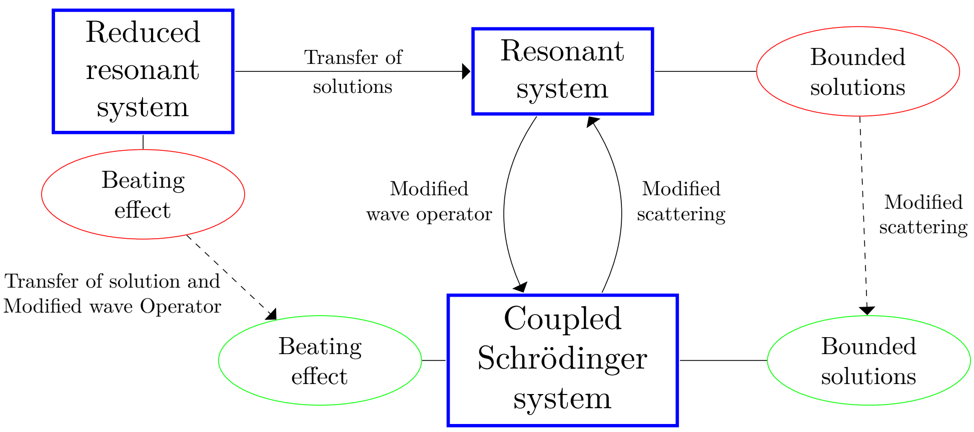
In particular, if $\varphi \equiv 1$ on an interval $I$, then

$$(\mathcal{F}W_u(t,\xi,p), \mathcal{F}W_v(t,\xi,p)) = (a_p(t), b_p(t)), \quad \forall t\in\mathbb{R} \quad \text{et} \quad \xi \in I.$$

We see through this method the introduction of the interval $I$ of Theorem F.

This strategy is summarized in the following sketch:

Figure 2.2: Sketch of the strategy



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