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Taoufik Hmidi Joan Mateu

Abstract

We study the existence of corotating and counter-rotating pairs of simply connected patches for Euler equations. From the numerical experiments implemented in [7, 16, 17] it is conjectured the existence of a curve of steady vortex pairs passing through the point vortex pairs. There are some analytical proofs based on variational principle [14, 18], however they do not give enough information about the pairs such as the uniqueness or the topological structure of each single vortex. We intend in this paper to give direct proofs confirming the numerical experiments. The proofs rely on the contour dynamics equations combined with a desingularization of the point vortex pairs and the application of the implicit function theorem.

1. Introduction

The present work deals with the dynamics of vortex pairs for two-dimensional Euler equations written in the formulation vorticity-velocity as follows,

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ v = \nabla^\perp\Delta^{-1}\omega, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{1.1}$$

Here $\nabla^\perp = (-\partial_2, \partial_1)$ and the velocity field v can be recovered from the vorticity according to Biot-Savart law:

$$v(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\omega(t,y)dy.$$

We shall be concerned with the motion of some special class of concentrated vortices, called vortex patches. For a single vortex patch, that is, $\omega_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain D, we know from Yudovich result [20] the existence of unique global solution in the patch form $\omega(t) = \chi_{D_t}$. In this case, the boundary motion of the domain D_t is described by the contour dynamics equation. Indeed, the Lagrangian parametrization $\gamma_t : \mathbb{T}\to\partial D_t$ obeys to the following integro-differential equations

$$\partial_t\gamma_t(w) = -\frac{1}{2\pi}\int_{\mathbb{T}}\log\big|\gamma_t(w)-\gamma_t(\xi)\big|\gamma_t'(\xi)d\xi.$$

The global persistence of the boundary regularity is established for Euler equations by Chemin [5], we refer also to the paper of Bertozzi and Constantin [1] for another proof. Let us mention that the contour dynamics equation remains globally well-posed when the domain of the initial data is composed of multiple patches with different magnitudes in each component. In this paper we shall focus on steady single and multiple patches moving without changing shape, called relative equilibria or V-states according to the terminology of Deem and Zabusky. Their dynamics is seemingly simple flow configurations described by rotating or translating motion but it is immensely rich and exhibits complex behaviors. There is abundant literature dealing with numerical and analytical structures for the isolated rotating patches

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and the first example goes back to Kirchhoff who proved for Euler equations that an ellipse of semi-axes a and b rotates uniformly with the angular velocity $\Omega = ab/(a+b)^2$. About one century later, Deem and Zabusky [7] provided strong numerical evidence for the existence of rotating patches with m-fold symmetry for the integers $m \in \{3,4,5\}$. Recall that a domain is said m-fold symmetric if it is invariant by the action of the dihedral group D_m. Few years later, Burbea gave in [2] an analytical proof and showed for any integer $m \geq 2$ the existence of a curve of V-states with m-fold symmetry bifurcating from Rankine vortex at the angular velocity $\frac{m-1}{2m}$. The proof relies on the use of complex analysis tools combined with the bifurcation theory. The regularity of the boundary close to Rankine vortices has been discussed very recently by the authors and Verdera in [12] and where it was proved that the boundary is C^∞ and convex. It seems that the boundary is actually analytic according to the recent result of Castro, Córdoba and Gómez-Serrano [3]. We also refer to the paper [19] where it is proved that corners with right angles is the only plausible scenario for the limiting V-states. In the same context, it turns out that for Euler equations a second bifurcation of countable branches from the ellipses occurs but the shapes have in fact less symmetry and being at most two-folds. This was first observed numerically in [13] and analytical proofs were recently discussed in [4, 10]. Another valuable investigation has been devoted to the existence of doubly connected V-states where the rotating patches have only one hole. In this case the boundary is comprised of two Jordan curves obeying to two coupled singular nonlinear equations and thereby the dynamics acquires more richness and significant behaviors. The existence of such structures was first accomplished for Euler equations in [6] by using bifurcation tools in the spirit of Burbea's approach. Roughly speaking, for higher symmetry m we get two branches of $m-$fold V-states bifurcating from the annulus $\{b < |z| < 1\}$ and numerical experiments about the limiting V-states reveal different plausible configurations depending on the size of the parameter b. It is worthy to mention that the bifurcation in the preceding cases is obtained under the transversality assumption of Crandall-Rabinowitz corresponding to simple nonlinear eigenvalues. However the bifurcation in the degenerate case where there is crossing eigenvalues is more complicate and has been recently solved in [11].

The main task of this paper is to deal with non connected V-states where the bifurcation arguments discussed above are out of use. To be more precise, we shall be concerned with vortex pairs moving without deformation. This is a fundamental and rich subject in vortex dynamics and they serve for instance to model trailing vortices behind the wings of aircraft in steady horizontal flight or to describe the interaction between isolated vortex and a solid wall. We point out that the literature is very abundant and it is by no means an easy task to collect and recall all the results done in this field. Therefore we shall restrict the discussion to the cases of counter-rotating and corotating vortices and recall some results that fit with our main goal. In the first case, the most common studied configuration is two symmetric vortex pairs with opposite circulations moving steadily with constant speed in a fixed direction. Notice that an explicit example is given by a pair of point vortices with opposite circulations which translates steadily with the speed $U_{sing} = \frac{\gamma}{2\pi d}$, where d is the distance separating the point vortices and γ is the magnitude, see for instance [15]. Another nontrivial explicit example of touching counter-rotating vortex pair was discovered by Lamb [15], where the vortex is not uniformly distributed but has a smooth compactly supported profile related to Bessel functions of the first kind. Later, Deem and Zabusky [7] and Pierrehumbert [16] provided numerically a class of translating vortex pairs of symmetric patches and they conjectured the existence of a curve of translating symmetric pair of simply connected patches emerging from two point vortices and ending with two touching patches at right angle. We mention that Keady [14] used a variational principle in order to explore the existence part and give asymptotic estimates for some significant functionals such as the excess kinetic energy and the speed of the pairs. The basic idea is to maximize the excess kinetic energy supplemented with some additional constraints and to show the existence of a maximizer taking the form of a pair of vortex patches in the spirit of the paper of Turkington [18]. However, this approach does not give sufficient information on the structure of the pairs. For example the uniqueness of the maximizer is left open and the topology of the patches is not well-explored, and it is not clear from the proof whether or not each single patch is simply connected as it is suggested numerically. Concerning the corotating vortex pair, which consists of two symmetric patches with the same circulations and rotating about the centroid of the system with constant angular velocity, it was investigated numerically by Saffman and Szeto in [17]. They showed that when far apart, the vortices are almost circular and when the distance between them decreases they become more deformed until they touch. We remark that a pair of point vortices far away at a distance d and with the same magnitude γ rotates steadily with the angular velocity $\Omega_{sing} = \frac{\gamma}{\pi d^2}$. By using variational principle, Turkington gave in [18] an analytic proof of the existence of corotating vortex pairs but this general approach does not give enough precision on the topological structure of each vortex patch similarly to the translating case commented before. Note that in the same direction Dritschel [9] calculated numerically V-states of vortex pairs with different shapes and discussed their linear stability. Very recently, Denisov established in [8] for a modified Euler equations the existence of corotating simply connected vortex patches and analyzed the contact point of the limiting V-states.

In the current paper we intend to give direct proofs for the existence of corotating and counter-rotating vortex pairs using the contour dynamics equations. Now we shall fix some notations before stating our main result. Let $0 < \varepsilon < 1$, $d > 2$ and take a small simply connected domain D_1 containing the origin and contained in the open ball $D(0, 2)$ centered at the origin and with radius 2. Define

$$\omega_{0,\varepsilon} = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} + \delta\frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}, \quad D_1^\varepsilon = \varepsilon D_1, \quad D_2^\varepsilon = -D_1^\varepsilon + 2d, \tag{1.2}$$

where the number δ is taken in $\{\pm 1\}$. As we can readily observe, this initial data is composed of symmetric pair of simply connected patches with equal or opposite circulations.
The main result of the paper is the following.

Main Theorem. *There exists $\varepsilon_0 > 0$ such that the following results hold true.*

(i) *Case $\delta = 1$. For any $\varepsilon \in (0, \varepsilon_0]$ there exists a strictly convex domain D_1^ε at least of class C^1 such that $\omega_{0,\varepsilon}$ in (1.2) generates a corotating vortex pair for (1.1).*

(ii) *Case $\delta = -1$. For any $\varepsilon \in (0, \varepsilon_0]$ there exists a strictly convex domain D_1^ε of class C^1 such that $\omega_{0,\varepsilon}$ generates a counter-rotating vortex pair for (1.1).*

Remark 1. *The domain D_1^ε is a small perturbation of the disc $D(0, \varepsilon)$, centered at zero and of radius ε. Moreover, it can be described by the conformal parametrization $\phi_\varepsilon : \mathbb{T} \to \partial D_1^\varepsilon$ which belongs to $C^{1+\beta}$ for any $\beta \in (0, 1)$, and satisfies*

$$\phi_\varepsilon(w) = \varepsilon w + \varepsilon^2 f_\varepsilon(w) \quad \textit{with} \quad \|f_\varepsilon\|_{C^{1+\beta}} \le 1.$$

Therefore the boundary of each V-state is at least C^1. Note that with slight modifications we can adapt the proofs and show that the domain D_1^ε belongs to $C^{n+\beta}$ for any fixed $n \in \mathbb{N}$. Of course, the size of ε_0 depends on the parameter n and cannot be uniform; it shrinks to zero as n grows to infinity. However, we expect the boundary to be analytic meaning that the conformal mapping possesses a holomorphic extension in $D(0, r)^c$ for some $0 < r < 1$. The ideas developed in the recent paper [4] might be useful to confirm such expectation.

Notation. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, we define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \equiv \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

where $d\tau$ stands for the complex integration.

2. Steady vortex pairs models

The aim of this section is to derive the equations governing co-rotating and translating symmetric pairs of patches using the conformal parametrization.

2.1. Corotating vortex pair

Let D_1 be a bounded simply connected domain containing the origin and contained in the ball $B(0, 2)$. For $\varepsilon \in]0, 1[$ and $d > 2$ we define the domains

$$D_1^\varepsilon = \varepsilon D_1 \quad \text{and} \quad D_2^\varepsilon = -D_1^\varepsilon + 2d.$$

Set

$$\omega_{0,\varepsilon} = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} + \frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}$$

and assume that this gives rise to a rotating pairs of patches about the centroid of the system $(d, 0)$ and with an angular velocity Ω. According to [6], this condition holds true if and only if

$$\mathrm{Re}\big(-i\,\Omega\,(\overline{z} - d)\,\vec{n}\big) = \mathrm{Re}\big(\overline{v(z)}\,\vec{n}\big), \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \tag{2.1}$$

where $\vec{n}$ is the exterior unit normal vector to the boundary of $D_1^\varepsilon \cup D_2^\varepsilon$ at the point z. It is well-known that the velocity can be recovered for the vorticity according to Biot-Savart law,

$$\overline{v(z)} = -\frac{i}{2\pi\,\varepsilon^2}\int_{D_1^\varepsilon}\frac{dA(\zeta)}{z-\zeta} - \frac{i}{2\pi\varepsilon^2}\int_{D_2^\varepsilon}\frac{dA(\zeta)}{z-\zeta}, \quad \forall z\in\mathbb{C}.$$

From Green-Stokes formula we record that

$$-\frac{1}{\pi}\int_D\frac{dA(\zeta)}{z-\zeta} = \fint_{\partial D}\frac{\overline{\xi}-\overline{z}}{\xi-z}\,d\xi, \quad \forall z\in\mathbb{C}.$$

Therefore

$$\operatorname{Re}\Big\{\Big(2\Omega\big(\overline{z}-d\big)+I(z)\Big)\,\vec{\tau}\Big\}=0, \quad \forall z\in\partial D_1^\varepsilon\cup\partial D_2^\varepsilon, \tag{2.2}$$

with $\vec{\tau}$ being the unit tangent vector to $\partial D_1^\varepsilon\cup\partial D_2^\varepsilon$ positively oriented and

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi+\frac{1}{\varepsilon^2}\fint_{\partial D_2^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi.$$

Changing in the last integral ξ to $-\xi+2d$, which sends ∂D_2^ε to ∂D_1^ε, we get

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi-\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}+\overline{z}-2d}{\xi+z-2d}d\xi.$$

We can check that if the equation (2.2) is satisfied for all $z\in\partial D_1^\varepsilon$, then it will be surely satisfied for all $z\in\partial D_2^\varepsilon$. This follows easily from the identity

$$I(-z+2d)=-I(z).$$

Now observe that when $z\in\partial D_1^\varepsilon$ then $-z+2d\notin\overline{D_1^\varepsilon}$ and thus residue theorem allows to get

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi-\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}}{\xi+z-2d}d\xi.$$

Denote $\Gamma_1=\partial D_1$ then by the change of variables $\xi\mapsto\varepsilon\xi$ and $z\mapsto\varepsilon z$ the equation (2.2) becomes

$$\operatorname{Re}\Big\{\Big(2\Omega\big(\varepsilon\overline{z}-d\big)+I_\varepsilon(z)\Big)\,\vec{\tau}\Big\}=0, \quad \forall z\in\Gamma_1.$$

with

$$\begin{aligned}
I_\varepsilon(z) &\equiv I(\varepsilon z)\\
&= \frac{1}{\varepsilon}\fint_{\Gamma_1}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi-\fint_{\Gamma_1}\frac{\overline{\xi}}{\varepsilon\xi+\varepsilon z-2d}d\xi\\
&\equiv I_\varepsilon^1(z)-I_\varepsilon^2(z).
\end{aligned}$$

We shall search for domains D_1 which are small perturbations of the unit disc with an amplitude of order ε. More precisely, we shall in the conformal parametrization $\phi:\mathbb{T}\to\partial D_1$ look for a solution in the form

$$\phi(w)=w+\varepsilon f(w), \text{with} \quad f(w)=\sum_{n\geq1}\frac{a_n}{w^n}, \quad a_n\in\mathbb{R}.$$

We remark that the assumption $a_n\in\mathbb{R}$ means that the domain D_1 is symmetric with respect to the real axis. Setting $z=\phi(w)$, then for $w\in\mathbb{T}$ a tangent vector to the boundary at the point z is given by

$$\vec{\tau}=i\,w\,\phi'(w)=i\,w\big(1+\varepsilon f'(w)\big).$$

Thus the steady vortex pairs equation becomes

$$\operatorname{Im}\Big\{\Big(2\Omega\Big[\varepsilon\overline{w}+\varepsilon^2 f(\overline{w})-d\Big]+I_\varepsilon(\phi(w))\Big)\,w\big(1+\varepsilon f'(w)\big)\Big\}=0, \quad \forall w\in\mathbb{T}. \tag{2.3}$$

Notice that we have used that f has real Fourier coefficients and thus $\overline{f(w)} = f(\overline{w})$. By using the notation $A = \tau - w$ and $B = f(\tau) - f(w)$ we can write for all $w \in \mathbb{T}$

$$\begin{aligned} I_\varepsilon^1(\phi(w)) &= \frac{1}{\varepsilon} \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon\big(f(\overline{\tau}) - f(\overline{w})\big)}{\tau - w + \varepsilon(f(\tau) - f(w))} \big(1 + \varepsilon f'(\tau)\big) d\tau \\ &= \fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon \overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau + \frac{1}{\varepsilon} \fint_{\mathbb{T}} \frac{\overline{A}}{A} d\tau \\ &= \fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon \overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau - \frac{1}{\varepsilon} \overline{w}, \end{aligned}$$

where we have used the obvious formula

$$\begin{aligned} \fint_{\mathbb{T}} \frac{\overline{A}}{A} d\tau &= -\overline{w} \fint_{\mathbb{T}} \frac{d\tau}{\tau} \\ &= -\overline{w}. \end{aligned}$$

This leads to a significant cancellation and the singular term will disappear from the full nonlinearity due in particular to the symmetry of the disc,

$$\begin{aligned} \mathrm{Im}\Big\{ I_\varepsilon^1(\phi(w))\, w\big(1 + \varepsilon f'(w)\big) \Big\} = \mathrm{Im}\Big\{ \Big(\fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon \overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau \Big) w \big[1 + \varepsilon f'(w)\big] \Big\} \\ - \mathrm{Im}(f'(w)), \quad \forall w \in \mathbb{T}. \end{aligned}$$

For the second term $I_\varepsilon^2(\phi(w)$ it takes the form

$$I_\varepsilon^2(\phi(w) = \fint_{\mathbb{T}} \frac{\overline{(\tau + \varepsilon f(\overline{\tau}))}(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2\big(f(\tau) + f(w)\big) - 2d} d\tau.$$

Hence the steady vortex pairs equation is equivalent to

$$G(\varepsilon, \Omega, f) \equiv \mathrm{Im}(F(\varepsilon, \Omega, f)) = 0 \tag{2.4}$$

with

$$\begin{aligned} F(\varepsilon, \Omega, f(w)) &= 2\Omega\Big(\varepsilon\overline{w} + \varepsilon^2 f(\overline{w}) - d\Big) w\big(1 + \varepsilon f'(w)\big) - f'(w) \\ &\quad + \Big(\fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon \overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau \Big) w\big(1 + \varepsilon f'(w)\big) \\ &\quad - \Big(\fint_{\mathbb{T}} \frac{\big(\overline{\tau} + \varepsilon f(\overline{\tau})\big)\big(1 + \varepsilon f'(\tau)\big)}{\varepsilon(\tau + w) + \varepsilon^2\big(f(\tau) + f(w)\big) - 2d} d\tau \Big) w\big(1 + \varepsilon f'(w)\big) \\ &\equiv F_1(\varepsilon, \Omega, f(w)) + F_2(\varepsilon, f(w)) + F_3(\varepsilon, f(w)). \end{aligned}$$

2.2. Counter-rotating vortex pair

Let D_1 be a bounded domain containing the origin and contained in the ball $B(0, 2)$. For $\varepsilon \in]0, 1[$ and $d > 2$ we define as before

$$D_1^\varepsilon = \varepsilon D_1 \quad \text{and} \quad D_2^\varepsilon = -D_1^\varepsilon + 2d.$$

Set

$$\omega_0 = \frac{1}{\varepsilon^2} \chi_{D_1^\varepsilon} - \frac{1}{\varepsilon^2} \chi_{D_2^\varepsilon}$$

and assume that θ_0 travels steadily in the (Oy) direction with uniform velocity U. Then in the moving frame the pair of the patches is stationary and consequently the analogous of the equation (2.1) is

$$\mathrm{Re}\big\{ \big(\overline{v(z)} + iU\big)\, \vec{n} \big\} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon. \tag{2.5}$$

One has from (2.5)

$$\mathrm{Re}\left\{ (2U + I(z))\ \vec{\tau} \right\} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \tag{2.6}$$

with

$$I(z) = \frac{1}{\varepsilon^2} \fint_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \frac{1}{\varepsilon^2} \fint_{\partial D_2^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi.$$

In the last integral changing ξ to $-\xi+2d$ which sends ∂D_2^ε to ∂D_1^ε we get

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi+\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}+\overline{z}-2d}{\xi+z-2d}d\xi.$$

We can check that if the equation (2.6) is satisfied for all $z\in\partial D_1^\varepsilon$ then it is also satisfied for all $z\in\partial D_2^\varepsilon$. This follows from the identity

$$I(-z+2d)=I(z).$$

Now observe that when $z\in\partial D_1^\varepsilon$ then $-z+2d\notin\overline{D_1^\varepsilon}$ and using residue theorem we obtain

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi+\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}}{\xi+z-2d}d\xi.$$

Let $\Gamma_1=\partial D_1$ then by change of variables $\xi\to\varepsilon\xi$ and $z\to\varepsilon z$. The equation (2.6) becomes

$$\mathrm{Re}\Big\{\Big(2U+I_\varepsilon(z)\Big)\,\vec{\tau}\Big\}=0,\quad\forall z\in\Gamma_1,$$

with

$$\begin{aligned}I_\varepsilon(z)&=I(\varepsilon z)\\&=\frac{1}{\varepsilon}\fint_{\Gamma_1}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi+\fint_{\Gamma_1}\frac{\overline{\xi}}{\varepsilon\xi+\varepsilon z-2d}d\xi\\&\equiv I_\varepsilon^1(z)+I_\varepsilon^2(z).\end{aligned}$$

we shall now use the conformal parametrization of the boundary Γ_1,

$$\phi(w)=w+\varepsilon f(w),\text{with}\quad f(w)=\sum_{n\geq1}\frac{a_n}{w^n},a_n\in\mathbb{R}.$$

Setting $z=\phi(w)$ and $\xi=\phi(\tau)$, then for $w\in\mathbb{T}$ a tangent vector at the point $\phi(w)$ is given by

$$\vec{\tau}=iw\,\phi'(w)=iw\big(1+\varepsilon f'(w)\big).$$

The V-states equation becomes

$$\mathrm{Im}\Big\{\Big(2U+I_\varepsilon(\phi(w))\Big)\,w\big(1+\varepsilon f'(w)\big)\Big\}=0,\quad\forall w\in\mathbb{T}.$$

As in the rotating case, with the notation $A=\tau-w$ and $B=f(\tau)-f(w)$ we get for $w\in\mathbb{T}$

$$I_\varepsilon^1(\phi(w))\quad=\quad\fint_{\mathbb{T}}\frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B}f'(\tau)d\tau+\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}d\tau-\frac{1}{\varepsilon}\overline{w}.$$

This yields

$$\begin{aligned}\mathrm{Im}\Big\{I_\varepsilon^1(\phi(w))\,w\big(1+\varepsilon f'(w)\big)\Big\}=\mathrm{Im}\Big\{\Big(\fint_{\mathbb{T}}\frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B}f'(\tau)d\tau+\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}d\tau\Big)w\big(1+\varepsilon f'(w)\big)\Big\}\\-\mathrm{Im}(f'(w)),\quad\forall w\in\mathbb{T}.\end{aligned}$$

For the second term $I_\varepsilon^2(\phi(w)$ it takes the form

$$I_\varepsilon^2(\phi(w))=\fint_{\mathbb{T}}\frac{(\overline{\tau}+\varepsilon\overline{f(\tau)})(1+\varepsilon f'(\tau))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}d\tau.$$

Hence the V-states equation becomes

$$G(U,\varepsilon,f)\equiv\mathrm{Im}(F(U,\varepsilon,f)=0$$

with

$$\begin{aligned}F(U,\varepsilon,f(w))\quad&=\quad2Uw\big(1+\varepsilon f'(w)\big)-f'(w)\\&\quad+\Big(\fint_{\mathbb{T}}\frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B}f'(\tau)d\tau+\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}d\tau\Big)w\big(1+\varepsilon f'(w)\big)\\&\quad+\Big(\fint_{\mathbb{T}}\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\big(1+\varepsilon f'(\tau)\big)d\tau\Big)w\big(1+\varepsilon f'(w)\big)\\&\equiv\quad F_1(U,\varepsilon,f(w))+F_2(\varepsilon,f(w))+F_3(\varepsilon,f(w)).\end{aligned}$$

3. Proof of corotating vortex pairs

In this section we will prove the results of the Main Theorem but we shall restrict the discussion to the corotating pairs. The first goal is to discuss the regularity of the functionals defining the V-states. In the subsection 3.2 we shall see how the angular velocity is uniquely determined through the geometry of the domain. Finally, in the subsection 3.3 we will get the existence of the vortex pairs as a consequence of the implicit function theorem on Banach spaces and discuss the arguments leading to the convexity of the each single vorticity.

3.1. Extension and regularity of the functional G

The main idea to prove the existence of rotating vortex pairs is to apply the implicit function theorem to the equation (2.4). For this purpose we have to check that the function G defined there satisfies some regularity conditions. First we need to fix some function spaces. Let $\beta \in]0,1[$ and consider the spaces

$$X = \Big\{ f \in C^{1+\beta}(\mathbb{T}), f(w) = \sum_{n\geq 1} a_n w^{-n} \Big\},$$

$$Y = \Big\{ f \in C^{\beta}(\mathbb{T}), f = \sum_{n\geq 1} a_n e_n,\, a_n \in \mathbb{R} \Big\}, \quad \widehat{Y} = \big\{ f \in Y, a_1 = 0 \big\}, \quad e_n(w) = \mathrm{Im}(w^n).$$

For $r > 0$ we denote by B_r the open ball of X centered at zero and of radius r. The next result deals with some properties of the function G.

Proposition 1. *The following assertions hold true.*

(i) *The function G can be extended from $]-\frac{1}{2}, \frac{1}{2}[\times\mathbb{R}\times B_1 \to Y$ as a C^1 function.*

(ii) *Two initial point vortex $\pi\delta_0$ and $\pi\delta_{(2d,0)}$ rotate uniformly about $(d,0)$ with the angular velocity*

$$\Omega_{sing} \equiv \frac{1}{4d^2}.$$

(iii) *For $\Omega \in \mathbb{R}$ and $h \in X$, we have*

$$\partial_f G(0,\Omega,0)h(w) = -\mathrm{Im}\{h'(w)\}.$$

(iv) *For any $\Omega \in \mathbb{R}$, the operator $\partial_f G(0,\Omega,0) : X \to \widehat{Y}$ is an isomorphism.*

Proof. (i) We will start with the regularity of the functional

$$G_1(\varepsilon,\Omega,f) = \mathrm{Im}\Big\{ 2\Omega\Big(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d\Big) w\big(1+\varepsilon f'(w)\big) - f'(w) \Big\}.$$

Clearly this function can be defined from the set $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1$ to Y because the function in the brackets is in $C^\beta(\mathbb{T})$, and is obtained as sums and products of functions with real coefficients. In order to prove its differentiability we have to compute the partial derivatives of G_1.

$$\partial_\varepsilon G_1(\varepsilon,\Omega,f) = \mathrm{Im}\Big\{ 2\Omega(\overline{w} + 2\varepsilon\overline{f(w)}) w(1+\varepsilon f'(w)) + 2\Omega(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d) w f'(w) \Big\},$$

and clearly this is a continuous function from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1$ to Y.
Taking now the derivative in Ω we get

$$\partial_\Omega G_1(\varepsilon,\Omega,f) = \mathrm{Im}\Big\{ 2\Big(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d\Big) w\big(1+\varepsilon f'(w)\big) \Big\},$$

which is continuous from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1$ to Y. Let's note that G_1 is a polynomial also in f and f' and consequently the derivative is also polynomial in f and f'. Thus, it is a continuous function from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1$ to Y. It is an easy computation to check that

$$\partial_f G_1(0,\Omega,0)(h) = -\mathrm{Im}\{h'(w)\}.$$

Let's take now

$$\begin{aligned} G_2(\varepsilon,f) &= \mathrm{Im}\Big\{ \Big(\fint_{\mathbb{T}} \frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A+\varepsilon B)} d\tau \Big) w\big(1+\varepsilon f'(w)\big) \Big\} \\ &= \mathrm{Im}\Big\{ \big(G_{21} + G_{22}\big) w\big(1+\varepsilon f'(w)\big) \Big\}. \end{aligned}$$

To prove that $G_2(\varepsilon, f)$ is a function from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to Y it is enough to verify that the functions $G_{21}(\varepsilon, f)$ and $G_{22}(\varepsilon, f)$ satisfies the same property. The function

$$G_{21}(\varepsilon, f) = \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau$$

is given by an integral operator. Since f is in $C^{1+\beta}(\mathbb{T})$, we will have that G_{21} is in the space $C^\beta(\mathbb{T})$ if the kernel

$$K(\tau, w) = \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))}$$

satisfies the hypothesis of the lemma below.

Lemma 1. *Consider a function $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ with the following properties. There exits $C_0 > 0$ such that,*

(i) *K is measurable on $\mathbb{T} \times \mathbb{T} \backslash \{(w, w),\, w \in \mathbb{T}\}$ and*

$$\big|K(w, \tau)\big| \leq C_0, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

(ii) *For each $\tau \in \mathbb{T}$, $w \mapsto K(w, \tau)$ is differentiable in $\mathbb{T} \backslash \{\tau\}$ and*

$$\big|\partial_w K(w, \tau)\big| \leq \frac{C_0}{|w - \tau|}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

Then the operator $\mathcal{T}$ is continuous from $L^\infty(\mathbb{T})$ to $C^\beta(\mathbb{T})$ for any $0 < \beta < 1$. That is, there exists a constant C_β depending only on β such that

$$\|\mathcal{T}(f)\|_\beta \leq C_\beta C_0 \|f\|_{L^\infty}.$$

Now , we note that for $\tau \neq w$

$$|K(\tau, w)| \leq 1,$$

and moreover

$$\begin{aligned} |\partial_w K(\tau, w)| &= \Big| \frac{(1 + \varepsilon f'(w)\big((\overline{\tau} - \overline{w}) + \varepsilon(f(\overline{\tau}) - f(\overline{w})\big)}{((\tau - w) + \varepsilon(f(\tau) - f(w)))^2} + \frac{1}{w^2} \frac{1 + \varepsilon f'(\overline{w})}{(\tau - w) + \varepsilon(f(\tau) - f(w))} \Big| \\ &\leq \frac{M^2 + M}{|\tau - w|}, \end{aligned}$$

where $M = \frac{1 + \varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}{1 - \varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}$. Now to check that this function has real coefficients we have to show that $\overline{G_{21}(\varepsilon, f)(w)} = G_{21}(\varepsilon, f)(\overline{w})$. Using the change of variable $\eta = \overline{\tau}$, it is an easy computation to see that

$$\begin{aligned} \overline{G_{21}(\varepsilon, f)(w)} &= -\fint_{\mathbb{T}} \frac{\tau - w + \varepsilon(f(\tau) - f(w))}{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))} f'(\overline{\tau}) d\overline{\tau} = \fint_{\mathbb{T}} \frac{\overline{\eta - \overline{w} + \varepsilon(f(\eta) - f(\overline{w}))}}{\eta - \overline{w} + \varepsilon(f(\eta) - f(\overline{w}))} f'(\eta) d\eta \\ &= G_{21}(\varepsilon, f)(\overline{w}). \end{aligned}$$

On the other hand the function

$$G_{22}(\varepsilon, f) = \fint_{\mathbb{T}} \frac{(\tau - w)\big(f(\overline{\tau}) - f(\overline{w})\big) - (\overline{\tau} - \overline{w})(f(\tau) - f(w))}{(\tau - w)\big((\tau - w) + \varepsilon(f(\tau) - f(w)\big)} d\tau$$

will be in the space $C^\beta(\mathbb{T})$ if the kernel

$$K(\tau, w) = \frac{(\tau - w)\big(f(\overline{\tau}) - f(\overline{w})\big) - (\overline{\tau} - \overline{w})(f(\tau) - f(w))}{(\tau - w)\big((\tau - w) + \varepsilon(f(\tau) - f(w)\big)}$$

satisfies the hypothesis of Lemma 1 for $\alpha = 0$. For $\tau \neq w$,

$$|K(\tau, w)| \leq \frac{2\|f\|_{C^{1+\alpha}}}{1 - \varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}.$$

Therefore

$$|\partial_w K(\tau, w) \leq \frac{C}{|\tau - w|},$$

where the constant C depends on ε and $\|f\|_{C^{1+\beta}(\mathbb{T})}$. To check that the function G_{22} has real coefficients one can repeat the same computations used for the function G_{21}.

Now we will verify that the function G_2 is of class C^1 from $(-\frac{1}{2},\frac{1}{2})\times\mathbb{R}\times B_1$ to Y. To do so, we will check the continuity of the partial derivatives of G_{21} and G_{22}. Simple computations prove that

$$\partial_\varepsilon G_{21}=\fint_{\mathbb{T}}\frac{f(\overline{\tau})-f(\overline{w})}{\tau-w+\varepsilon(f(\tau)-f(w))}f'(\tau)d\tau-\fint_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}+\varepsilon(f(\overline{\tau})-f(\overline{w}))}{(\tau-w+\varepsilon(f(\tau)-f(w))^2}(f(\tau)-f(w))f'(\tau)d\tau$$

and

$$\partial_\varepsilon G_{22}=-2i\fint_{\mathbb{T}}\frac{\mathrm{Im}\Big\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\Big\}}{(\tau-w)\big(\tau-w+\varepsilon(f(\tau)-f(w))\big)^2}(f(\tau)-f(w))d\tau.$$

The existence and the continuity of this partial derivative can be obtained proving that the kernels that appear in the integral operators satisfy the conditions of Lemma 1. For $h\in X$ we will compute the Gâteaux derivative in the direction h of the function G_2. For it we only need to calculate the Gâteaux derivatives of the functions G_{21} and G_{22}.

$$\begin{aligned}\partial_f G_{21}(\varepsilon,f)h(w) \quad &\equiv \quad \fint_{\mathbb{T}}\frac{\varepsilon(h(\overline{\tau})-h(\overline{w}))}{\tau-w+\varepsilon(f(\tau)-f(w))}f'(\tau)d\tau+\fint_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}+\varepsilon(f(\overline{\tau})-f(\overline{w}))}{\tau-w+\varepsilon(f(\tau)-f(w))}h'(\tau)d\tau\\ &-\fint_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}+\varepsilon(f(\overline{\tau})-f(\overline{w}))}{\big(\tau-w+\varepsilon(f(\tau)-f(w))\big)^2}\varepsilon\big(h(\tau)-h(w)\big)f'(\tau)d\tau.\end{aligned}$$

Moreover the Gâteaux derivative of the G_{22} in the direction h is given by

$$\begin{aligned}\partial_f G_{22}(\varepsilon,f)h(w) \quad &= \quad 2i\fint_{\mathbb{T}}\frac{\mathrm{Im}\big\{(\tau-w)(h(\overline{\tau})-h(\overline{w}))\big\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w))}d\tau\\ &-2i\varepsilon\fint_{\mathbb{T}}\frac{\mathrm{Im}\big\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\big\}}{(\tau-w)\big((\tau-w+\varepsilon(f(\tau)-f(w))\big)^2}(h(\tau)-h(w))d\tau.\end{aligned}$$

Again Lemma 1 applied to the kernels that appear in the Gâteaux derivatives of the functions G_{21} and G_{22} will give the existence and the continuity of the functions $\partial_f G_{21}$ and $\partial_f G_{22}$. On the other hand,

$$\partial_f G_2(0,0)(h)=\mathrm{Im}\Big\{\Big(\partial_f G_{21}(0,0)(h)-\partial_f G_{22}(0,0)(h)\Big)w\Big\}.$$

Moreover, by the residue theorem, we can compute explicitly the partial derivatives at $(0,0)$,

$$\partial_f G_{21}(0,0)(h)=\fint_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}}{\tau-w}h'(\tau)d\tau=0$$

and

$$\partial_f G_{22}(0,0)(h)=2i\fint_{\mathbb{T}}\frac{\mathrm{Im}\{(\tau-w)(h(\overline{\tau})-h(\overline{w}))\}}{(\tau-w)^2}d\tau=0.$$

Consequently $\partial_f G_2(0,0)(h)=0$. Let's now study the function

$$\begin{aligned}G_3(\varepsilon,f) \quad &= \quad -\mathrm{Im}\Big\{\Big(\fint_{\mathbb{T}}\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}(1+\varepsilon f'(\tau))d\tau\Big)w(1+\varepsilon f'(w))\Big\}\\ &= \quad -\mathrm{Im}\Big\{G_{31}(\varepsilon,f)w(1+\varepsilon f'(w)\Big\}.\end{aligned}$$

So, the regularity of the function G_3 is equivalent to the regularity of the function G_{31}. Now, this function is given by an integral operator with kernel

$$K(\tau,w)=\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}.$$

It is clear that $|K(\tau,w)|\leq C$ and moreover

$$|\partial_w K(\tau,w)|=\Big|\frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))\big(\varepsilon+\varepsilon^2 f'(w)\big)}{\big(\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d\big)^2}\Big|\leq C.$$

Since $1+\varepsilon f'(\tau)$ is in $C^\beta(\mathbb{T})$ and applying Lemma 1 to the above kernel we get that G_{31} is a function in $C^\beta(\mathbb{T})$. To prove that G_{31} has real coefficients one only has to repeat the arguments given in the case of the function G_{21}. Now, to check that that the function G_{31} is in C^1 we have to compute its partial derivatives

$$\begin{aligned}\partial_\varepsilon G_{31} &= \fint_{\mathbb{T}} \frac{f(\overline{\tau})(1+\varepsilon f'(\tau))}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}d\tau + \fint_{\mathbb{T}} \frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))f'(\tau)}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}d\tau \\ &\quad - \fint_{\mathbb{T}} \frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))(\tau+w+2\varepsilon(f(\tau)+f(w))}{\big(\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d\big)^2}(1+\varepsilon f'(\tau))d\tau.\end{aligned}$$

Easy computations, using Lemma 1, prove that these operators are continuous from $(-\frac{1}{2},\frac{1}{2})\times\mathbb{R}\times B_1$ to $C^\beta(\mathbb{T})$. Since they are functions with real coefficients we can conclude that $\partial_\varepsilon G_3$ is continuous from $(-\frac{1}{2},\frac{1}{2})\times\mathbb{R}\times B_1$ to Y. On the other hand, we can compute the Gâteaux derivative of G_{31} in a given direction $h\in X$

$$\begin{aligned}\partial_f G_{31}(\varepsilon,f)(h) &= \varepsilon \fint_{\mathbb{T}} \frac{h(\overline{\tau})(1+\varepsilon f'(\tau))}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-d}d\tau \\ &\quad + \varepsilon \fint_{\mathbb{T}} \frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))h'(\tau)}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-d}d\tau \\ &\quad - \varepsilon^2 \fint_{\mathbb{T}} \frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))(h(\tau)+h(w))}{\big(\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-d\big)^2}(1+\varepsilon f'(\tau))d\tau.\end{aligned}$$

Again it is an easy computation to verify that the integral operators defined by these partial derivatives are continuous and so we obtain that $\partial_f G_3$ is continuous from $(-\frac{1}{2},\frac{1}{2})\times\mathbb{R}\times B_1$ to Y. Moreover we have that $\partial_f G_{31}(0,0)(h)=0$, and consequently

$$\partial_f G_{31}(0,0)(h)=0.$$

Therefore (i) and (iii) are proved. Note that when $\varepsilon=0$ one should get the two point vortices. Indeed, we can easily check that

$$G(0,\Omega,0)=\operatorname{Im}\Big\{\big(-2\Omega d+\frac{1}{2d}\big)w\Big\}$$

and therefore $G(0,\Omega,0)=0$ if and only if

$$\Omega=\Omega_{sing}=\frac{1}{4d^2},$$

and so (ii) is obtained.
To prove (iv) we use that $\partial_f G(0,\Omega,0)(h)=-\operatorname{Im}\{h'\}$, therefore we can conclude that the linear mapping $\partial_f G(0,\Omega,0): X\to\widehat{Y}$ is an isomorphism. □

3.2. Lagrangian multiplier

As we have seen in Proposition 1 the linear operator $\partial_f G(0,\Omega,0)$ is an isomorphism from X to $\widehat{Y}$ and not to the space Y. However the functional G has its range in Y which contains strictly $\widehat{Y}$. The strategy will be to use Ω as a Lagrangian multiplier in order to guarantee that the range of G is contained in $\widehat{Y}$. This condition is equivalent to

$$\fint_{\mathbb{T}} F\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw-\fint_{\mathbb{T}} F\big(\Omega,\varepsilon,f(w)\big)dw=0. \tag{3.1}$$

We recall that F was defined in (2.4). Then using residue theorem we get

$$\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw=2\Omega\Big(-d+\varepsilon^3\fint_{\mathbb{T}} f(\overline{w})\overline{w}f'(w)dw\Big)$$

and

$$\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)dw=2\Omega\Big(-d\varepsilon\fint_{\mathbb{T}} wf'(w)dw+\varepsilon^3\fint_{\mathbb{T}} f(\overline{w})wf'(w)dw\Big).$$

This last identity can be written in the form

$$\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)dw=2\Omega\Big(d\varepsilon\fint_{\mathbb{T}} f(w)dw+\varepsilon^3\fint_{\mathbb{T}} f(\overline{w})wf'(w)dw\Big).$$

Consequently

$$\begin{aligned}\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw-\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)dw &= 2\Omega\bigg(-d\Big[1+\varepsilon\fint_{\mathbb{T}} f(w)dw\Big]\\ &\quad+\varepsilon^3\fint_{\mathbb{T}} f(\overline{w})f'(w)\big(\overline{w}-w\big)dw\bigg).\end{aligned}$$

Now we shall look for the contribution of F_3. First

$$F_3(\varepsilon,f(w))=-\widetilde{F_3}(w)w\big(1+\varepsilon f'(w)\big),$$

with

$$\widetilde{F_3}(\varepsilon,f(w))\equiv\fint_{\mathbb{T}}\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\big(1+\varepsilon f'(\tau)\big)d\tau.$$

We write

$$\begin{aligned}\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d} &= -\frac{\overline{\tau}}{2d}+\varepsilon\frac{f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\\ &\quad+\frac{\varepsilon}{2d}\frac{\tau+w+\varepsilon(f(\tau)+f(w))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\overline{\tau}\\ &\equiv -\frac{\overline{\tau}}{2d}+\varepsilon g_3(\varepsilon,\tau,w).\end{aligned}$$

Thus

$$\widetilde{F_3}(\varepsilon,f(w))=-\frac{1}{2d}+\varepsilon\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)d\tau.$$

Hence

$$\begin{aligned}&\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw=\frac{1}{2d}-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)\overline{w}\big(1+\varepsilon f'(w)\big)d\tau dw\\ &\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)dw=-\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)w\big(1+\varepsilon f'(w)\big)d\tau dw.\end{aligned}$$

Consequently

$$\begin{aligned}&\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw-\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)dw\\ &\qquad=\frac{1}{2d}+\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)d\tau dw.\end{aligned}$$

On the other hand using residue theorem we get

$$\begin{aligned}F_2(\varepsilon,f(w)) &= \varepsilon\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}f'(\tau)d\tau\, w\big(1+\varepsilon f'(w)\big)\\ &\quad+\varepsilon\fint_{\mathbb{T}}\frac{\big(\overline{A}B-A\overline{B}\big)B}{A^2(A+\varepsilon B)}d\tau\, w\big(1+\varepsilon f'(w)\big)\\ &\equiv \varepsilon g_2(\varepsilon,w)w\big(1+\varepsilon f'(w)\big).\end{aligned}$$

Thus

$$\fint_{\mathbb{T}} F_2\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw-\fint_{\mathbb{T}} F_2\big(\Omega,\varepsilon,f(w)\big)dw=\varepsilon\fint_{\mathbb{T}} g_2(\varepsilon,w)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)dw.$$

The equation (3.1) becomes

$$\begin{aligned}
2\Omega\Big(d\Big[1+\varepsilon\fint_{\mathbb{T}} f(w)dw\Big] \quad &- \quad \varepsilon^3\fint_{\mathbb{T}} f(\overline{w})f'(w)\big(\overline{w}-w\big)dw\Big)\\
&= \quad \frac{1}{2d}+\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau\\
&\quad\ +\varepsilon\fint_{\mathbb{T}} g_2(\varepsilon,w)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)dw\\
&\quad\ +\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)(w-\overline{w})\big(1+\varepsilon f'(w)\big)d\tau dw\\
&\equiv \quad \frac{1}{2d}+\frac{\varepsilon}{2d}T_1(\varepsilon,f)
\end{aligned}$$

which can be written in the form

$$\begin{aligned}
\Omega \quad &= \quad \Omega(\varepsilon,f)\\
&= \quad \frac{1}{4d^2}\frac{1+\varepsilon T_1(\varepsilon,f)}{1-\varepsilon T_2(\varepsilon,f)}\\
&= \quad \Omega_{sing}+\frac{\varepsilon}{4d^2}\frac{T_1(\varepsilon,f)+T_2(\varepsilon,f)}{1-\varepsilon T_2(\varepsilon,f)} \qquad (3.2)
\end{aligned}$$

with

$$T_2(\varepsilon,f)=-\fint_{\mathbb{T}} f(w)dw+\frac{\varepsilon^2}{d}\fint_{\mathbb{T}} f(\overline{w})f'(w)\big(\overline{w}-w\big)dw.$$

Now we intend to discuss the regularity of Ω.

Proposition 2. *The function $\Omega:(\frac{1}{2},\frac{1}{2})\times B_1\longrightarrow\mathbb{R}$ defined in (3.2) is a C^1 function.*

Proof. It is enough to check that the functions $T_1(\varepsilon,f)$ and $T_2(\varepsilon,f)$ are C^1 functions and moreover $|T_2(\varepsilon,f)|<2$. Since f has real coefficients it is clear that $T_2(\varepsilon,f)\in\mathbb{R}$ and

$$|T_2(\varepsilon,f)|\le\|f\|_{C^{1+\beta}(\mathbb{T})}+\frac{\varepsilon^2}{d}2\|f\|^2_{C^{1+\beta}(\mathbb{T})}<2.$$

On the other hand, T_2 is polynomial in ε, f and f' and so its derivatives. Thus, we can conclude that T_2 is a C^1 function from $(\frac{1}{2},\frac{1}{2})\times B_1$ to $\mathbb{R}$. Let's take now the functional

$$\begin{aligned}
T_1(\varepsilon,f) \quad &= \quad \fint_{\mathbb{T}} f(\tau)d\tau+2d\fint_{\mathbb{T}} g_2(\varepsilon,w)(\overline{w}-w)(1+\varepsilon f'(w))dw\\
&\quad +2d\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)(1+\varepsilon f'(\tau))(w-\overline{w})(1+\varepsilon f'(\tau))d\tau dw,
\end{aligned}$$

where

$$g_2(\varepsilon,f)=\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}f'(\tau)d\tau+\fint_{\mathbb{T}}\frac{(\overline{A}B-A\overline{B})B}{A^2(A+\varepsilon B)}d\tau,$$

with $A=\tau-w, B=f(\tau)-f(w)$ and

$$g_3(\varepsilon,f)=\frac{f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}+2d\frac{\tau+w+\varepsilon(f(\tau)+f(w))}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}\overline{\tau}.$$

Since $|\varepsilon|<\frac{1}{2}$ and $\|f\|_{C^{1+\beta}}<1$ we get that g_3 is a bounded function. Moreover

$$\begin{aligned}
|g_2(\varepsilon,f)(w)| \quad &\le \quad 2\int_{\mathbb{T}}\Big|\frac{\mathrm{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))}f'(\tau)\Big||d\tau|\\
&\quad +2\int_{\mathbb{T}}\Big|\frac{\mathrm{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}(f(\tau)-f(w))}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))}\Big||d\tau| \ \le\ C,
\end{aligned}$$

where in the last inequality we use again that $|\varepsilon|<\frac{1}{2}$ and $\|f\|_{C^{1+\beta}}<1$. To prove that T_1 is a C^1 function it is enough to check that the partial derivatives of $g_2(\varepsilon,f)$ and $g_3(\varepsilon,f)$ are continuous functions on$(-\frac{1}{2},\frac{1}{2})\times B_1$. Observe that,

$$
\begin{aligned}
\partial_\varepsilon g_2(\varepsilon,f) = & - 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(f(\tau)-f(w))f'(\tau)d\tau\\
& - 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(f(\tau)-f(w))^2d\tau.
\end{aligned}
$$

It is easy to verify that the kernels involved in the above integral operators satisfy the conditions of Lemma 1 and so we can conclude that $\partial_\varepsilon g_2(\varepsilon,f)$ is a continuous function from $(-\frac{1}{2},\frac{1}{2})\times B_1$ to $\mathbb{R}$. For any direction $h\in X$ straightforward computations yield

$$
\begin{aligned}
\partial_f g_2(\varepsilon,f)(h) \;=\; & 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\tau-w)(h(\overline{\tau})-h(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))}f'(\tau)d\tau\\
& + 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))}h'(\tau)d\tau\\
& - 2i\varepsilon\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(h(\tau)-h(w))f'(\tau)d\tau\\
& + 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\overline{\tau}-\overline{w})(h(\tau)-h(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))}(f(\tau)-f(w))d\tau\\
& + 2i\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))}(h(\tau)-h(w))d\tau.\\
& - 2i\varepsilon\fint_{\mathbb{T}} \frac{\mathrm{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(f(\tau)-f(w))(h(\tau)-h(w))d\tau.
\end{aligned}
$$

Again the kernels involved in the integral operators satisfy the conditions in Lemma 1 and so $\partial_f g_2(\varepsilon,f)(h)$ defines a continuous function from $(-\frac{1}{2},\frac{1}{2})\times B_1$ to $\mathbb{R}$. Reproducing similar computations one can prove that $g_3(\varepsilon,f)$ is a C^1 function from $(-\frac{1}{2},\frac{1}{2})\times B_1$ to $\mathbb{R}$.

$\square$

3.3. Existence of the pairs

In this section we will finish the proof of the existence of corotating vortex pairs and show the convexity of each single vortex forming the vortex pair. Recall that the equation of the V-states is given by

$$
\widehat{G}(\varepsilon,f(w)) \quad\equiv\quad \mathrm{Im}\Big\{F(\Omega(\varepsilon,f),\varepsilon,f(w))\Big\}=0,\quad \forall w\in\mathbb{T}.
$$

Our goal is to prove the following result.

Proposition 3. *The following holds true.*

(i) *The linear operator $\partial_f\widehat{G}(0,0):X\to\widehat{Y}$ is an isomorphism and*

$$
\partial_f\widehat{G}(0,0)h(w)=-\mathrm{Im}\{h'(w)\}.
$$

(ii) *There exists $\varepsilon_0>0$ such that the set*

$$
\Big\{(\varepsilon,f)\in[-\varepsilon_0,\varepsilon_0]\times B_1,\ s.t.\quad \widehat{G}(\varepsilon,f)=0\Big\}
$$

is parametrized by one-dimensional curve $\varepsilon\in[-\varepsilon_0,\varepsilon_0]\mapsto(\varepsilon,f_\varepsilon)$ and

$$
\forall\varepsilon\in[-\varepsilon_0,\varepsilon_0]\backslash\{0\},\ f_\varepsilon\neq 0.
$$

(iii) *If (ε,f) is a solution then $(-\varepsilon,\tilde{f})$ is also a solution, where*

$$
\forall w\in\mathbb{T},\ \tilde{f}(w)=f(-w)
$$

(iv) *For all $\varepsilon\in[-\varepsilon_0,\varepsilon_0]\backslash\{0\}$, the domain D_1^ε is strictly convex.*

Proof. **(i)** From the composition rule

$$\partial_f \widehat{G}(0,0)h(w) = \partial_\Omega G(0,\Omega_{sing},0)\,\partial_f \Omega(0,0)h + \partial_f G(0,\Omega_{sing},0)h(w).$$

From the formula of $\Omega(\varepsilon, f)$ in Proposition 2 we deduce that

$$\begin{aligned}\partial_f \Omega(0,0) &= \frac{d}{dt}\Omega(0, th(w))_{|t=0}\\ &= 0\end{aligned}$$

and therefore

$$\partial_f \widehat{G}(0,0)h(w) = \partial_f G(0,\Omega_{sing},0)h(w).$$

Combining this identity with Proposition 1 we obtain the desired result.
(ii) As we have seen before $\widehat{G} :]-\frac{1}{2},\frac{1}{2}[\times B_1 \to \widehat{Y}$ is C^1 and $\partial_f \widehat{G}(0,0) : X \to \widehat{Y}$ is an isomorphism. Thus we can apply the implicit function theorem. More precisely, there exist $\varepsilon_0 > 0$ and a C^1 function $f : (-\varepsilon_0, \varepsilon_0) \to B_1$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the function f_ε satisfies

$$\mathrm{Im}\Big\{F(\Omega(\varepsilon, f_\varepsilon), \varepsilon, f_\varepsilon(w))\Big\} = 0, \quad \forall w \in \mathbb{T},$$

and so we can assert that f defines a rotating vortex pair. It remains to check that $f_\varepsilon \neq 0$ for $\varepsilon \neq 0$. To this end, we will prove that for any ε small enough and any Ω we can not get a vortex pair with $f = 0$. So, it means that

$$G(\varepsilon, \Omega, 0) \neq 0.$$

It is easy to check from (2.4) that

$$F_1(\varepsilon, \Omega, 0) = 2\Omega\big(\varepsilon - d\,w\big) \quad \text{and} \quad F_2(\varepsilon, 0) = 0.$$

However to compute F_3 we proceed by Taylor expansion as follows,

$$\begin{aligned}F_3(\varepsilon, 0) &= -w \fint_{\mathbb{T}} \frac{\overline{\tau}}{\varepsilon(\tau + w) - 2d} d\tau\\ &= w \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{(2d)^{n+1}} \fint_{\mathbb{T}} \overline{\tau}(\tau + w)^n\, d\tau\\ &= \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{(2d)^{n+1}} w^{n+1},\end{aligned}$$

which gives in turn

$$F_3(\varepsilon, 0) = \frac{w}{2d - \varepsilon w}\cdot \tag{3.3}$$

Consequently

$$G(\varepsilon, \Omega, 0) = \mathrm{Im}\Big\{-2d\Omega w + \frac{w}{2d - \varepsilon w}\Big\}.$$

and this quantity is not zero if $\varepsilon \neq 0$ is small enough.
(iii) Using the definition of $\tilde{f}$ one can check that $T_i(-\varepsilon, \tilde{f}) = -T_i(\varepsilon, f)$, for $i = 1, 2$ and so by (3.2) we obtain that

$$\Omega(\varepsilon, f) = \Omega(-\varepsilon, \tilde{f}).$$

Taking the decomposition of $F = F_1 + F_2 + F_3$ given in (2.4) we only need to check that $F_i(\varepsilon, \Omega, f)(-w) = -F_i(-\varepsilon, \Omega, \tilde{f})(w)$, for $i = 1, 2, 3$. Since $\tilde{f}'(w) = -f'(-w)$ we have

$$\begin{aligned}F_1(-\varepsilon, \Omega, \tilde{f})(w) &= 2\Omega\big(-\varepsilon\overline{w} + \varepsilon^2 \tilde{f}(\overline{w}) - d)w(1 - \varepsilon \tilde{f}'(w)\big) - \tilde{f}'(w)\\ &= -\big[2\Omega\big(\varepsilon(-\overline{w}) + \varepsilon^2 f(-\overline{w}) - d)(-w)(1 + \varepsilon f'(-w)\big) - f'(-w)\big]\\ &= -F_1(\varepsilon, \Omega, f)(-w).\end{aligned}$$

Straightforward computations will lead to the same properties for the functions F_2 and F_3. Consequently,

$$F(\varepsilon, \Omega, f)(w) - F(-\varepsilon, \Omega, \tilde{f})(-w)$$

and therefore $(-\varepsilon, \tilde{f})$ defines a curve of solutions for $0 < \varepsilon < \varepsilon_0$.

(iv) First we shall make the following comment. As it was mentioned in Remark 1 one can reproduce the preceding proofs when we replace β by $n + \beta$ with $n \in \mathbb{N}$. Therefore the implicit function theorem gives that the function f_ε belongs to $C^{n+1+\beta}$ for any fixed n. Of course, the size of ε_0 is not uniform with respect to n and it shrinks to zero as n grows to infinity. Now to prove the convexity of the domain D_1^ε

we shall reproduce the same arguments of [12]. Recall that the outside conformal mapping associated to this domain is given by

$$\phi(w) = \varepsilon w + \varepsilon^2 f_\varepsilon(w)$$

and the curvature can be expressed by the formula

$$\kappa(\theta) = \frac{1}{|\phi'(w)|} \operatorname{Re}\Big(1 + w\frac{\phi''(w)}{\phi'(w)}\Big).$$

It is plain that

$$1 + w\frac{\phi''(w)}{\phi'(w)} = 1 + \varepsilon w \frac{f''(w)}{1+\varepsilon f'(w)}$$

and so

$$\operatorname{Re}\Big(1 + w\frac{\phi''(w)}{\phi'(w)}\Big) \geq 1 - |\varepsilon|\frac{|f''(w)|}{1-|\varepsilon||f'(w)|} \geq 1 - \frac{|\varepsilon|}{1-|\varepsilon|},$$

which is non-negative if $|\varepsilon| < 1/2$. Thus the curvature is strictly positive and therefore the domain is strictly convex. □

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