Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Roscoff, 30 mai–3 juin 2016

Riccardo Montalto

A note on KAM for gravity-capillary water waves

J. É. D. P. (2016), Exposé n^{o} VII, 18 p.

<http://jedp.cedram.org/item?id=JEDP_2016____A7_0>

Journées *Équations aux dérivées partielles*
Roscoff, 30 mai–3 juin 2016
GDR 2434 (CNRS)

A note on KAM for gravity-capillary water waves

Riccardo Montalto

Abstract

We present the result and the ideas of the recent paper [8] (obtained in collaboration with M. Berti) concerning the existence of Cantor families of small amplitude time *quasi-periodic* standing wave solutions (i.e. periodic and even in the space variable x) of a 2-dimensional ocean, with infinite depth, in irrotational regime, under the action of gravity and surface tension at the free boundary. These quasi-periodic solutions are linearly stable.

1. Introduction

Consider a 2-dimensional ocean, with infinite depth, filled by an incompressible fluid, in irrotational regime, under the action of gravity and capillarity at the surface. The fluid satisfies periodic boundary conditions and occupies the free boundary region

$$\mathcal{D}_\eta := \big\{(x,y) \in \mathbb{T} \times \mathbb{R} \, : \, y < \eta(t,x) \, , \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})\big\} \, .$$

Since the velocity field is irrotational, it is the gradient of a velocity potential $\Phi(t,x,y)$. The incompressibility condition means that Φ is an harmonic function on $\mathcal{D}_\eta$. In this context, the Euler equation for the motion of the fluid reduces to the Bernoulli equation. The water waves equations are

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = \kappa \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{at } y = \eta(x) \\ \Delta\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \nabla\Phi \to 0 & \text{as } y \to -\infty \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } y = \eta(x) \end{cases} \tag{1.1}$$

where g is the acceleration of gravity, $\kappa \in [\kappa_1, \kappa_2]$, $\kappa_1 > 0$, is the surface tension coefficient and

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)$$

is the mean curvature of the free surface. The unknowns of the problem are the free surface $y = \eta(x)$ and the velocity potential $\Phi : \mathcal{D}_\eta \to \mathbb{R}$. The first equation in (1.1) is the Bernoulli condition (also called dynamics condition) according to which the jump of pressure across the free surface is proportional to the mean curvature. The last equation in (1.1) (also called kynematic condition) expresses that the velocity of the free surface coincides with the one of the fluid particles, and therefore the fluid particles on the free surface $y = \eta(x,t)$ remain on it along the fluid evolution. In the sequel we shall assume (with no loss of generality) that the gravity constant $g = 1$.

Following Zakharov [35] and Craig-Sulem [14], the evolution problem (1.1) may be written as an infinite dimensional Hamiltonian system. At each time $t \in \mathbb{R}$ the profile $\eta(t,x)$ of the fluid and the value

$$\psi(t,x) = \Phi(t,x,\eta(t,x))$$

MSC 2000: 76B15, 37K55, 76D45, 37K50.
Keywords: KAM for PDEs, water waves, quasi-periodic solutions.

of the velocity potential Φ restricted to the free boundary uniquely determine the velocity potential Φ in the whole $\mathcal{D}_\eta$, solving (at each t) the elliptic problem

$$\Delta\Phi = 0 \quad \text{in } \mathcal{D}_\eta, \quad \Phi(x+2\pi, y) = \Phi(x,y)\,, \quad \Phi|_{y=\eta} = \psi, \quad \nabla\Phi(x,y) \to 0 \text{ as } y \to -\infty\,.$$

As proved in [35], [14], system (1.1) is then equivalent to the system

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + \eta + \frac{1}{2}\psi_x^2 - \frac{1}{2}\frac{\big(G(\eta)\psi + \eta_x\psi_x\big)^2}{1+\eta_x^2} = \kappa \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \end{cases} \tag{1.2}$$

where $G(\eta)$ is the so-called Dirichlet–Neumann operator defined by

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\partial_y\Phi)(x,\eta(x)) - \eta_x(x)\,(\partial_x\Phi)(x,\eta(x))$$

(we denote by η_x the space derivative $\partial_x\eta$.) The operator $G(\eta)$ is linear in ψ, self-adjoint with respect to the L^2 scalar product and semi positive definite, actually its Kernel are only the constants. It depends in a analytic way with respect to the free boundary $\eta(x)$ and its derivative with respect to η is

$$d_\eta G(\eta)[\hat\eta]\psi = -G(\eta)(B\hat\eta) - \partial_x(V\hat\eta) \tag{1.3}$$

where

$$B := B(\eta,\psi) := \frac{\eta_x\psi_x + G(\eta)\psi}{1+\eta_x^2}\,, \qquad V := V(\eta,\psi) := \psi_x - B\eta_x. \tag{1.4}$$

The vector $(V,B) = \nabla_{x,y}\Phi$ is the velocity field evaluated at the free surface $y = \eta(x)$. It is well known since Calderon that the Dirichlet-Neumann operator $G(\eta)$ is a *pseudo-differential* operator with principal symbol $|D|$, actually $G(\eta) - |D| \in OPS^{-\infty}$, if η is $\mathcal{C}^\infty$.

The equations (1.2) are the Hamiltonian system (see [35], [14])

$$\partial_t\eta = \nabla_\psi H(\eta,\psi)\,, \quad \partial_t\psi = -\nabla_\eta H(\eta,\psi)$$

$$\partial_t u = J\nabla_u H(u)\,, \quad u := \begin{pmatrix}\eta\\ \psi\end{pmatrix}\,, \quad J := \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}\,, \tag{1.5}$$

where ∇ denotes the L^2-gradient, and the Hamiltonian

$$H(\eta,\psi) := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} + \int_{\mathbb{T}} \frac{\eta^2}{2}\,dx + \kappa\int_{\mathbb{T}}\sqrt{1+\eta_x^2}\,dx \tag{1.6}$$

is the sum of the kinetic energy

$$K := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} = \frac{1}{2}\int_{\mathcal{D}_\eta}|\nabla\Phi|^2(x,y)dxdy\,,$$

the potential energy and the energy of the capillarity forces (area surface integral) expressed in terms of the variables (η,ψ). In light of (1.5) the variables (η,ψ) are symplectic "Darboux coordinates" and the symplectic structure is the standard Darboux 2-form

$$\mathcal{W}(u_1,u_2) := (u_1, Ju_2)_{L^2(\mathbb{T}_x)} = (\eta_1,\psi_2)_{L^2(\mathbb{T}_x)} - (\psi_1,\eta_2)_{L^2(\mathbb{T}_x)} \tag{1.7}$$

for all $u_1 = (\eta_1,\psi_1)$, $u_2 = (\eta_2,\psi_2)$.

The water-waves system (1.2)-(1.5) exhibits several symmetries. First of all, the *mass*

$$\int_{\mathbb{T}} \eta\,dx$$

is a prime integral of (1.2). Moreover

$$\partial_t\int_{\mathbb{T}}\psi\,dx = -\int_{\mathbb{T}}\eta\,dx - \int_{\mathbb{T}}\nabla_\eta K\,dx = -\int_{\mathbb{T}}\eta\,dx$$

because $\int_{\mathbb{T}}\nabla_\eta K\,dx = 0$. This follows because $\mathbb{R}\ni c\mapsto K(c+\eta,\psi)$ is constant (the bottom of the ocean is at $-\infty$) and so $0 = d_\eta K(\eta,\psi)[1] = (\nabla_\eta K, 1)_{L^2(\mathbb{T})}$. As a consequence the subspace

$$\int_{\mathbb{T}}\eta\,dx = \int_{\mathbb{T}}\psi\,dx = 0 \tag{1.8}$$

is invariant under the evolution of (1.2) and we shall restrict to solutions satisfying (1.8).

In addition, the subspace of functions which are *even* in x,

$$\eta(x) = \eta(-x)\,, \quad \psi(x) = \psi(-x)\,, \tag{1.9}$$

is invariant under (1.2). Thus we restrict (η, ψ) to the phase space of 2π-periodic even functions with zero mean, i.e. which admit the Fourier expansion

$$\eta(x) = \sum\nolimits_{j\geq 1} \eta_j \cos(jx)\,, \quad \psi(x) = \sum\nolimits_{j \geq 1} \psi_j \cos(jx)\,. \tag{1.10}$$

In this case also the velocity potential $\Phi(x, y)$ is even and 2π-periodic in x and so the x-component of the velocity field $v = (\Phi_x, \Phi_y)$ vanishes at $x = k\pi$, $\forall k \in \mathbb{Z}$. Hence there is no flux of fluid through the lines $x = k\pi$, $k \in \mathbb{Z}$, and a solution of (1.2) satisfying (1.10) physically describes the motion of a liquid confined between two walls.

Another important symmetry of the capillary-gravity water waves system is *reversibility*, namely the equations (1.2)-(1.5) are reversible with respect to the involution $\rho : (\eta, \psi) \mapsto (\eta, -\psi)$, or, equivalently, the Hamiltonian is even in ψ:

$$H \circ \rho = H\,, \quad H(\eta, \psi) = H(\eta, -\psi)\,, \quad \rho : (\eta, \psi) \mapsto (\eta, -\psi)\,. \tag{1.11}$$

As a consequence it is natural to look for solutions of (1.2) satisfying

$$u(-t) = \rho u(t)\,, \quad i.e. \quad \eta(-t, x) = \eta(t, x)\,,\ \psi(-t, x) = -\psi(t, x)\,,\ \forall t, x \in \mathbb{R}\,, \tag{1.12}$$

namely η is even in time and ψ is odd in time. Solutions of the water waves equations (1.2) satisfying (1.10) and (1.12) are called capillary-gravity *standing water waves*.

Existence of small amplitude time periodic pure gravity (without surface tension) standing wave solutions has been proved by Iooss, Plotnikov, Toland in [23], see also [19], [20], and in [30] in finite depth. Existence of time periodic capillary-gravity standing wave solutions has been recently proved by Alazard-Baldi [1]. The above results are proved via a Lyapunov Schmidt decomposition combined with a Nash-Moser iterative scheme.

In [8] we have extended this result proving the existence of also time *quasi-periodic* capillary-gravity standing wave solutions of (1.2) as well as their linear stability. This is the result presented in Theorem 1.1. The reducibility of the linearized equations at the quasi-periodic solutions is not only an interesting dynamical information but it is also the key for the existence proof.

We also mention that existence of small amplitude 2-d travelling gravity water wave solutions dates back to Levi-Civita [24] (standing waves are not traveling because they are even in space, see (1.9)). Existence of small amplitude 3-d traveling gravity-capillary water wave solutions with space periodic boundary conditions has been proved by Craig-Nicholls [13] (it is not a small divisor problem) and by Iooss-Plotinikov [21]-[22] in the case of zero surface tension (in such a case it is a small divisor problem).

The first existence results of quasi-periodic solutions of PDEs with unbounded perturbations (i.e. the nonlinearity contains derivatives) has been obtained by Kuksin [28] for KdV, see also Kappeler-Pöschel [26], by Liu-Yuan [25], Zhang-Gao-Yuan [36] for derivative NLS, by Berti-Biasco-Procesi [9]-[10] for derivative NLW. All these previous results still refer to semilinear perturbations, i.e. the order of the derivatives in the nonlinearity is strictly lower than the order of the constant coefficient (integrable) linear differential operator.

For quasi-linear, also fully nonlinear, perturbations the first KAM results have been recently proved by Baldi-Berti-Montalto in [3], [5], [6] (see also [2], [4]) for Hamiltonian perturbations of Airy, KdV and mKdV equations. These techniques have been applied by Feola-Procesi [18] also to quasi-linear perturbations of 1-d Schrödinger equations and by Montalto [29] to the Kirchhoff equation.

The gravity-capillary water waves system (1.2) is indeed a quasi-linear PDE. In suitable complex coordinates (having introduced the good unknown of Alinach) it can be written in the symmetric form

$$\mathtt{u}_t = \mathrm{i}T(D)\mathtt{u} + N(\mathtt{u}, \bar{\mathtt{u}})\ ,\ \mathtt{u} \in \mathbb{C}\,,$$

where

$$T(D) := |D|^{1/2}(1 - \kappa\partial_{xx})^{1/2}$$

is the Fourier multiplier which describes the linear dispersion relation of the water waves equations linearized at $(\eta, \psi) = 0$ (see (1.13)-(1.16)), and the nonlinearity $N(\mathtt{u}, \bar{\mathtt{u}})$ depends on the highest order term $|D|^{3/2}\mathtt{u}$ as well.

1.1. Main result

We look for small amplitude quasi-periodic solutions of (1.2), and therefore it is of main importance the dynamics of the linearized system at the equilibrium $(\eta, \psi) = (0, 0)$ (flat ocean and fluid at rest), namely

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + \eta = \kappa \eta_{xx} \end{cases} \tag{1.13}$$

where $G(0) = |D_x|$ is the Dirichlet-Neumann operator at the flat surface $\eta = 0$, namely

$$|D_x| \cos(jx) = |j| \cos(jx), \quad |D_x| \sin(jx) = |j| \sin(jx)\,, \ \forall j \in \mathbb{Z}\,.$$

In compact Hamiltonian form, the system (1.13) reads

$$\partial_t u = J\Omega u\,, \quad \Omega := \begin{pmatrix} 1 - \kappa \partial_{xx} & 0 \\ 0 & G(0) \end{pmatrix}, \tag{1.14}$$

which is the Hamiltonian system generated by the quadratic Hamiltonian (see (1.6))

$$H_L := \frac{1}{2}(u, \Omega u)_{L^2(\mathbb{T}_x)} = \frac{1}{2}(\psi, G(0)\psi)_{L^2(\mathbb{T}_x)} + \frac{1}{2}\int_{\mathbb{T}} \left(\eta^2 + \kappa \eta_x^2\right) dx\,. \tag{1.15}$$

The standing wave solutions of the linear system (1.13) are

$$\eta(t, x) = \sum\nolimits_{j \geq 1} a_j \cos(\omega_j t) \cos(jx), \quad \psi(t, x) = -\sum\nolimits_{j \geq 1} a_j j^{-1} \omega_j \sin(\omega_j t) \cos(jx)\,,$$

where $a_j \in \mathbb{R}$, and the linear frequencies of oscillations are

$$\omega_j := \omega_j(\kappa) := \sqrt{j(1 + \kappa j^2)}\,, \quad j \geq 1\,. \tag{1.16}$$

Fix an arbitrary finite subset $\mathbb{S}^+ \subset \mathbb{N}^+ := \{1, 2, \ldots\}$ (tangential sites) and consider the linear standing wave solutions

$$\eta(t, x) = \sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \cos(\omega_j t) \cos(jx), \quad \psi(t, x) = -\sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} j^{-1} \omega_j \sin(\omega_j t) \cos(jx)\,, \ \xi_j > 0\,, \tag{1.17}$$

which are Fourier supported in $\mathbb{S}^+$. The main result of [8] proves that such linear standing wave solutions can be continued to solutions of the nonlinear water-waves Hamiltonian system (1.2) for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$. Theorem 1.1 below states the existence of quasi-periodic solutions

$$u(\tilde{\omega} t, x) = (\eta, \psi)(\tilde{\omega} t, x)$$

of (1.2), with frequency $\tilde{\omega} := (\tilde{\omega}_j)_{j \in \mathbb{S}^+}$ (to be determined), close to the solutions (1.17) of (1.13), for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$.

Let $\nu := |\mathbb{S}^+|$ denote the cardinality of $\mathbb{S}^+$. The function $u(\varphi, x) = (\eta, \psi)(\varphi, x)$, $\varphi \in \mathbb{T}^\nu$, belongs to the Sobolev spaces of $(2\pi)^{\nu+1}$-periodic real functions

$$H^s(\mathbb{T}^{\nu+1}, \mathbb{R}^2) := \left\{ u = (\eta, \psi) : \eta, \psi \in H^s \right\}$$

$$H^s := H^s(\mathbb{T}^{\nu+1}, \mathbb{R}) = \Big\{ f = \sum_{(\ell, j) \in \mathbb{Z}^{\nu+1}} \widehat{f}_{\ell j}\, e^{\mathrm{i}(\ell \cdot \varphi + jx)} : \ \|f\|_s^2 := \sum_{(\ell, j) \in \mathbb{Z}^{\nu+1}} |\widehat{f}_{\ell j}|^2 \langle \ell, j \rangle^{2s} < +\infty \Big\} \tag{1.18}$$

where $\langle \ell, j \rangle := \max\{1, |\ell|, |j|\}$ with $|\ell| := \max_{i=1,\ldots,\nu} |\ell_i|$. For

$$s \geq s_0 := \left[\frac{\nu + 1}{2}\right] + 1 \in \mathbb{N}$$

the Sobolev spaces $H^s \subset L^\infty(\mathbb{T}^{\nu+1})$ are an algebra with respect to the product of functions.

Theorem 1.1. (KAM for capillary-gravity periodic standing water waves [8]) *For every choice of finitely many tangential sites $\mathbb{S}^+ \subset \mathbb{N}^+$, there exists $\bar{s} > s_0$, $\varepsilon_0 \in (0,1)$ such that for every $|\xi| \le \varepsilon_0^2$, $\xi := (\xi_j)_{j\in\mathbb{S}^+}$, there exists a Cantor like set $\mathcal{G} \subset [\kappa_1, \kappa_2]$ with asymptotically full measure as $\xi \to 0$, i.e.*

$$\lim_{\xi\to 0} |\mathcal{G}| = \kappa_2 - \kappa_1 \,,$$

such that, for any surface tension coefficient $\kappa \in \mathcal{G}$, the capillary-gravity system (1.2) *has a time quasi-periodic standing wave solution $u(\tilde{\omega}t, x) = (\eta(\tilde{\omega}t, x), \psi(\tilde{\omega}t, x))$, with Sobolev regularity $(\eta, \psi)(\varphi, x) \in H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$, of the form*

$$\begin{aligned} \eta(\tilde{\omega}t, x) &= \sum\nolimits_{j\in\mathbb{S}^+} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|}), \\ \psi(\tilde{\omega}, x) &= -\sum\nolimits_{j\in\mathbb{S}^+} \sqrt{\xi_j} j^{-1} \omega_j \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|}) \end{aligned} \tag{1.19}$$

with a diophantine frequency vector $\tilde{\omega} := \tilde{\omega}(\kappa, \xi) \in \mathbb{R}^\nu$ satisfying $\tilde{\omega}_j - \omega_j(\kappa) \to 0$, $j \in \mathbb{S}^+$, as $\xi \to 0$. The terms $o(\sqrt{|\xi|})$ are small in $H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$. In addition these quasi-periodic solutions are linearly stable.

Let us make some comments.

1. No global in time existence results concerning the initial value problem of the water waves equations (1.2) under *periodic* boundary conditions are known so far. The present Nash-Moser-KAM iterative procedure selects many values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$ which give rise to the quasi-periodic solutions (1.19), which are defined for all times.

   The fact that we find quasi-periodic solutions restricting to a proper subset of parameters is not a technical issue. The capillary-gravity water-waves equations (1.2) are not expected to be integrable (albeit a rigorous proof is still lacking): yet the third order Birkhoff normal form possesses multiple resonant triads (Wilton ripples), see Craig-Sulem [15].

2. In the proof of Theorem 1.1 all the estimates depend on the surface tension coefficient $\kappa > 0$ and the result does not hold at the limit of zero surface tension $\kappa \to 0$. Because of capillarity the linear frequencies (1.16) grow asymptotically $\sim \sqrt{\kappa} j^{3/2}$ as $j \to +\infty$. Without surface tension the linear frequencies grow asymptotically as $\sim j^{1/2}$ and a different proof is required.

3. The quasi-periodic solutions (1.19) are mainly supported in Fourier space on the tangential sites $\mathbb{S}^+$. The dynamics of the water waves equations (1.2) restricted to the symplectic subspaces

$$H_{\mathbb{S}^+} := \Big\{ v = \sum_{j\in\mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \Big\} \,, \quad H_{\mathbb{S}^+}^\perp := \Big\{ z = \sum_{j\in\mathbb{N}\setminus\mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \in H_0^1(\mathbb{T}_x) \Big\} \,, \tag{1.20}$$

   is quite different. We call $v \in H_{\mathbb{S}^+}$ the *tangential* variable and $z \in H_{\mathbb{S}^+}^\perp$ the *normal* one. On the finite dimensional subspace $H_{\mathbb{S}^+}$ we shall describe the dynamics by introducing the action-angle variables $(\theta, I) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$ as in (2.2). The quasi-periodic solutions (1.19) of (1.2) are therefore close to $\mathbb{T}^\nu \times \{\xi\} \times \{z = 0\}$, $\xi \in \mathbb{R}^\nu_+$. On the infinite dimensional subspace $H_{\mathbb{S}^+}^\perp$ the solution stays forever close to the elliptic equilibrium $z = 0$, in some Sobolev norm.

   A first key observation is that, for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$, the unperturbed linear frequencies (1.16), regrouped into the tangential and normal components

$$\vec{\omega}(\kappa) := (\omega_j(\kappa))_{j\in\mathbb{S}^+} \,, \quad \vec{\Omega}(\kappa) := (\Omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} := (\omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} \,, \tag{1.21}$$

   respectively, are diophantine, namely

$$|\vec{\omega}(\kappa) \cdot \ell| \ge \frac{\gamma}{|\ell|^\tau} \,, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\} \,,$$

   and satisfy also non-resonance conditions between the tangential and the normal frequencies, called first and second order Melnikov non-resonance conditions, see (2.11). For such values of $\kappa \in [\kappa_1, \kappa_2]$, the solutions (1.17) of the linear equation (1.13) are already sufficiently good approximate quasi-periodic solutions of the nonlinear water waves system (1.2). Since the parameter space $[\kappa_1, \kappa_2]$ is fixed, the small divisor constant γ can be taken $\gamma = o(\varepsilon^a)$ with $a > 0$ small as needed. As a consequence for proving the continuation of (1.17) to solutions of the nonlinear water waves system (1.2), all the terms which are at least quadratic in (1.2) are already perturbative (i.e. in (2.1) it is sufficient to regard the vector field $\varepsilon X_{P_\varepsilon}$ as a perturbation of the linear vector field $J\Omega$).

Linear stability. The quasi-periodic solutions $u(\tilde{\omega}t) = (\eta(\tilde{\omega}t), \psi(\tilde{\omega}t))$ found in Theorem 1.1 are linearly stable. This is not only a dynamically relevant information but also an essential ingredient of the existence proof (it is not necessary for time periodic solutions as in [1], [19], [20], [23]). Let us state precisely the result. Around each invariant torus there exist symplectic coordinates

$$(\phi, y, w) = (\phi, y, \eta, \psi) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}$$

(see [11]) in which the water waves Hamiltonian reads

$$\omega \cdot y + \frac{1}{2} K_{20}(\phi) y \cdot y + \big(K_{11}(\phi) y, w\big)_{L^2(\mathbb{T}_x)} + \frac{1}{2}\big(K_{02}(\phi) w, w\big)_{L^2(\mathbb{T}_x)} + K_{\geq 3}(\phi, y, w)$$

where $K_{\geq 3}$ collects the terms at least cubic in the variables (y, w). In these coordinates the quasi-periodic solution reads $t \mapsto (\omega t, 0, 0)$ (for simplicity we denote the frequency $\tilde{\omega}$ of the quasi-periodic solution by ω) and the corresponding linearized water waves equations are

$$\begin{cases} \dot{\phi} = K_{20}(\omega t)[y] + K_{11}^T(\omega t)[w] \\ \dot{y} = 0 \\ \dot{w} = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y]\,. \end{cases}$$

Thus the actions $y(t) = y(0)$ do not evolve in time and the third equation reduces to the forced PDE

$$\dot{w} = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y(0)]\,. \tag{1.22}$$

The self-adjoint operator $K_{02}(\omega t)$ is, up to a finite dimensional remainder, the restriction to $H^\perp_{\mathbb{S}^+}$ of the linearized water-waves vector field $\partial_u \nabla H(u(\omega t))$, which is explicitly computed in (2.17).

Denote $H^s_\perp := H^s_\perp(\mathbb{T}_x) := H^s(\mathbb{T}_x) \cap H^\perp_{\mathbb{S}}$ (real or complex valued). We prove the existence of bounded and invertible maps $\mathbf{W}_{m,\infty}(\varphi)$, $m = 1, 2$ such that $\forall \varphi \in \mathbb{T}^\nu$, $s \geq s_0$,

$$\mathbf{W}_{m,\infty}(\varphi) : \Big(H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C})\Big) \cap H^\perp_{\mathbb{S}_+} \to \Big(H^s(\mathbb{T}_x, \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{T}_x, \mathbb{R})\Big) \cap H^\perp_{\mathbb{S}_+}\,, \tag{1.23}$$

$$\mathbf{W}^{-1}_{m,\infty}(\varphi) : \Big(H^s(\mathbb{T}_x, \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{T}_x, \mathbb{R})\Big) \cap H^\perp_{\mathbb{S}_+} \to \Big(H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C})\Big) \cap H^\perp_{\mathbb{S}_+}\,, \tag{1.24}$$

and such that, under the quasi-periodic in time change of variables

$$w = (\eta, \psi) = \mathbf{W}_{1,\infty}(\omega t) w_\infty\,, \quad w_\infty = (\mathtt{w}_\infty, \overline{\mathtt{w}}_\infty)\,,$$

the equation (1.22) transforms into the diagonal system

$$\partial_t w_\infty = -\mathrm{i}\mathbf{D}_\infty w_\infty + f_\infty(\omega t)\,, \quad f_\infty(\omega t) := \mathbf{W}_{2,\infty}(\varphi)(\omega t)^{-1} J K_{11}(\omega t)[y(0)] = \begin{pmatrix} \mathtt{f}_\infty(\omega t) \\ \overline{\mathtt{f}}_\infty(\omega t) \end{pmatrix} \tag{1.25}$$

where, denoting $\mathbb{S}_0 := \mathbb{S}_+ \cup (-\mathbb{S}_+) \cup \{0\} \subseteq \mathbb{Z}$,

$$\mathbf{D}_\infty := \begin{pmatrix} D_\infty & 0 \\ 0 & -D_\infty \end{pmatrix}, \quad D_\infty := \mathrm{diag}_{j \in \mathbb{S}_0^c}\{\mu_j^\infty\}\,, \quad \mu_j^\infty \in \mathbb{R}\,, \tag{1.26}$$

is a Fourier multiplier operator of the form

$$\mu_j^\infty := \mathtt{m}_3^\infty \sqrt{|j|(1 + \kappa j^2)} + \mathtt{m}_1^\infty |j|^{\frac{1}{2}} + r_j^\infty\,,\ j \in \mathbb{S}_0^c\,, \quad r_j^\infty = r_{-j}^\infty\,, \tag{1.27}$$

where, for some $\mathtt{a} > 0$,

$$\mathtt{m}_3^\infty = 1 + O(\varepsilon^{\mathtt{a}})\,,\ \mathtt{m}_1^\infty = O(\varepsilon^{\mathtt{a}})\,,\ \sup_{j \in \mathbb{S}_0^c} |r_j^\infty| = O(\varepsilon^{\mathtt{a}})\,,$$

see (2.12)-(2.13). The $\mathrm{i}\mu_j^\infty$ are the *Floquet exponents* of the quasi-periodic solution. The fact that they are purely imaginary is a consequence of the reversible structure of the water waves equations.

The second equation of system (1.25) is actually the complex conjugated of the first one, and (1.25) reduces to the infinitely many decoupled scalar equations

$$\partial_t \mathtt{w}_{\infty,j} = -\mathrm{i}\mu_j^\infty \mathtt{w}_{\infty,j} + \mathtt{f}_{\infty,j}(\omega t)\,, \quad \forall j \in \mathbb{S}_0^c\,.$$

By variation of constants the solutions are

$$\mathtt{w}_{\infty,j}(t) = c_j e^{-\mathrm{i}\mu_j^\infty t} + \mathtt{v}_{\infty,j}(t) \qquad \text{where} \qquad \mathtt{v}_{\infty,j}(t) := \sum_{\ell\in\mathbb{Z}^\nu} \frac{\mathtt{f}_{\infty,j,\ell}\, e^{\mathrm{i}\omega\cdot\ell t}}{\mathrm{i}(\omega\cdot\ell + \mu_j^\infty)}\,, \quad \forall j \in \mathbb{S}_0^c\,. \tag{1.28}$$

Since the first Melnikov conditions (2.11) are satisfied at a solution, the denominators in (1.28) are different from zero and $\mathtt{v}_{\infty,j}(t)$ is well defined. Moreover (1.23) implies $\|f_\infty(\omega t)\|_{H_x^s\times H_x^s} \leq C|y(0)|$. As a consequence the Sobolev norm of the solution of (1.25) with initial condition $w_\infty(0) \in H^{\mathfrak{s}_0}(\mathbb{T}_x)$, $\mathfrak{s}_0 < s$ (in a suitable range of values), satisfies

$$\|w_\infty(t)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0}} \leq C(s)(|y(0)| + \|w_\infty(0)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0}})\,,$$

and, for all $t\in\mathbb{R}$, using (1.23), (1.24), we get

$$\|(\eta,\psi)(t)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0-\frac12}} \leq \|(\eta(0),\psi(0))\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0-\frac12}}$$

which proves the linear stability of the quasi-periodic solution. Note that the profile $\eta \in H^{\mathfrak{s}_0}(\mathbb{T}_x)$ is more regular than the velocity potential $\psi \in H^{\mathfrak{s}_0-\frac12}(\mathbb{T}_x)$, as it is expected in presence of surface tension.

Clearly a crucial point is the diagonalization of (1.22) into (1.26). With respect to [1] this requires to analyze more in detail the pseudo-differential nature of the operators obtained after each conjugation and to implement a KAM scheme with second order Melnikov non-resonance conditions, as we shall explain in detail below. We now present the main ideas of the proof.

Acknowledgements. This research was partially supported by the Swiss National Foundation (SNF).

2. Ideas of the proof

We prove Theorem 1.1 by a Nash-Moser iterative scheme in Sobolev spaces formulated as a "Theorem of hypothetic conjugation" á la Herman (section 2.1) plus a degenerate KAM theory argument to perform the measure estimates with respect to the 1-dimensional parameter $\kappa\in[\kappa_1,\kappa_2]$. The core of the Nash-Moser scheme is to prove that the linearized operators obtained at any approximate solution are invertible, with an inverse that satisfies tame estimates in Sobolev spaces. We explain how to prove this property in section 2.3.

First of all, instead of working in a shrinking neighborhood of the origin, it is a convenient devise to rescale the variable $u\mapsto\varepsilon u$ with $u = O(1)$, writing (1.2), (1.5) as

$$\partial_t u = J\Omega u + \varepsilon X_{P_\varepsilon}(u) \tag{2.1}$$

where $J\Omega$ is defined in (1.14) and $X_{P_\varepsilon}(u)$ is the Hamiltonian vector field generated by the Hamiltonian

$$\mathcal{H}_\varepsilon(u) := \varepsilon^{-2}H(\varepsilon u) = H_L(u) + \varepsilon P_\varepsilon(u)$$

where H is the water-waves Hamiltonian (1.6) and H_L is defined in (1.15).

We decompose the phase space as in (1.20),

$$H^1_{0,\mathrm{even}} = H_{\mathbb{S}^+} \oplus H^\perp_{\mathbb{S}^+}\,,$$

and we introduce action-angle variables on the tangential sites by setting

$$\eta_j := \sqrt{\frac{2}{\pi}}\,\Lambda_j^{1/2}\sqrt{\xi_j + I_j}\cos(\theta_j),\ \psi_j := -\sqrt{\frac{2}{\pi}}\,\Lambda_j^{-1/2}\sqrt{\xi_j+I_j}\,\sin(\theta_j)\,,$$

$$\Lambda_j := \sqrt{j(1+\kappa j^2)^{-1}}\,,\ j\in\mathbb{S}^+\,, \tag{2.2}$$

where $\xi_j > 0$, $j = 1,\ldots,\nu$, are positive constants, and $|I_j| < \xi_j$. The symplectic 2-form in (1.7) then reads

$$\mathcal{W} := \Big(\sum\nolimits_{j\in\mathbb{S}^+} d\theta_j\wedge dI_j\Big) \oplus \mathcal{W}_{|H^\perp_{\mathbb{S}^+}}$$

and the Hamiltonian system (2.1) transforms into the new Hamiltonian system

$$\dot\theta = \partial_I H_\varepsilon(\theta, I, z)\,, \ \ \dot I = -\partial_\theta H_\varepsilon(\theta, I, z)\,, \quad z_t = J\nabla_z H_\varepsilon(\theta, I, z) \tag{2.3}$$

generated by the Hamiltonian

$$H_\varepsilon := \mathcal{H}_\varepsilon \circ A = \varepsilon^{-2} H \circ \varepsilon A \tag{2.4}$$

where

$$A(\theta, I, z) := v(\theta, I) + z := \sum_{j\in\mathbb{S}^+} \sqrt{\frac{2}{\pi}} \begin{pmatrix} \Lambda_j^{1/2}\sqrt{\xi_j + I_j}\,\cos(\theta_j) \\ -\Lambda_j^{-1/2}\sqrt{\xi_j + I_j}\,\sin(\theta_j) \end{pmatrix} \cos(jx) + z\,. \tag{2.5}$$

We denote by

$$X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\theta H_\varepsilon, J\nabla_z H_\varepsilon)$$

the Hamiltonian vector field in the variables $(\theta, I, z) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}$. The involution ρ in (1.11) becomes

$$\tilde\rho : (\theta, I, z) \mapsto (-\theta, I, \rho z)\,. \tag{2.6}$$

By (1.6) and (2.4) the Hamiltonian H_ε reads (up to a constant which is irrelevant for the dynamics)

$$H_\varepsilon = \mathcal{N} + \varepsilon P\,, \quad \mathcal{N} := H_L \circ A = \vec\omega(\kappa)\cdot I + \frac{1}{2}(z, \Omega z)_{L^2_x}\,, \quad P := P_\varepsilon \circ A\,, \tag{2.7}$$

where $\vec\omega(\kappa)$ is the vector of the tangential linear frequencies defined in (1.21) and Ω is the linear operator in (1.14). We look for an embedded invariant torus

$$i : \mathbb{T}^\nu \to \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}, \quad \varphi \mapsto i(\varphi) := (\theta(\varphi), I(\varphi), z(\varphi))$$

of the Hamiltonian system (2.3) filled by quasi-periodic solutions with some diophantine frequency $\omega \in \mathbb{R}^\nu$.

2.1. Nash-Moser theorem of hypothetic conjugation

The Hamiltonian H_ε in (2.7) is a perturbation of the isochronous system with Hamiltonian $\mathcal{N}$. The expected quasi-periodic solutions of the corresponding Hamiltonian system (2.3) will have a shifted frequency vector ω -to be found- close to the vector $\vec\omega(\kappa)$ of the linear frequencies in (1.21), which depend on the nonlinear term P and the amplitudes ξ_j.

In view of that we introduce the family of Hamiltonians

$$H_\alpha := \mathcal{N}_\alpha + \varepsilon P\,, \quad \mathcal{N}_\alpha := \alpha\cdot I + \frac{1}{2}(z, \Omega z)_{L^2_x}\,, \quad \alpha \in \mathbb{R}^\nu\,, \tag{2.8}$$

which depend on a constant vector $\alpha \in \mathbb{R}^\nu$. For the value $\alpha = \vec\omega(\kappa)$ we have $H_\alpha = H_\varepsilon$. Then we look for a zero (i, α) of the nonlinear operator

$$\begin{aligned} \mathcal{F}(i,\alpha) &:= \mathcal{F}(i,\alpha,\omega,\kappa,\varepsilon) := \omega\cdot\partial_\varphi i(\varphi) - X_{H_\alpha} = \omega\cdot\partial_\varphi i(\varphi) - (X_{\mathcal{N}_\alpha} + \varepsilon X_P)(i(\varphi)) \\ &:= \begin{pmatrix} \omega\cdot\partial_\varphi\theta(\varphi) - \alpha - \varepsilon\partial_I P(i(\varphi)) \\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\theta P(i(\varphi)) \\ \omega\cdot\partial_\varphi z(\varphi) - J(\Omega z + \varepsilon\nabla_z P(i(\varphi))) \end{pmatrix} \end{aligned} \tag{2.9}$$

for some diophantine vector $\omega \in \mathbb{R}^\nu$. If $\mathcal{F}(i,\alpha) = 0$ then $\varphi \mapsto i(\varphi)$ is an embedded torus, invariant for the Hamiltonian vector field X_{H_α}, filled by quasi-periodic solutions with frequency ω.

Since each Hamiltonian H_α in (2.8) is reversible, we look for reversible solutions of $\mathcal{F}(i,\alpha) = 0$, namely satisfying $\tilde\varrho i(\varphi) = i(-\varphi)$ (see (2.6)), i.e.

$$\theta(-\varphi) = -\theta(\varphi)\,, \quad I(-\varphi) = I(\varphi)\,, \quad z(-\varphi) = (\varrho z)(\varphi)\,. \tag{2.10}$$

The Sobolev norm of the periodic component of the embedded torus

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0, 0) := (\Theta(\varphi), I(\varphi), z(\varphi))\,, \quad \Theta(\varphi) := \theta(\varphi) - \varphi\,,$$

is

$$\|\mathfrak{I}\|_s := \|\Theta\|_{H^s_\varphi} + \|I\|_{H^s_\varphi} + \|z\|_s$$

where $\|z\|_s := \|z\|_{H^s_{\varphi,x}} = \|\eta\|_s + \|\psi\|_s$, see (1.18). Our goal is to prove that, for ε small enough, for "most" parameters (ω, κ) in a suitable Cantor-like $\mathcal{C}^\gamma_\infty$, there exists a value of the constants $\alpha := \alpha_\infty(\omega,\kappa,\varepsilon) \simeq \omega$ and a ν-dimensional embedded torus $\mathcal{T} = i_\infty(\mathbb{T}^\nu)$ close to $\mathbb{T}^\nu \times \{0\} \times \{0\}$, invariant for the Hamiltonian vector field $X_{H(\alpha_\infty(\omega,\kappa,\varepsilon),\cdot)}$ and supporting quasi-periodic solutions with frequency ω. This is equivalent to look for zeros of the nonlinear operator $\mathcal{F}(i,\alpha,\omega,\kappa,\epsilon) = 0$

defined in (2.9). This equation is solved by a Nash-Moser iterative scheme. The value of $\alpha := \alpha_\infty(\omega,\kappa,\varepsilon)$ is adjusted along the iteration in order to control the average of the first component of the Hamilton equation (2.9).

The set of parameters $(\omega,\kappa) \in \mathcal{C}^\gamma_\infty$ for which the invariant torus exists has an explicit form and it depends only on the *final torus* i_∞. Its explicit expression is given by

$$\begin{aligned} \mathcal{C}^\gamma_\infty := \Big\{ & (\omega,\kappa) \in \Omega \times [\kappa_1,\kappa_2] \,:\, |\omega\cdot\ell| \geq \gamma\langle\ell\rangle^{-\tau},\ \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}, \\ & |\omega\cdot\ell + \mu_j^\infty(\omega,\kappa)| \geq 4\gamma j^{\frac{3}{2}}\langle\ell\rangle^{-\tau},\ \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+ \ \text{(1-Melnikov conditions)} \\ & |\omega\cdot\ell + \mu_j^\infty(\omega,\kappa) - \varsigma\mu_{j'}^\infty(\omega,\kappa)| \geq \frac{4\gamma|j^{\frac{3}{2}} - \varsigma j'^{\frac{3}{2}}|}{\langle\ell\rangle^\tau}, \forall\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+, \varsigma=\pm1\,, \\ & \text{(2-Melnikov)}\Big\} \end{aligned} \tag{2.11}$$

where

$$\Omega := \Big\{\omega\in\mathbb{R}^\nu \,:\, \mathrm{dist}\big(\omega, \vec{\omega}[\kappa_1,\kappa_2]\big) < \delta\,,\ \delta>0\Big\}$$

and $\mu_j^\infty : \Omega\times[\kappa_1,\kappa_2]\to\mathbb{R}$, $j\in\mathbb{N}^+\setminus\mathbb{S}^+$ are k_0-times differentiable functions of the form

$$\mu_j^\infty(\omega,\kappa) = \mathtt{m}_3^\infty(\omega,\kappa) j^{\frac{1}{2}}(1+\kappa j^2)^{\frac{1}{2}} + \mathtt{m}_1^\infty(\omega,\kappa) j^{\frac{1}{2}} + r_j^\infty(\omega,\kappa) \tag{2.12}$$

satisfying

$$\mathtt{m}_3^\infty - 1, \mathtt{m}_1^\infty = O(\varepsilon)\,, \qquad \sup_{j\in\mathbb{S}^c} |r_j^\infty| = O(\varepsilon)\,, \tag{2.13}$$

where $k_0\in\mathbb{N}$ is an absolute constant fixed by the degenerate KAM theory of Section 2.2.
In order to prove the existence of quasi-periodic solutions of the water waves equation (1.2), and not only of the system with modified Hamiltonian H_α with $\alpha := \alpha_\infty(\omega,\kappa,\varepsilon)$, we have then to prove that the curve of the unperturbed linear frequencies

$$[\kappa_1,\kappa_2] \ni \kappa \mapsto \vec{\omega}(\kappa) := \big(\sqrt{j(1+\kappa j^2)}\big)_{j\in\mathbb{S}^+} \in \mathbb{R}^\nu$$

intersects the image $\alpha_\infty(\mathcal{C}^\gamma_\infty)$, under the map α_∞ of the Cantor set $\mathcal{C}^\gamma_\infty$, for "most" values of $\kappa\in[\kappa_1,\kappa_2]$, namely we have to estimate the measure of the set

$$\mathcal{G}_\varepsilon := \big\{\kappa\in[\kappa_1,\kappa_2] \,:\, \alpha_\infty^{-1}(\vec{\omega}(\kappa),\kappa) \in \mathcal{C}^\gamma_\infty\big\} \tag{2.14}$$

This is proved by using degenerate KAM theory. For any value of the parameter κ in $\mathcal{G}_\varepsilon$, we have found a quasi-periodic solution of (1.2) with diophantine frequency $\omega_\varepsilon(\kappa) := \alpha_\infty^{-1}(\vec{\omega}(\kappa),\kappa)$.

The above functional setting perspective is in the spirit of the Théoréme de conjugaison hypothétique of Herman proved by Fejoz [17] for finite dimensional Hamiltonian systems, see also the discussion in [11]. A relevant difference is that in [17], in addition to α, also the normal frequencies are introduced as independent parameters, unlike in our strategy. Actually for PDEs it seems more convenient the present formulation: it is a major point of the work to know the asymptotic expansion (1.27) of the Floquet exponents.

2.2. Degenerate KAM theory

A first key observation is that, for most values of the surface tension parameter $\kappa\in[\kappa_1,\kappa_2]$, the unperturbed linear frequencies (1.16) are diophantine and satisfy also first and second order Melnikov non-resonance conditions. More precisely the unperturbed tangential frequency vector $\vec{\omega}(\kappa) := (\omega_j(\kappa))_{j\in\mathbb{S}^+}$ satisfies

$$|\vec{\omega}(\kappa)\cdot\ell| \geq \gamma\langle\ell\rangle^{-\tau},\ \forall\ell\in\mathbb{Z}^\nu\setminus\{0\},$$

and it is non-resonant with the normal frequencies

$$\vec{\Omega}(\kappa) := (\Omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} = (\omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+},$$

i.e.

$$|\vec{\omega}(\kappa)\cdot\ell + \Omega_j(\kappa)| \geq \gamma j^{\frac{3}{2}}\langle\ell\rangle^{-\tau},\ \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+\,,$$
$$|\vec{\omega}(\kappa)\cdot\ell + \Omega_j(\kappa) \pm \Omega_{j'}(\kappa)| \geq \gamma|j^{\frac{3}{2}} \pm j'^{\frac{3}{2}}|\langle\ell\rangle^{-\tau}, \forall\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+\,.$$

This is a problem of diophantine approximation on submanifolds. It can be solved by degenerate KAM theory (explained below) exploiting that the linear frequencies $\kappa \mapsto \omega_j(\kappa)$ are *analytic*, simple, grow asymptotically as $j^{3/2}$ and are *non-degenerate* in the sense of Bambusi-Berti-Magistrelli [7]. This is verified as in [7] by using a Van der Monde determinant. We deduce that $\exists k_0 > 0$, $\rho_0 > 0$ such that, for all $\kappa \in [\kappa_1, \kappa_2]$,

$$\max_{0 \leq k \leq k_0} \big| \partial_\kappa^k \big(\vec{\omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa) \big) \big| \geq \rho_0 \langle \ell \rangle \,, \quad \forall (\ell, j, j') \neq (0, j, j), \ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+ \,, \tag{2.15}$$

and similarly for the 0-th, 1-th and the 2-th order Melnikov non-resonance condition with the sign $+$. For such values of $\kappa \in [\kappa_1, \kappa_2]$, the solutions (1.17) of the linear equation (1.13) are already sufficiently good approximate quasi-periodic solutions of the nonlinear water waves system (1.2). Since the parameter space $[\kappa_1, \kappa_2]$ is fixed, the small divisor constant γ can be taken $\gamma = o(\varepsilon^a)$ with $a > 0$ small as needed. As a consequence for proving the continuation of (1.17) to solutions of the nonlinear water waves system (1.2), all the terms which are at least quadratic in (1.2) are already perturbative (in (2.1) it is sufficient to regard the vector field $\varepsilon X_{P_\varepsilon}$ as a perturbation of the linear vector field $J\Omega$). Actually we need to verify, *for most* parameters $\kappa \in [\kappa_1, \kappa_2]$, the Melnikov conditions for the perturbed frequencies μ_j^∞ (see (2.12), (2.13)), namely

$$|\vec{\omega}(\kappa) \cdot \ell + \mu_j^\infty(\vec{\omega}(\kappa), \kappa)| \geq \gamma j^{\frac{3}{2}} \langle \ell \rangle^{-\tau}, \ \forall \ell \in \mathbb{Z}^\nu, \ j \in \mathbb{N}^+ \setminus \mathbb{S}^+ \,,$$
$$|\vec{\omega}(\kappa) \cdot \ell + \mu_j^\infty(\vec{\omega}(\kappa), \kappa) \pm \mu_{j'}^\infty(\vec{\omega}(\kappa), \kappa)| \geq \gamma |j^{\frac{3}{2}} \pm j'^{\frac{3}{2}}| \langle \ell \rangle^{-\tau}, \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+ \,.$$

This follows since the perturbed frequencies satisfy

$$\max_{0 \leq k \leq k_0} \big| \partial_\kappa^k \big(\vec{\omega}(\kappa) \cdot \ell + \mu_j^\infty(\vec{\omega}(\kappa), \kappa) - \mu_{j'}^\infty(\vec{\omega}(\kappa), \kappa) \big) \big| \geq \rho_0 \langle \ell \rangle \,,$$
$$\forall (\ell, j, j') \neq (0, j, j), \ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+ \,, \tag{2.16}$$

(0-th, 1-th and the 2-th order Melnikov non-resonance condition with the sign $+$). This can be proved by a perturbative argument, using (2.15). As a consequence, by applying a classical Rüssmann lemma (Theorem 17.1 in [33]) we prove that the Cantor like set of non-resonant parameters $\mathcal{G}_\varepsilon$ has measure $\kappa_2 - \kappa_1 - O(\gamma^{1/k_0})$.

2.3. Analysis of the linearized operators

In addition to the previous bifurcation analysis, the other key step for the Nash-Moser iterative scheme is to prove that the operator $d_{\alpha,i}\mathcal{F}$ obtained linearizing (2.9) at any approximate solution is, for most values of the parameters (ω, κ), invertible, and that its inverse satisfies *tame* estimates in Sobolev spaces.

The linearized operator $d_{\alpha,i}\mathcal{F}$ is quite complicated because all the (θ, I, z) components in the system (2.9) are coupled among them. Therefore we first implement the procedure developed in Berti-Bolle [11], and used in [5], [6], which consists in introducing a convenient set of symplectic variables near the approximate invariant torus such that the linearized equations become (approximately) decoupled in the action-angle components and the normal direction. As a consequence, the problem reduces to "almost-approximately" invert a quasi-periodic linear operator restricted to the normal directions. Actually, since this symplectic change of variables modifies, up to a translation, only the finite dimensional action component, this operator turns out to be just the linearized water-waves system in the original coordinates, restricted to the normal directions. More precisely

$$\Pi_{\mathbb{S}^+}^\perp \mathcal{L}_{|H_{\mathbb{S}^+}^\perp} \qquad \text{where} \qquad \mathcal{L} := \omega \cdot \partial_\varphi \mathbb{I}_2 - J \partial_u \nabla_u H(U(\varphi))$$

is obtained linearizing (1.2), (1.5) at an approximate solution $U(\varphi) = (\eta, \psi)(\varphi, x)$, changing $\partial_t \rightsquigarrow \omega \cdot \partial_\varphi$, and denoting the 2×2-identity matrix by

$$\mathbb{I}_2 := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} .$$

Using formula (1.3) the linearized operator $\mathcal{L}$ is computed to be

$$\mathcal{L} = \omega \cdot \partial_\varphi \mathbb{I}_2 + \begin{pmatrix} \partial_x V + G(\eta) B & -G(\eta) \\ (1 + B V_x) + B G(\eta) B - \kappa \partial_x c \partial_x & V \partial_x - B G(\eta) \end{pmatrix} \tag{2.17}$$

where the functions $B := B(\varphi, x)$, $V := V(\varphi, x)$ are defined by (1.4) and $c := c(\varphi, x) := (1 + \eta_x^2)^{-3/2}$. The operator $\mathcal{L}$ is real, even and reversible.

Notation. In (2.17) and hereafter any function a is identified with the corresponding multiplication operators $h \mapsto ah$, and, where there is no parenthesis, composition of operators is understood. For example, $\partial_x c \partial_x$ means: $h \mapsto \partial_x(c\partial_x h)$.

The key part of the analysis consists now in diagonalizing (actually it is sufficient to "almost" diagonalize) the quasi-periodic linear operator $\mathcal{L}$, via linear changes of variables close to the identity, which map Sobolev spaces into itself and satisfy tame estimates. These changes of variables have two well different tasks:

1. Transform $\mathcal{L}$ to an operator of the form (2.18) which has constant coefficients up to pseudo-differential remainders of order zero (actually more regularizing on the off-diagonal terms). These steps are exposed in sections 2.3.1-2.3.5.

2. Reduce quadratically the size of the perturbative terms $\mathcal{R}$, $\mathcal{Q}$, see section 2.3.6.

For the search of periodic solutions, i.e. [1], [23], [19], [20], [30], there is no need to perform the task 2, because it is possible to invert the linearized operator in (2.18) simply by a Neumann-argument. Indeed, for periodic solutions, a sufficiently regularizing operator in the space variable is also regularizing in the time variable, on the characteristic Fourier indices which correspond to the small divisors. This is clearly not true for quasi-periodic solutions. That is why we will completely diagonalize the linear operator in (2.18) by a KAM scheme. For that we need to analyze more in detail the pseudo-differential nature of the remainders after each conjugation step.

The approximate solution $U(\varphi, x)$ at which we linearize is assumed to be bounded in a low Sobolev norm (as it is satisfied by any approximate solutions along the Nash-Moser iteration). Moreover $U(\varphi, x)$ is supposed to be $\mathcal{C}^\infty(\mathbb{T}^\nu_\varphi \times \mathbb{T}_x)$ because, along the Nash-Moser iteration, each approximate solution is actually a trigonometric polynomial in (φ, x) (with clearly more and more harmonics). As a consequence, all the coefficients of the linearized operator $\mathcal{L}$ in (2.17) are $\mathcal{C}^\infty$. This allows to work in the usual framework of $\mathcal{C}^\infty$ pseudo-differential symbols. For the Nash-Moser convergence we shall then perform quantitative estimates in Sobolev spaces.

2.3.1. Reduction of $\mathcal{L}$ to constant coefficients in decreasing symbols

The goal of the first steps is to reduce $\mathcal{L}$ to a quasi-periodic linear operator of the form

$$(h, \bar{h}) \mapsto \big(\omega\cdot\partial_\varphi + \mathrm{i}\mathtt{m}_3 T(D) + \mathrm{i}\mathtt{m}_1 |D|^{\frac{1}{2}}\big)h + \mathcal{R}h + \mathcal{Q}\bar{h}\,, \quad h \in \mathbb{C}\,, \tag{2.18}$$

where $\mathtt{m}_3, \mathtt{m}_1 \in \mathbb{R}$ are constants coefficients, satisfying $\mathtt{m}_3 \approx 1$, $\mathtt{m}_1 \approx 0$, the principal symbol operator

$$T(D) = |D|^{1/2}(1 - \kappa\partial_{xx})^{1/2}\,,$$

and the remainders $\mathcal{R} := \mathcal{R}(\varphi)$, $\mathcal{Q} := \mathcal{Q}(\varphi)$ are small $O(\varepsilon)$ bounded operators acting in the Sobolev spaces H^s, which satisfy tame estimates. More precisely, in view of a KAM reducibility scheme that will completely diagonalize the operator (2.18) (section 2.3.6) we need that all the derivatives

$$\partial_\varphi^\beta \partial_{\omega,\kappa}^k \mathcal{R}\,, \quad \partial_\varphi^\beta \partial_{\omega,\kappa}^k \mathcal{Q}\,, \quad |\beta| \le \beta_0\,,\ |k| \le k_0\,, \tag{2.19}$$

for β_0 large enough (depending on the diophantine exponent τ), satisfy tame estimates.

2.3.2. Symmetrization and space-time reduction of $\mathcal{L}$ at the highest order

The first part of the analysis is similar to Alazard-Baldi [1]. We first conjugate the linear operator $\mathcal{L}$ in (2.17) by the change of variable

$$\mathcal{Z} := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad \mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

obtaining

$$\mathcal{L}_0 := \mathcal{Z}^{-1}\mathcal{L}\mathcal{Z} = \omega\cdot\partial_\varphi \mathbb{I}_2 + \begin{pmatrix} \partial_x V & -G(\eta) \\ a - \kappa\partial_x c \partial_x & V\partial_x \end{pmatrix} \tag{2.20}$$

for some function $a(\varphi, x)$. This step amounts to introduce (a linearized version of) the "good unknown of Alinhac", $\omega = \psi - B\eta$.

As a second step we conjugate $\mathcal{L}_0$ with an operator of the form $\mathcal{SQB}$ where $\mathcal{B}$ is a change of variable

$$(\mathcal{B}h)(\varphi, x) := h(\varphi, x + \beta(\varphi, x)) \tag{2.21}$$

induced by a φ-dependent family of diffeomorphisms of the torus

$$y = x + \beta(\varphi, x) \qquad \Leftrightarrow \qquad x = y + \tilde{\beta}(\varphi, y)\,, \tag{2.22}$$

$\mathcal{Q}$ is a matrix valued multiplication operator

$$\mathcal{Q} := \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \qquad \mathcal{Q}^{-1} := \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix},$$

for a function $q(\varphi, x)$ close to 1, and $\mathcal{S}$ is the vector valued Fourier multiplier

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}, \quad G := |D|^{-\frac{1}{2}}(1 - \kappa\partial_{xx})^{\frac{1}{2}} \in OPS^{1/2}\,.$$

Choosing properly the small periodic functions $\beta(\varphi, x)$ and $q(\varphi, x) - 1$ one gets

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{S}^{-1}\mathcal{Q}^{-1}\mathcal{B}^{-1}\mathcal{L}_0\mathcal{BQS} = \omega\cdot\partial_\varphi \mathbb{I}_2 \\ &+ \begin{pmatrix} a_1\partial_x + a_2 & -m_3(\varphi)T(D) + \sqrt{\kappa}\, a_3\mathcal{H}|D|^{\frac{1}{2}} + \mathcal{R}_{1,B} \\ m_3(\varphi)T(D) - \frac{a_4}{\sqrt{\kappa}}|D|^{\frac{1}{2}}\mathcal{H} + m_3(\varphi)\pi_0 + \mathcal{R}_{1,C} & a_1\partial_x + \mathcal{R}_{1,D} \end{pmatrix} \end{aligned} \tag{2.23}$$

for suitable functions a_1, a_2, a_3, a_4 and pseudo-differential operators $\mathcal{R}_{1,B}$, $\mathcal{R}_{1,C}$, $\mathcal{R}_{1,D} \in OPS^0$ which are $O(\varepsilon)$ small, in low Sobolev norm. All the coefficients and the operators depend in a tame way, i.e. at most linearly, in the high Sobolev norm of the approximate solution $\|U\|_{s+\sigma}$ with a possible fixed loss of derivatives σ. Note that the coefficient $m_3(\varphi)$ of the highest order operator $\mathcal{L}_1$ in (2.23) is independent of the space variable. The operator π_0 is the L^2 projector on the constants, that, for simplicity of exposition, we neglect in the sequel.

We then write $\mathcal{L}_1$ as an operator acting on the complex variables

$$h := \eta + \mathrm{i}\psi\,, \quad \bar{h} := \eta - \mathrm{i}\psi\,,$$

obtaining

$$\mathcal{L}_1 = \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathrm{i}m_3(\varphi)\mathbf{T}(D) + \mathbf{A}_1(\varphi, x)\partial_x + \mathrm{i}(\mathbf{A}_0^{(I)}(\varphi, x) + \mathbf{A}_0^{(II)}(\varphi, x))\mathcal{H}|D|^{\frac{1}{2}} + \mathbf{R}_1^{(I)} + \mathbf{R}_1^{(II)} \tag{2.24}$$

where

$$\mathbf{T}(D) := \begin{pmatrix} T(D) & 0 \\ 0 & -T(D) \end{pmatrix}, \quad \mathbf{A}_1(\varphi, x) := \begin{pmatrix} a_1(\varphi, x) & 0 \\ 0 & a_1(\varphi, x) \end{pmatrix},$$

$$\mathbf{A}_0^{(I)}(\varphi, x) := \begin{pmatrix} a_5(\varphi, x) & 0 \\ 0 & -a_5(\varphi, x) \end{pmatrix}, \quad \mathbf{A}_0^{(II)}(\varphi, x) := \begin{pmatrix} 0 & a_6(\varphi, x) \\ -a_6(\varphi, x) & 0 \end{pmatrix},$$

and

$$\mathbf{R}_1^{(I)} := \begin{pmatrix} r_1^{(I)}(x, D) & 0 \\ 0 & \overline{r_1^{(I)}(x, D)} \end{pmatrix}, \; \mathbf{R}_1^{(II)} := \begin{pmatrix} 0 & r_1^{(II)}(x, D) \\ \overline{r_1^{(II)}(x, D)} & 0 \end{pmatrix} \in OPS^0 \tag{2.25}$$

are $O(\varepsilon)$-pseudo-differential operators. Note that $\mathcal{L}_1$ in (2.24) is block-diagonal (in $(u, \bar{u})$) up to order $|D|^{1/2}$.

The next step is to remove the dependence on φ from the highest order term $\mathrm{i}m_3(\varphi)\mathbf{T}(D)$, by applyling a quasi periodic time reparametrization

$$\mathcal{P}\,\mathbb{I}_2 = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{P} \end{pmatrix}, \quad (\mathcal{P}h)(\varphi, x) := h(\varphi + \omega p(\varphi), x)\,,$$

induced by the diffeomorphism

$$\vartheta := \varphi + \omega p(\varphi) \quad \Leftrightarrow \quad \varphi = \vartheta + \omega\tilde{p}(\vartheta)$$

where $p(\varphi)$ is a small periodic function. Choosing properly p and assuming ω to be diophantine, we get

$$\mathcal{L}_2 := \omega\cdot\partial_\varphi \mathbb{I}_2 + \mathrm{im}_3 \mathbf{T}(D) + \mathbf{B}_1(\varphi,x)\partial_x + \mathrm{i}\big(\mathbf{B}_0^{(I)}(\varphi,x) + \mathbf{B}_0^{(II)}(\varphi,x)\big)\mathcal{H}|D|^{\frac12} + \mathbf{R}_2^{(I)} + \mathbf{R}_2^{(II)} \tag{2.26}$$

where

$$\mathbf{B}_1 = \begin{pmatrix} a_7(\varphi,x) & 0 \\ 0 & a_7(\varphi,x) \end{pmatrix}, \quad \mathbf{B}_0^{(I)} = \begin{pmatrix} a_8(\varphi,x) & 0 \\ 0 & -a_8(\varphi,x) \end{pmatrix},$$

$$\mathbf{B}_0^{(II)} = \begin{pmatrix} 0 & a_9(\varphi,x) \\ a_9(\varphi,x) & 0 \end{pmatrix}$$

and $\mathbf{R}_2^{(I)}, \mathbf{R}_2^{(II)}$ are $O(\varepsilon)$ pseudo differential operators in OPS^0, of the same form as in (2.25). All the previous transformations are real, even, and reversibility preserving, so that $\mathcal{L}_2$ is a real, even and reversible operator.

From this point we have to proceed quite differently with respect to [1].

2.3.3. Block-decoupling

The next step is to conjugate the operator $\mathcal{L}_2$ in (2.26) to an operator of the form

$$\mathcal{L}_M := \mathbf{\Phi}_M^{-1}\mathcal{L}_2\mathbf{\Phi}_M = \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathrm{im}_3\mathbf{T}(D) + \mathbf{B}_1(\varphi,x)\partial_x + \mathrm{i}\mathbf{B}_0^{(I)}(\varphi,x)\mathcal{H}|D|^{\frac12} + \mathbf{R}_M^{(I)} + \mathbf{R}_M^{(II)} \tag{2.27}$$

where the remainders

$$\mathbf{R}_M^{(I)} := \begin{pmatrix} r_M^{(I)}(\varphi,x,D) & 0 \\ 0 & \overline{r_M^{(I)}(\varphi,x,D)} \end{pmatrix} \in OPS^0, \quad \mathbf{R}_M^{(II)} := \begin{pmatrix} 0 & \mathcal{R}_M^{(II)} \\ \overline{\mathcal{R}}_M^{(II)} & 0 \end{pmatrix} \in OPS^{\frac12 - M}$$

are $O(\varepsilon)$ small. This is achieved by applying iteratively M-times a conjugation map which transforms the off-diagonal block operators into 1-smoother ones. Notice that the operator $\mathcal{L}_M$ in (2.27) is block-diagonal up to the smoothing remainder $\mathbf{R}_M^{(II)} \in OPS^{\frac12 - M}$. The coefficients of $\mathbf{R}_M^{(II)}$ depend $O(M)$-derivatives of the approximate solution. In any case, the number of regularizing steps M will be fixed (independently on s, depending just on the diophantine exponent τ), determined by the KAM reducibility scheme.

2.3.4. Egorov analysis. Space reduction of the order ∂_x.

The goal is now to eliminate the first order vector field $\mathbf{B}_1(\varphi,x)\partial_x$ from $\mathcal{L}_M$. We conjugate $\mathcal{L}_M$ by the flow

$$\mathbf{\Phi}(\varphi,t) := \begin{pmatrix} \Phi(\varphi,t) & 0 \\ 0 & \overline{\Phi}(\varphi,t) \end{pmatrix}$$

generated by the system

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \mathrm{i}\begin{pmatrix} a(\varphi,x) & 0 \\ 0 & -a(\varphi,x) \end{pmatrix} |D|^{\frac12} \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \tag{2.28}$$

where $a(\varphi,x)$ is a small *real* valued function to be determined. Thus $\Phi(\varphi,t)$ is the flow of the scalar linear pseudo-PDE

$$\partial_t u = \mathrm{i}a(\varphi,x)|D|^{\frac12}u\,. \tag{2.29}$$

Conjugating the operator $\mathcal{L}_M$ in (2.27) by the time one flow operator $\mathbf{\Phi}(\varphi) := \mathbf{\Phi}(\varphi,1)$ we get

$$\mathcal{L}_M^{(1)} = \mathbf{\Phi}\mathcal{L}_M\mathbf{\Phi}^{-1} = \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathbf{\Phi}(\varphi)\mathbf{P}_0(\varphi,x,D)\mathbf{\Phi}(\varphi)^{-1} + \mathbf{\Phi}(\varphi)\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\} + \mathbf{\Phi}\mathbf{R}_M^{(II)}\mathbf{\Phi}^{-1}$$

where we have denoted

$$\mathbf{P}_0(\varphi,x,D) = \mathrm{im}_3\mathbf{T}(D) + \mathbf{B}_1(\varphi,x)\partial_x + \mathrm{i}\mathbf{B}_0^{(I)}(\varphi,x)\mathcal{H}|D|^{\frac12} + \mathbf{R}_M^{(I)}$$

the diagonal part of $\mathcal{L}_M$. Note that the terms $\mathbf{\Phi}(\varphi)\mathbf{P}_0(\varphi,x,D)\mathbf{\Phi}(\varphi)^{-1}$ and $\mathbf{\Phi}(\varphi)\,\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\}$ are block-diagonal. They are classical pseudo-differential operators and can be analyzed by an

Egorov type argument. On the other hand the off-diagonal term $\mathbf{\Phi}\mathbf{R}_M^{(II)}\mathbf{\Phi}^{-1}$ is very regularizing and satisfy tame estimates. Let us see how evolves the operator

$$\mathbf{P}(\varphi,t)=\mathbf{\Phi}(\varphi,t)\mathbf{P}_0\mathbf{\Phi}(\varphi,t)^{-1}=\begin{pmatrix}P(\varphi,t) & 0\\ 0 & \overline{P}(\varphi,t)\end{pmatrix},$$
$$P(\varphi,t):=\Phi(\varphi,t)p_0(\varphi,x,D)\Phi^{-1}(\varphi,t)\,. \quad (2.30)$$

under the flow of (2.28). The operator $P(\varphi,t)$ solves the usual Heisenberg equation

$$\begin{cases}\partial_t P(\varphi,t)=\mathrm{i}[A(\varphi),P(\varphi,t)]\\ P(\varphi,0)=P_0:=p_0(\varphi,x,D)\end{cases}\qquad \text{where} \qquad A(\varphi)=a(\varphi,x)|D|^{\frac12}\,. \quad (2.31)$$

The equation (2.31) can be solved in decreasing symbols using the fact that the order of the commutator $[A(\varphi),Q(\varphi)]$ with a classical pseudo-differential operator Q is strictly less than the order of Q. More precisely (2.31) has an approximate solution $Q(\varphi,t):=q(t,\varphi,x,D)$ expanded in decreasing orders

$$q(t,\varphi,x,\xi)=\sum_{n=0}^{M}q_n(t,\varphi,x,\xi)\,,\quad q_n(t,\varphi,x,\xi)\in S^{\frac12(3-n)}\,,\quad \forall n=0,\dots,M \quad (2.32)$$

where $q_0=p_0$ and the other lower order symbols q_n are recursively computed. This shows that the diagonal term $P(\varphi,t)$ remains pseudo-differential along the conjugation. More precisely we prove that $P(\varphi,t)$ is equal to $Q(\varphi,t)$ plus a term which is very regularizing and satisfies tame estimates in Sobolev spaces. One can analyze $\mathbf{\Phi}(\varphi)\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\}$ in the same way.

As an outcome, choosing properly the function $a(\varphi,x)$, and using the fact that the operator $\mathcal{L}_2$ is even, one can eliminate the order ∂_x getting an operator of the form

$$\mathcal{L}_M^{(1)}=\omega\cdot\partial_\varphi\mathbb{I}_2+\mathrm{i}\mathtt{m}_3\mathbf{T}(D)+\mathrm{i}\big(\mathbf{C}_1(\varphi,x)+\mathbf{C}_0(\varphi,x)\mathcal{H}\big)|D|^{\frac12}+\mathbf{R}_M^{(1)}+\mathbf{Q}_M^{(1)} \quad (2.33)$$

where

$$\mathbf{C}_1(\varphi,x):=\begin{pmatrix}a_{10} & 0\\ 0 & -a_{10}\end{pmatrix},\quad \mathbf{C}_0(\varphi,x):=\begin{pmatrix}a_{11} & 0\\ 0 & -a_{11}\end{pmatrix}.$$

Remark 2.1. Alazard-Baldi [1] uses a semi-Fourier integral operator $\mathrm{Op}(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}})\in OPS^0_{\frac12,\frac12}$. The use of the flow $\Phi(\varphi)$ of (2.29) is simpler because the proof that Φ, as well as its inverse Φ^{-1}, is a bounded operator on Sobolev spaces H^s and satisfies tame estimates, follows by simple energy estimates (the vector field $\mathrm{i}a(\varphi,x)|D|^{1/2}$ is skew-adjoint at the highest order, see Appendix of [8]). □

The fact that the diagonal terms on the conjugated operator (2.33) are still pseudo-differential is a relevant information. Indeed the flow $\Phi(\varphi)\sim\mathrm{Op}(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}})$ maps Sobolev spaces in itself, but each derivative

$$\partial_\varphi\Phi(\varphi)\sim\mathrm{Op}\big(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}}\,\mathrm{i}\partial_\varphi a(\varphi,x)\sqrt{|\xi|}\big)$$

is an unbounded operator which loses $|D|^{1/2}$ derivatives. Actually $\partial^k_{\omega,\kappa}\partial^\beta_\varphi\Phi(\varphi)$ loses $|D|^{\frac{|\beta|+|k|}{2}}$ derivatives. Since the conjugated operator

$$P_+(\varphi):=\Phi(\varphi)P_0\Phi(\varphi)^{-1}=\mathrm{Op}(c(\varphi,x,\xi))\,,\quad c(\varphi,x,\xi)\in S^m\,, \quad (2.34)$$

is a classical pseudo-differential operator, the differentiated operator

$$\partial_\varphi P_+(\varphi)=\mathrm{Op}(\partial_\varphi c(\varphi,x,\xi))\in OPS^m$$

is still a pseudo-differential operator of the same order of P_0 with just a symbol $\partial_\varphi c$ less regular in φ. The loss of regularity for $\partial_\varphi c$ may be compensated by the usual Nash-Moser smoothing procedure in φ. This is the reason why we require that the diagonal remainder $\mathcal{R}_M^{(I)}\in OPS^0$ is just of order zero.

On the other hand, the off-diagonal term $\mathcal{R}_M^{(II)}\in OPS^{-M}$ evolves, under the flow $\mathbf{\Phi}(\varphi,t)$, according to the "skew-Heisenberg" equation obtained replacing in (2.31) the commutator with the skew-commutator. As a consequence the symbol of $(\mathcal{R}_M^{(II)})_+:=\Phi(\varphi)\mathcal{R}_M^{(II)}\overline{\Phi}(\varphi)^{-1}$ assumes the form $e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}}q(\varphi,x,\xi)$ where $q(\varphi,x,\xi)\in S^{-M}$ is a classical symbol (actually we do not prove it explicitly because it is not needed). Thus the action of each ∂_φ on $(\mathcal{R}_M^{(II)})_+$ produces an operator which loses $|D|^{\frac{1}{2}}$ derivatives in space more than $\mathcal{Q}_M$. This is why we have performed previously a large number M of regularizing steps for the off-diagonal components $\mathcal{R}_M^{(II)}$.

2.3.5. Space reduction of the order $|D|^{1/2}$

Finally we eliminate the x-dependence of the coefficient in front of $|D|^{\frac{1}{2}}$ in the operator $\mathcal{L}_M^{(1)}$ in (2.33), conjugating $\mathcal{L}_M^{(1)}$ by a matrix valued multiplication operator of the form

$$\mathbf{V} := \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \overline{\mathcal{V}} \end{pmatrix}, \qquad \mathcal{V} := \mathrm{Op}(v), \qquad v := v(\varphi, x, \xi) \in S^0 .$$

Choosing properly the function $v(\varphi, x)$ one finally gets

$$\mathcal{L}_M^{(2)} := \omega\cdot\partial_\varphi \mathbb{I}_2 + \mathrm{i}\mathtt{m}_3 \mathbf{T}(D) + \mathrm{i}\mathtt{m}_1 \mathbf{\Sigma}|D|^{\frac{1}{2}} + \mathbf{R}_M^{(2)} + \mathbf{Q}_M^{(2)}, \quad \text{where} \quad \mathbf{\Sigma} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.35}$$

which has the form stated in (2.18). We remark that all the previous transformations are real, even, and reversibility preserving, so that $\mathcal{L}_M^{(2)}$ is real, even and reversible.

2.3.6. KAM-reducibility scheme

We are now in position to apply an iterative quadratic scheme to reduce the *size* of the the terms $\mathbf{R}_M^{(2)}$ and $\mathbf{Q}_M^{(2)}$ (if possible) of the operator in (2.35). Let us explain the main idea. Consider a linear real, even, and reversible operator acting on $H_{\mathbb{S}^+}^{\perp}$ (recall (1.20)),

$$\mathbf{L} = \omega\cdot\partial_\varphi \mathbb{I}_2^{\perp} + \mathbf{D} + \varepsilon \mathbf{P}, \tag{2.36}$$

with diagonal part

$$\mathbf{D} = \begin{pmatrix} \mathrm{i}\Lambda & 0 \\ 0 & -\mathrm{i}\Lambda \end{pmatrix}, \quad \Lambda = \mathrm{diag}_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} \mu_j , \quad \mu_j = \mathtt{m}_3\sqrt{j(1+\kappa j^2)} + \mathtt{m}_1 j^{\frac{1}{2}} + r_j ,$$

with

$$\mathtt{m}_3, \mathtt{m}_1 \in \mathbb{R},\ \mathtt{m}_3 - 1,\ \mathtt{m}_1 = O(\varepsilon), \quad r_j \in \mathbb{R}, \quad \forall j \in \mathbb{N}^+ \setminus \mathbb{S}^+, \qquad \sup_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} |r_j| = O(\varepsilon),$$

(at the first step $\mu_j = \mathtt{m}_3\sqrt{j(1+\kappa j^2)} + \mathtt{m}_1 j^{\frac{1}{2}}$ by (2.35)), and a bounded perturbation

$$\mathbf{P} = \begin{pmatrix} P_1 & P_2 \\ \overline{P_2} & \overline{P_1} \end{pmatrix}.$$

Transform $\mathbf{L}$ under the flow $\mathbf{\Phi}(\varphi)$ having the form

$$\mathbf{\Phi}(\varphi) = \mathbb{I}_\perp + W(\varphi), \quad \mathbb{I}_\perp = \begin{pmatrix} \mathrm{Id}_{H_{\mathbb{S}^+}^{\perp}} & 0 \\ 0 & \mathrm{Id}_{H_{\mathbb{S}^+}^{\perp}} \end{pmatrix}, \qquad \mathbf{W}(\varphi) = \begin{pmatrix} W_1 & W_2 \\ \overline{W_2} & \overline{W_1} \end{pmatrix}$$

is a bounded map, to be determined. The conjugated operator $\mathbf{L}_+ = \mathbf{\Phi}(\varphi)^{-1}\mathbf{L}\mathbf{\Phi}(\varphi)$ is given by

$$\mathbf{L}_+ = \mathbf{L} + [\varepsilon\mathbf{W}(\varphi), \mathbf{L}(0)] + O(\varepsilon^2) = \omega\cdot\partial_\varphi + \mathbf{D} + \varepsilon\big\{\mathbf{P} + \omega\cdot\partial_\varphi\mathbf{W}(\varphi) + [\mathbf{W}, \mathbf{D}]\big\} + O(\varepsilon^2).$$

The goal is to eliminate the ε-term $\omega\cdot\partial_\varphi\mathbf{W}(\varphi) + [\mathbf{W}, \mathbf{D}] + \mathbf{P}$ (if possible). This amounts to solve

$$\begin{aligned} &\omega\cdot\partial_\varphi W_1(\varphi) + \mathrm{i}(W_1(\varphi)\Lambda - \Lambda W_1(\varphi)) + P_1(\varphi) = [\![P_1(\varphi)]\!] \\ &\omega\cdot\partial_\varphi W_2(\varphi) + \mathrm{i}(W_2(\varphi)\Lambda + \Lambda W_2(\varphi)) + P_2(\varphi) = 0, \end{aligned}$$

where $[\![P_1(\varphi)]\!] := \mathrm{diag}([P_1(\varphi)]_j^j)$. Expanding in Fourier series $W_i(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} W_i(\ell)e^{\mathrm{i}\ell\cdot\varphi}$ we are led to the following equations

$$\begin{aligned} &\mathrm{i}\omega\cdot\ell W_1(\ell) + \mathrm{i}(W_1(\ell)\Lambda - \Lambda W_1(\ell)) + P_1(\ell) = [\![P_1(\ell)]\!] \\ &\mathrm{i}\omega\cdot\ell W_2(\ell) + \mathrm{i}(W_2(\ell)\Lambda + \Lambda W_1(\ell)) + P_2(\ell) = 0. \end{aligned}$$

Representing $W_i(\ell) = \big([W_i(\ell)]_k^j\big)_{j,k\in\mathbb{N}^+\setminus\mathbb{S}^+}$ as a matrix with respect to the basis $\{\cos(jx) : j \in \mathbb{N}^+ \setminus \mathbb{S}^+\}$ for even functions in x, we get the infinitely many scalar equations

$$\begin{aligned}
&\mathrm{i}\omega\cdot\ell[W_1(\ell)]_k^j + \mathrm{i}[W_1(\ell)]_k^j(\mu_j - \mu_k) + [P_1(\ell)]_k^j = [P_1(\ell)]_j^k \\
&\mathrm{i}\omega\cdot\ell[W_2(\ell)]_k^j + \mathrm{i}[W_2(\ell)]_k^j(\mu_j + \mu_k) + [P_2(\ell)]_k^j = 0\,.
\end{aligned}$$

These equations admit the solutions

$$\begin{aligned}
[W_1(\ell)]_k^j &= \frac{[P_1(\ell)]_k^j}{\mathrm{i}(\omega\cdot\ell + \mu_j - \mu_k)}\,, \quad \forall(\ell,j,k)\neq(0,j,j)\,,\\
[W_2(\ell)]_k^j &= \frac{[P_2(\ell)]_k^j}{\mathrm{i}(\omega\cdot\ell + \mu_j + \mu_k)}\,, \quad \forall(\ell,j,k)
\end{aligned}$$

if the corresponding denominators do not vanish. We actually require a quantitative lower bound for the denominators as

$$|\omega\cdot\ell + \mu_j - \mu_k| \geq \frac{|j^{3/2} - k^{3/2}|}{\gamma\langle\ell\rangle^\tau}\,, \quad |\omega\cdot\ell + \mu_j + \mu_k| \geq \frac{|j^{3/2} + k^{3/2}|}{\gamma\langle\ell\rangle^\tau}\,.$$

These conditions are called the second order Melnikov non-resonance conditions and appear in the Cantor set (2.11). After this conjugation step we have obtained a linear operator of the same form (2.36), but with a smaller $O(\varepsilon^2)$ perturbation and a new diagonal part corrected by the matrix $[\![P_1(0)]\!] = \mathrm{diag}[P_1(0)]_j^j$. Since $\mathbf{P}$ is reversible then $[P_1(0)]_j^j$ are purely imaginary. In order to apply the above classical KAM reducibility scheme to the operator $\mathcal{L}_M^{(2)}$ in (2.35) a difficulty is that the remainders $\mathbf{R}_M^{(2)}$, $\mathbf{Q}_M^{(2)}$ satisfy tame estimates. For technical details of the proof we refer to [8]. Here we just mention that for the convergence we need the tame conditions (2.19). In conclusion the operator $\mathcal{L}_M^{(2)}$ defined in (2.35) may be conjugated to a diagonal operator of the form

$$\omega\cdot\partial_\varphi\mathbb{I}_2^\perp + \mathrm{i}\mathbf{D}_\infty\,, \quad \mathbf{D}_\infty = \begin{pmatrix}\mathrm{i}\Lambda & 0\\ 0 & -\mathrm{i}\Lambda_\infty\end{pmatrix}\,,$$

with

$$\Lambda_\infty = \mathrm{diag}_{j\in\mathbb{N}^+\setminus\mathbb{S}^+}\mu_j^\infty\,, \quad \mu_j^\infty = \mathtt{m}_3\sqrt{j(1+\kappa j^2)} + \mathtt{m}_1 j^{\frac12} + r_j^\infty\,, \quad \sup_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} r_j^\infty = O(\varepsilon)\,.$$

It is then sufficient to require the first order Melnikov conditions

$$|\omega\cdot\ell + \mu_j^\infty| \geq 2\gamma j^{\frac32}\langle\ell\rangle^{-\tau}\,, \quad \forall\ell\in\mathbb{Z}^\nu\,,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+$$

to prove the invertibility of the diagonal operators $\omega\cdot\partial_\varphi\mathbb{I}_2^\perp + \mathrm{i}\mathbf{D}_\infty$. These conditions appear in (2.11).

Since all the transformations that we have performed in sections 2.3.2-2.3.6 above are bounded map between Sobolev spaces (of high norm) we get the required tame estimates for the inverse of the original operator $\mathcal{L}$ defined in (2.17).

References

[1] Alazard T., Baldi P. Gravity capillary standing water waves. *Arch. Rat. Mech. Anal.*, 217, 3, 741-830, 2015.

[2] Baldi P., Berti M., Montalto R. A note on KAM theory for quasi-linear and fully nonlinear KdV. *Rend. Lincei Mat. Appl.*, 24, 437-450, 2013.

[3] Baldi P., Berti M., Montalto R. KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. *Math. Annalen*, 359, 1-2, 471-536, 2014.

[4] Baldi P., Berti M., Montalto R. *KAM for quasi-linear KdV*, C. R. Acad. Sci. Paris, Ser. I 352, 603-607, 2014.

[5] Baldi P., Berti M., Montalto R. KAM for autonomous quasi-linear perturbations of KdV. *Ann. I. H. Poincaré (C) Anal. Non Linéaire*, 33, 1589-1638, 2016.

[6] Baldi P., Berti M., Montalto R. KAM for autonomous quasi-linear perturbations of mKdV. *Bollettino Unione Matematica Italiana*, 9, 143-188, 2016.

[7] Bambusi D., Berti M., Magistrelli E. Degenerate KAM theory for partial differential equations. *Journal Diff. Equations*, 250, 8, 3379-3397, 2011.

[8] Berti M., Montalto R. Quasi-periodic standing wave solutions of gravity-capillary water waves, arXiv:1602.02411 .

[9] Berti M., Biasco L., Procesi M. KAM theory for the Hamiltonian derivative wave equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(2):301-373, 2013.

[10] Berti M., Biasco L., Procesi M. KAM for Reversible Derivative Wave Equations. *Arch. Rat. Mech. Anal.*, 212(3):905-955, 2014.

[11] Berti M., Bolle P. A Nash-Moser approach to KAM theory. *Fields Institute Communications*, 255-284, special volume "Hamiltonian PDEs and Applications", 2015.

[12] Brezis H., Coron J.-M., Nirenberg L., Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz, *Comm. Pure Appl. Math.* 33, no. 5, 667–684, 1980.

[13] Craig W., Nicholls D. Travelling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.*, 32(2):323–359 (electronic), 2000.

[14] Craig W., Sulem C. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73-83, 1993.

[15] Craig W., Sulem C. Normal form transformations for capillary-gravity water waves. *Field Institute Communications*, 73-110, special volume "Hamiltonian PDEs and Applications", 2015.

[16] Craig W., Wayne C.E., Newton's method and periodic solutions of nonlinear wave equation, *Comm. Pure Appl. Math.* 46, 1409-1498, 1993.

[17] Fejoz J. Démonstration du théoréme d' Arnold sur la stabilité du systéme planétaire (d' aprés Herman). *Ergodic Theory Dynam. Systems* 24 (5), 1521-1582, 2004.

[18] Feola R., Procesi M. Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations, *J. Diff. Eq.*, 259, no. 7, 3389-3447, 2015.

[19] Iooss G., Plotnikov P. Existence of multimodal standing gravity waves. *J. Math. Fluid Mech.*, 7, 349–364, 2005.

[20] Iooss G., Plotnikov P. Multimodal standing gravity waves: a completely resonant system. *J. Math. Fluid Mech.*, 7(suppl. 1), 110-126, 2005.

[21] Iooss G., Plotnikov P. Small divisor problem in the theory of three-dimensional water gravity waves. *Mem. Amer. Math. Soc.*, 200(940):viii+128, 2009.

[22] Iooss G., Plotnikov P. Asymmetrical tridimensional travelling gravity waves. *Arch. Rat. Mech. Anal.*, 200(3):789-880, 2011.

[23] Iooss G., Plotnikov P., Toland J. Standing waves on an infinitely deep perfect fluid under gravity. *Arch. Rat. Mech. Anal.*, 177(3):367–478, 2005.

[24] Levi-Civita T. Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.*, 93 , pp. 264-314, 1925.

[25] Liu J., Yuan X. A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations. *Comm. Math. Phys.*, 307(3), 629–673, 2011.

[26] Kappeler T., Pöschel J. KAM and KdV, *Springer*, 2003.

[27] Kuksin S., Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum, *Funktsional. Anal. i Prilozhen.* 21, no. 3, 22–37, 95, 1987.

[28] Kuksin S. Analysis of Hamiltonian PDEs, volume 19 of Oxford Lecture Series in Mathematics and its Applications. *Oxford University Press*, Oxford, 2000.

[29] Montalto R. Quasi-periodic solutions of forced Kirchhoff equation. *NoDEA, Nonlinear Differ. Equ. Appl.*, in press, DOI: 10.1007/s00030-017-0432-3, 2017.

[30] Plotnikov P., Toland J. Nash-Moser theory for standing water waves. *Arch. Rat. Mech. Anal.*, 159(1):1–83, 2001.

[31] Rabinowitz P., Periodic solutions of nonlinear hyperbolic partial differential equations, Part I. *Comm. Pure Appl. Math.* 20, 145–205 (1967)

[32] Rabinowitz P., Periodic solutions of nonlinear hyperbolic partial differential equations. Part II. *Comm. Pure Appl. Math.* 22, 15–39 (1969)

[33] Rüssmann H. Invariant tori in non-degenerate nearly integrable Hamiltonian systems. *Regul. Chaotic Dyn.* 6 (2), 119-204, 2001.

[34] Wayne E., Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.* 127, 479-528, 1990.

[35] Zakharov V. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

[36] Zhang J., Gao M., Yuan X. KAM tori for reversible partial differential equations. *Nonlinearity*, 24(4):1189-1228, 2011.

University of Zürich
Winterthurerstrasse 190
CH-8057, Zürich
Switzerland
riccardo.montalto@math.uzh.ch