Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Roscoff, 1–5 juin 2015

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Abstract

A classical result in kinetic theory establishes that if $f(x,v)$ and $v \cdot \nabla_x f(x,v)$ both belong to $L^2\left(\mathbb{R}^n_x \times \mathbb{R}^n_v\right)$, then $\int_K f\,dv \in H^{\frac{1}{2}}\left(\mathbb{R}^n_x\right)$, for any compact set $K \subset \mathbb{R}^n_v$. Such regularity statements are known as velocity averaging lemmas and have important implications in the analysis of kinetic equations.

It was asked in [2] whether other settings of velocity averaging could produce a similar maximal gain of regularity of half a derivative. This question, motivated by an earlier work of Pierre-Emmanuel Jabin and Luis Vega [17] on the subject, turns out to be surprisingly rich and difficult, and it is, for the moment, far from being fully understood.

In this article, after recalling some classical results in the field, we survey the recent developments from [2], where new settings of velocity averaging lemmas were investigated. We also formulate a few conjectures, mainly derived from a dimensional analysis and by analogy with known results, thus delimiting the possibilities for other new settings of velocity averaging.

Velocity averaging lemmas concern the regularity theory of solutions to the kinetic transport equation

$$\left(\partial_t + v \cdot \nabla_x\right) f(t,x,v) = g(t,x,v), \tag{0.1}$$

where $(t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, or its stationary counterpart

$$v \cdot \nabla_x f(x,v) = g(x,v), \tag{0.2}$$

where $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, with $n \geq 1$.

Variants of the above equations are also relevant. Indeed, different spatial and velocity domains, as well as non-linear velocity fields (consider the relativistic case), are sometimes studied. Nevertheless, for the sake of simplicity, we will mainly focus on the Euclidean stationary setting (0.2), which, we believe, captures the essential features of kinetic transport (at least as far as velocity averaging is concerned). We will nevertheless make brief references to the non-stationary case (0.1) in Section 2 below without expanding on the subject.

After describing a modern viewpoint on classical velocity averaging in Sections 1, 2 and 3, we intend to survey, in the remainder of this text, recent theorems and conjectures on the subject. Most results presented here are taken directly from [2] and we will systematically sketch proofs of the asserted results. Nevertheless, we refer the reader to [2] and the references therein for further details and complete justifications.

1. Classical velocity averaging, the Hilbertian case

The classical Hilbertian case of velocity averaging is contained in the following result. It is the starting point of the regularity theory of kinetic transport equations and has been established first in [12]. Note, however, that such regularity results had already been suggested in weaker forms in [1, 13].

Theorem 1.1 ([12]). *Let $f, g \in L^2\left(\mathbb{R}^n_x \times \mathbb{R}^n_v\right)$ satisfy the transport relation* (0.2). *Then, the velocity averages of f enjoy the following regularization:*

$$\int_{\mathbb{R}^n} f(x, v)\varphi(v)dv \in H^{\frac{1}{2}}\left(\mathbb{R}^n_x\right),$$

for any given $\varphi \in L^\infty_c\left(\mathbb{R}^n\right)$ (i.e. measurable functions bounded almost everywhere with compact support).

Sketch of proof. This classical and essential result is based on a simple microlocal decomposition. More precisely, denoting the Fourier transform in the space variable only by

$$\hat{f}(\eta, v) = \mathcal{F}_x f(\eta, v) = \int_{\mathbb{R}^n} e^{-i\eta \cdot x} f(x, v)dx,$$

it holds that

$$iv \cdot \eta \hat{f}(\eta, v) = \hat{g}(\eta, v).$$

Then, introducing some cutoff function $\rho \in C^\infty_c(\mathbb{R})$ such that $\rho(0) = 1$, we may decompose

$$\hat{f}(\eta, v) = \rho\left(v \cdot \eta\right)\hat{f}(\eta, v) + \tau\left(v \cdot \eta\right)\hat{g}(\eta, v), \tag{1.1}$$

where $\tau\left(v \cdot \eta\right) = \frac{1 - \rho(v \cdot \eta)}{iv \cdot \eta}$. Loosely speaking, we are using here some partial ellipticity of the transport operator, for we invert $v \cdot \eta$ wherever it is non-vanishing.

Thus, the result is a direct consequence of the boundedness of the operator $T : L^2_{x,v} \to \dot{H}^{\frac{1}{2}}_x$ defined by

$$Tf(x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}_x \rho\left(v \cdot \eta\right)\mathcal{F}_x f(x, v)\varphi(v)dv, \tag{1.2}$$

and of a similar operator where ρ is replaced by τ. In fact, this continuity is easily obtained, using the Plancherel theorem, from the estimate

$$\|Tf\|_{\dot{H}^{\frac{1}{2}}_x} = \frac{1}{(2\pi)^{\frac{n}{2}}}\left\||\eta|^{\frac{1}{2}}\widehat{Tf}\right\|_{L^2_\eta} = \frac{1}{(2\pi)^{\frac{n}{2}}}\left\||\eta|^{\frac{1}{2}}\int_{\mathbb{R}^n} \rho\left(v \cdot \eta\right)\hat{f}\varphi dv\right\|_{L^2_\eta}$$

$$\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\left\||\eta|^{\frac{1}{2}}\left\|\hat{f}\right\|_{L^2_v}\|\rho\left(v \cdot \eta\right)\varphi\|_{L^2_v}\right\|_{L^2_\eta},$$

upon noticing that

$$\sup_{\eta \in \mathbb{R}^n}|\eta|\int_{\mathbb{R}^n}\rho\left(v \cdot \eta\right)^2\varphi(v)^2 dv \leq C_\alpha\left\|(1 + |v|)^\alpha\varphi(v)\right\|^2_{L^\infty(\mathbb{R}^n)}\|\rho\|^2_{L^2(\mathbb{R})}, \tag{1.3}$$

where $C_\alpha > 0$ only depends on $\alpha > \frac{n-1}{2}$.

A similar estimate holds with the cutoff τ in place of ρ, which concludes the justification of the theorem. $\qquad\square$

Remark. Note that, by duality, the boundedness of the operator T established above is equivalent to the continuity of the adjoint operator $T^* : L^2\left(\mathbb{R}^n_x\right) \to \dot{H}^{\frac{1}{2}}\left(\mathbb{R}^n_x; L^2\left(\mathbb{R}^n_v\right)\right)$ defined by

$$T^*h(x, v) = \mathcal{F}^{-1}_x \rho\left(v \cdot \eta\right)\mathcal{F}_x h(x)\varphi(v). \tag{1.4}$$

2. Classical velocity averaging, the Hilbertian non-stationary case

Applying the method of proof of Theorem 1.1 to the non-stationary kinetic transport equation (0.1) (employing this time a Fourier transform in both t and x so that (0.1) is equivalent to $i\left(\tau + v \cdot \eta\right)\hat{f}(\tau, \eta, v) = \hat{g}(\tau, \eta, v)$ and then introducing a cutoff $\rho\left(\tau + v \cdot \eta\right)$) yields the following similar result, which establishes regularization in both time and space variables.

Theorem 2.1 ([12]). *Let $f, g \in L^2\left(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v\right)$ satisfy the transport relation* (0.2). *Then, the velocity averages of f enjoy the following regularization:*

$$\int_{\mathbb{R}^n} f(t, x, v)\varphi(v)dv \in H^{\frac{1}{2}}\left(\mathbb{R}_t \times \mathbb{R}^n_x\right),$$

for any given $\varphi \in L^\infty_c\left(\mathbb{R}^n\right)$.

Recall now that the solution $f(t, x, v)$ of (0.2) may be expressed according to Duhamel's representation formula:

$$
\begin{aligned}
f(t, x, v) &= f_0(x - tv, v) + \int_0^t g(s, x - (t - s)v, v)ds \\
&= \mathcal{F}_x^{-1} e^{-itv \cdot \eta} \mathcal{F}_x f_0(x, v) + \int_0^t \mathcal{F}_x^{-1} e^{-i(t-s)v \cdot \eta} \mathcal{F}_x g(s, x, v)ds \qquad (2.1) \\
&= e^{-tv \cdot \nabla_x} f_0(x, v) + \int_0^t e^{-(t-s)v \cdot \nabla_x} g(s, x, v)ds,
\end{aligned}
$$

where $f_0(x, v)$ is the initial data.

The following result shows that it is also possible to formulate the regularity of velocity averages with respect to the initial data and the source term, thus establishing the average smoothing effect of the kinetic flow $e^{-tv \cdot \nabla_x}$.

Theorem 2.2. *Let $f_0 \in L^2\left(\mathbb{R}_x^n \times \mathbb{R}_v^n\right)$ and $g \in L^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_x^n \times \mathbb{R}_v^n\right)\right)$. Then, the velocity averages of f, defined by (2.1), enjoy the following regularization:*

$$
\int_{\mathbb{R}^n} f(t, x, v)\varphi(v)dv \in L^2\left(\mathbb{R}_t; \dot{H}^{\frac{1}{2}}\left(\mathbb{R}_x^n\right)\right),
$$

for any given $\varphi \in L_c^\infty\left(\mathbb{R}^n\right)$.

Sketch of proof. For any $h \in L^2\left(\mathbb{R}_t \times \mathbb{R}_x^n\right)$, we obtain first, using Plancherel's theorem and estimate (1.3), that

$$
\begin{aligned}
\left\| \int_{\mathbb{R}} h(t, x + tv)\varphi(v)dt \right\|_{\dot{H}_x^{\frac{1}{2}} L_v^2} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\| |\eta|^{\frac{1}{2}} \int_{\mathbb{R}} e^{itv \cdot \eta} \mathcal{F}_x h(t, \eta)\varphi(v)dt \right\|_{L_{\eta, v}^2} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\| |\eta|^{\frac{1}{2}} \mathcal{F}_{t, x} h(-v \cdot \eta, \eta)\varphi(v) \right\|_{L_{\eta, v}^2} \\
&\leq \frac{C_\alpha \left\| (1 + |v|)^\alpha \varphi \right\|_{L^\infty}}{(2\pi)^{\frac{n}{2}}} \left\| \mathcal{F}_{t, x} h(\tau, \eta) \right\|_{L_{\tau, \eta}^2} \\
&= C_\alpha (2\pi)^{\frac{1}{2}} \left\| (1 + |v|)^\alpha \varphi \right\|_{L^\infty} \left\| h(t, x) \right\|_{L_{t, x}^2},
\end{aligned}
$$

where $C_\alpha > 0$ only depends on $\alpha > \frac{n-1}{2}$.

It then follows, by duality, that

$$
\left\| \int_{\mathbb{R}^n} f_0(x - tv, v)\varphi(v)dv \right\|_{L_t^2 \dot{H}_x^{\frac{1}{2}}} \leq C \left\| f_0(x, v) \right\|_{L_{x, v}^2},
$$

and, consequently, that

$$
\begin{aligned}
\left\| \int_{\mathbb{R}^n} \int_0^t g(s, x - (t - s)v, v)ds \varphi(v)dv \right\|_{L_t^2 \dot{H}_x^{\frac{1}{2}}} &\leq \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^n} g(s, x - tv, v)\varphi(v)dv \right\|_{L_t^2 \dot{H}_x^{\frac{1}{2}}} ds \\
&\leq C \left\| g(x, v) \right\|_{L_t^1 L_{x, v}^2},
\end{aligned}
$$

which, in view of Duhamel's representation formula (2.1), concludes the justification of the theorem. $\square$

Remark. The preceding formulation of velocity averages regularity with respect to the kinetic transport semi-group $e^{-tv \cdot \nabla_x}$ and its proof are strikingly reminiscent of the local smoothing effect for the Schrödinger equation established in [8] (see Theorems 1.1, 2.1 and 3.1 therein). Note, however, that the above result is global in time and space, whereas the smoothing effect for the Schrödinger equation is only local in t and x. It is necessary here to tame large velocites, though.

3. Classical velocity averaging, the general case

Interpolation theory (see [4]) provides us with powerful tools to conveniently extend the Hilbertian case of Theorem 1.1 to a general setting in $L_{x,v}^p$, with $1 \leq p \leq \infty$. The next theorem provides the regularity of velocity averages in this context.

Theorem 3.1 ([2, 5, 9, 10, 12])**.** *Let $f, g \in L^p(\mathbb{R}_x^n \times \mathbb{R}_v^n)$, with $1 < p < \infty$, satisfy the transport relation (0.2). Then, the velocity averages of f enjoy the following regularization:*

$$\int_{\mathbb{R}^n} f(x,v)\varphi(v)dv \in W^{s,p}(\mathbb{R}_x^n),$$

where $s = \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, for any given $\varphi \in L_c^\infty(\mathbb{R}^n)$.

Remark. Under the same hypotheses, at least for velocity weights $\varphi \in C_c^\infty(\mathbb{R}^n)$, Theorem 3.1 has been first established in [12] for values $s < \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$. It was then improved in [10] to show that velocity averages belong to the Besov spaces $B_{p,2}^{\frac{1}{p'}}(\mathbb{R}^n)$, when $1 < p \leq 2$, and $B_{p,p}^{\frac{1}{p}}(\mathbb{R}^n)$, when $2 \leq p < \infty$. For values $1 < p \leq 2$ only, it was eventually shown in [5], through a convoluted analysis in Hardy spaces on product domains, that Theorem 3.1 holds as stated above with velocity averages lying in the Sobolev space $W^{\frac{1}{p'},p}(\mathbb{R}^n)$. It was later proven in [9] with refined interpolation methods that, for values $1 < p < 2$, velocity averages even belong to $B_{p,p}^{\frac{1}{p'}}(\mathbb{R}^n)$, which is a proper subset of $W^{\frac{1}{p'},p}(\mathbb{R}^n)$. The question of whether velocity averages belong to $W^{\frac{1}{p},p}(\mathbb{R}^n)$, when $2 < p < \infty$, was finally settled by the affirmative as a simple by-product of the careful analysis conducted in [2].

Sketch of proof. The justification of this result merely consists in interpolating the Hilbertian case from Theorem 1.1 with the degenerate $L_{x,v}^1$ and $L_{x,v}^\infty$ cases (corresponding to $p = 1$ and $p = \infty$ in the statement of the theorem). This interpolation procedure is carried out on the operator T and its adjoint T^* defined by (1.2) and (1.4), respectively.

To this end, we use the elementary yet crucial trick of expressing the cutoff defining T and T^* as $\rho(v \cdot \eta) = \int_{\mathbb{R}} e^{-itv\cdot\eta} \tilde{\rho}(t)dt$, where $\tilde{\rho}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itr}\rho(r)dr$ is the inverse Fourier transform of ρ, so that T and T^* can be recast as

$$Tf(x) = \int_{\mathbb{R}^n} \mathcal{F}_x^{-1} \rho(v \cdot \eta) \mathcal{F}_x f(x,v)\varphi(v)dv = \int_{\mathbb{R}^n}\int_{\mathbb{R}} f(x - tv, v)\tilde{\rho}(t)\varphi(v)dtdv,$$

$$T^*h(x,v) = \mathcal{F}_x^{-1} \rho(v \cdot \eta) \mathcal{F}_x h(x)\varphi(v) = \int_{\mathbb{R}} h(x - tv)\tilde{\rho}(t)\varphi(v)dt. \tag{3.1}$$

Using these representations, it is readily seen that the operators $T : L_{x,v}^p \to L_x^p$ and $T^* : L_x^p \to L_{x,v}^p$ are bounded, for any $1 \leq p \leq \infty$.

Therefore, interpolating (we use here both real and complex interpolation procedures; see [2] for full details) the bounds $T : L_{x,v}^2 \to H_x^{\frac{1}{2}}$ and $T^* : H_x^{-\frac{1}{2}} \to L_{x,v}^2$ (from the proof of Theorem 1.1) with $T : L_{x,v}^1 \to L_x^1$ and $T^* : L_x^1 \to L_{x,v}^1$, respectively, we deduce that $T : L_{x,v}^p \to W_x^{\frac{1}{p'},p}$ and $T^* : W_x^{-\frac{1}{p'},p} \to L_{x,v}^p$ are bounded, for any $1 \leq p \leq 2$.

Finally, by duality, we conclude that $T : L_{x,v}^p \to W_x^{s,p}$, where $s = \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, is bounded, for all $1 \leq p \leq \infty$. Hence, considering decomposition (1.1) and since a similar estimate holds for T where the cutoff $\rho(r)$ is replaced by $\tau(r) = \frac{1-\rho(r)}{ir}$, the theorem is proven. $\qquad\square$

4. Maximal regularity and dimensional analysis

We address now the question of the optimality of Theorem 3.1. Under the hypotheses stated therein, it was shown in [18], through a straightforward dimensional analysis, which we reproduce and generalize below, that the regularity index $s = 1 - \frac{1}{p}$ is optimal when $1 \leq p \leq 2$. As for the case $p > 2$, it was also argued in [18] that the regularity of velocity averages cannot be improved beyond the value $s = \frac{1}{p}$, but this optimality argument remains incomplete for it requires the use of a larger class of velocity weights $\varphi(v)$ with unbounded support. In fact, it turns out that, in general, the

value $s = \frac{1}{p}$ is not optimal in the range $2 < p < \infty$ for compactly supported velocity weights, for it is possible to largely improve this regularity index beyond the value $s = \frac{1}{p}$ in dimension $n = 1$, as stated in the one-dimensional Theorem 5.1 from Section 5 below. The question of the optimality of the value $s = \frac{1}{p}$, when $p > 2$, in higher dimensions $n \geq 2$, for compactly supported velocity weights $\varphi(v)$, was finally settled in [9, Theorem 1.3] where a convoluted counterexample shows the necessity of the constraint $s \leq \frac{1}{p}$ whenever $p > 2$ and $n \geq 2$.

Thus, on the whole, the maximal gain of regularity in the $L^p(\mathbb{R}_x^n \times \mathbb{R}_v^n)$ setting, with $n \geq 2$, clearly happens for the value $p = 2$ where half a derivative is gained by averaging in velocity. This naturally raises the question addressed in [2]:

Are there other settings where velocity averages gain half a derivative in regularity?

In fact, there already exist generalizations of the aforementioned theorems where velocity averages are regularized by half a derivative or more. For instance, under the hypotheses of Theorem 1.1, if one further assumes that $(-\Delta_v)^{\frac{\beta}{2}} f \in L^2_{x,v}$, for some $\beta > 0$, then it is possible to show, at least for velocity weights $\varphi \in C_c^\infty(\mathbb{R}^n)$, that the velocity averages of f belong to $H_x^{\frac{1+2\beta}{2+2\beta}}$ (see [6, Corollary 1.2], for instance). However, such statements result from the combination of velocity averaging methods with the hypoelliptic phenomenon in kinetic transport equations (as discovered in [6]; see also [3]), which essentially allows to transfer regularity from v to x. Hypoellipticity is well understood (at least in the context of velocity averaging), we are therefore going to avoid it altogether by restricting our study to a setting where the densities f and g in (0.2) belong to Lebesgue spaces. In order to further contain our discussion, we will always assume that f and g belong to the same space (of course, the case where f and g belong to distinct spaces is also interesting and should be investigated). In this context, we may rephrase the above interrogation as follows:

For what values of $1 \leq p, q, r \leq \infty$ does it hold that $f, v \cdot \nabla_x f \in L^p(\mathbb{R}_x^n; L^q(\mathbb{R}_v^n))$ implies $\int_{\mathbb{R}^n} f\varphi dv \in W^{s,r}(\mathbb{R}_x^n)$, for every $0 \leq s < \frac{1}{2}$, where $\varphi \in L_c^\infty(\mathbb{R}^n)$ is given?

Note that, by the closed graph theorem, this can only happen if, for all $0 \leq s < \frac{1}{2}$, there is a constant $C > 0$, which only depends on $1 \leq p, q, r \leq \infty$, $0 \leq s < \frac{1}{2}$ and $\varphi \in L_c^\infty(\mathbb{R}^n)$, such that one has an estimate

$$\left\| \int_{\mathbb{R}^n} f\varphi dv \right\|_{W_x^{s,r}} \leq C\left(\|f\|_{L_x^p L_v^q} + \|v \cdot \nabla_x f\|_{L_x^p L_v^q} \right). \tag{4.1}$$

At this stage, it is very natural to conduct a dimensional analysis of the above inequality, which will restrict the range of parameters $1 \leq p, q, r \leq \infty$ and $0 \leq s < \frac{1}{2}$ for which such an estimate may hold. We perform now this dimensional analysis by generalizing the procedure from [18]. To this end, let $d \in \{1, \ldots, n\}$ and denote $x = (x', x'') = (x_1, \ldots, x_d, x_{d+1}, \ldots, x_n)$ and $v = (v', v'') = (v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$. For some given $\psi(x, v) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varepsilon > 0$ and $0 < \delta < 1$, we define

$$\tilde{f}(x, v) = \psi\left(\frac{x'}{\varepsilon}, x'', \frac{v'}{\delta}, v'' \right).$$

We suppose now that (4.1) holds for some $\varphi(v) \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi(0, v'') \not\equiv 0$, which we test with the function $\tilde{f}$. This yields

$$\varepsilon^{\frac{d}{r} - s} \delta^d \left\| \int_{\mathbb{R}^n} \psi(x, v) \varphi(\delta v', v'') \, dv \right\|_{\dot{W}_x^{s,r}}$$

$$\leq \left\| \int_{\mathbb{R}^n} \tilde{f}\varphi dv \right\|_{W_x^{s,r}} \leq C\left(\|\tilde{f}\|_{L_x^p L_v^q} + \|v \cdot \nabla_x \tilde{f}\|_{L_x^p L_v^q} \right)$$

$$\leq C\left(\varepsilon^{\frac{d}{p}} \delta^{\frac{d}{q}} \|\psi\|_{L_x^p L_v^q} + \varepsilon^{\frac{d}{p}-1} \delta^{\frac{d}{q}+1} \|v' \cdot \nabla_{x'} \psi\|_{L_x^p L_v^q} + \varepsilon^{\frac{d}{p}} \delta^{\frac{d}{q}} \|v'' \cdot \nabla_{x''} \psi\|_{L_x^p L_v^q} \right).$$

Therefore, we conclude that (4.1) can possibly hold only if, for some $C > 0$,

$$\varepsilon^{d\left(\frac{1}{r} - \frac{1}{p}\right)+1-s} \delta^{d\left(1-\frac{1}{q}\right)} \leq C(\varepsilon + \delta), \tag{4.2}$$

for every $0 \leq s < \frac{1}{2}$, which places some restrictions on the parameters $1 \leq p, q, r \leq \infty$.

Indeed, setting now $d = 1$ and $\delta = \varepsilon$, we find, letting $\varepsilon \to 0$ in (4.2), that necessarily $\frac{1}{r} - \frac{1}{p} + 1 - s - \frac{1}{q} \geq 0$. Further letting $s \to \frac{1}{2}$, we deduce the first constraint

$$\frac{1}{p} - \frac{1}{r} \leq \frac{1}{2} - \frac{1}{q}. \tag{4.3}$$

Next, setting $\delta = 1$ in (4.2) and the letting $\varepsilon \to \infty$, we obtain $d \left(\frac{1}{r} - \frac{1}{p} \right) - s \leq 0$. Further letting $s \to 0$, we infer the second constraint

$$\frac{1}{p} - \frac{1}{r} \geq 0. \tag{4.4}$$

Finally, setting $d = n$ and $\delta = 1$ in (4.2) and then letting $\varepsilon \to 0$, we conclude that necessarily $n \left(\frac{1}{r} - \frac{1}{p} \right) + 1 - s \geq 0$, which, as $s \to \frac{1}{2}$, implies the third constraint

$$\frac{1}{p} - \frac{1}{r} \leq \frac{1}{2n}. \tag{4.5}$$

Since the regularity results discussed here are all local in v, the estimate (4.1) will be the strongest when the parameter q assumes its lowest possible value. Therefore, according to (4.3), we consider $1 \leq q \leq \infty$ such that $\frac{1}{p} - \frac{1}{r} = \frac{1}{2} - \frac{1}{q}$. Then, in view of (4.4) and (4.5), it is enough to focus on the extreme values $\frac{1}{p} - \frac{1}{r} = 0$ and $\frac{1}{p} - \frac{1}{r} = \frac{1}{2n}$, for the remaining cases can be obtained by interpolation, at least formally.

Thus, on the whole, we distinguish two kinds of expected results. The first one corresponds to the case $\frac{1}{p} - \frac{1}{r} = 0$, whence $q = 2$, and takes the form of an estimate, valid for all $0 \leq s < \frac{1}{2}$,

$$\left\| \int_{\mathbb{R}^n} f \varphi \, dv \right\|_{W_x^{s,r}} \leq C \left(\|f\|_{L_x^r L_v^2} + \|v \cdot \nabla_x f\|_{L_x^r L_v^2} \right), \tag{4.6}$$

where $1 \leq r \leq \infty$.

The second one corresponds to the case $\frac{1}{p} - \frac{1}{r} = \frac{1}{2n}$, whence $q = \frac{2n}{n-1}$, and takes the form of an estimate, valid for all $0 \leq s < \frac{1}{2}$,

$$\left\| \int_{\mathbb{R}^n} f \varphi \, dv \right\|_{W_x^{s,r}} \leq C \left(\|f\|_{L_x^{\frac{2nr}{2n+r}} L_v^{\frac{2n}{n-1}}} + \|v \cdot \nabla_x f\|_{L_x^{\frac{2nr}{2n+r}} L_v^{\frac{2n}{n-1}}} \right), \tag{4.7}$$

where $\frac{2n}{2n-1} \leq r \leq \infty$ (so that $\frac{2nr}{2n+r} \geq 1$).

Note that, in dimensions $n \geq 2$, the range $r > 2$ for estimate (4.6) is automatically ruled out by the optimality of Theorem 3.1 in the range $p > 2$ provided by the aforementioned counterexample from [9, Theorem 1.3] (recall that this counterexample is local in velocity). As for estimate (4.7), we do not expect to be able to beat the Hilbertian case of Theorem 1.1 in regularity or in local integrability, in large dimensions $n \geq 2$. Therefore, we only consider the range $\frac{2n}{2n-1} \leq r \leq 2$ in (4.7). It is possible that variations of the counterexample from [9, Theorem 1.3] could establish the sharpness of the condition $r \leq 2$ for (4.7), when $n \geq 2$, as well.

Estimates (4.6) and (4.7) have been considered in [2]. They are both established therein in full generality in one dimension (see Section 5 below).

In higher dimensions, estimate (4.6) is shown in [2] to hold in two dimensions in the range $\frac{4}{3} \leq r \leq 2$ (see Section 6) and partial results are obtained when $n \geq 3$ (see Section 7). As for estimate (4.7), it is established in [2] for the value $r = 2$ in any dimension $n \geq 2$ (see Section 8).

Finally, the possibility of a more general range of validity for both (4.6) and (4.7) is addressed through corresponding conjectures formulated in Section 9.

5. The one-dimensional case

It turns out that, in one dimension, it is possible to obtain a wide range of velocity averaging results. This is due to the fact that, in this setting, spatial frequencies are always parallel to velocities. The next theorem provides a general regularity statement for velocity averages in one dimension.

Theorem 5.1 ([2]). *In dimension $n = 1$, let $1 < p < \infty$ and $1 < q \le \infty$. Then, for any $f, g \in L^p(\mathbb{R}_x; L^q(\mathbb{R}_v))$ such that (0.2) holds true, one has that*

$$\int_{\mathbb{R}} f(x, v)\varphi(v)dv \in W^{s,p}(\mathbb{R}_x),$$

for all $0 \le s < 1 - \frac{1}{q}$ and any $\varphi \in L_c^\infty(\mathbb{R})$.

Remark. Notice that, setting $q = 2$ in the preceding theorem validates the full range (except endpoint cases) of estimates predicted in (4.6) when $n = 1$.

Note also that, setting $q = \infty$ and $p = \frac{2r}{2+r}$, with $2 < r \le \infty$, in the above theorem, we obtain, upon noticing that $W^{s, \frac{2r}{2+r}}(\mathbb{R})$ is continuously embedded into $W^{s-\frac{1}{2}, r}(\mathbb{R})$, the full range (except endpoint cases) of estimates predicted in (4.7) in one dimension $n = 1$.

Sketch of proof. As in the proofs of Theorems 1.1 and 3.1, this result follows from the boundedness of the velocity averaging operator $T : L_x^p L_v^q \to W_x^{s,p}$ defined by (1.2), where $\rho(r)$ is a smooth cutoff function such that $\rho(0) = 1$, and of a similar operator where $\rho(r)$ is replaced by $\tau(r) = \frac{1-\rho(r)}{ir}$. For simplicity, we only deal with the cutoff $\rho(r)$ and refer to [2] for full details.

Then, by duality, the boundedness of $T : L_x^p L_v^q \to W_x^{s,p}$, for any $1 < p < \infty$, $1 < q \le \infty$ and $0 \le s < 1 - \frac{1}{q}$, is a consequence of the boundedness of the adjoint operator $T^* : L_x^p \to W_x^{s,p} L_v^q$ defined in (1.4), for any $1 < p < \infty$, $1 \le q < \infty$ and $0 \le s < \frac{1}{q}$.

Thus, in order to establish the boundedness of T^*, employing the representation formula (3.1) for T^* and assuming that $\tilde{\rho}(s)$ is compactly supported inside $\{|s| \le r_0\}$, we write that

$$|T^*h(x, v)| = \left| \int_{\mathbb{R}} h(x - tv)\tilde{\rho}(t)dt\varphi(v) \right| = \left| \frac{1}{v} \int_{\mathbb{R}} h(x - r)\tilde{\rho}\left(\frac{r}{v}\right)dr\varphi(v) \right|$$

$$\le \|\tilde{\rho}\|_{L^\infty} \frac{1}{|v|} \int_{\{|r| \le |v|r_0\}} |h(x - r)|\,dr\,|\varphi(v)| \le 2r_0 \|\tilde{\rho}\|_{L^\infty} Mh(x)\,|\varphi(v)|,$$

where Mh denotes the Hardy-Littlewood maximal function of h defined by

$$Mh(x) = \sup_{\delta > 0} \frac{1}{2\delta} \int_{\{|y| \le \delta\}} |h(x - y)|dy.$$

Recalling now that the Hardy-Littlewood maximal operator is bounded over $L^p(\mathbb{R})$, for any $1 < p < \infty$, we deduce that

$$\|T^*h\|_{L_x^p L_v^q} \le C \|h\|_{L_x^p}, \tag{5.1}$$

for any $1 < p < \infty$ and $1 \le q \le \infty$.

Next, we further compute, exploiting the one-dimensional structure of the operators, for any $0 < \alpha < 1$,

$$(-\Delta_x)^{\frac{\alpha}{2}} T^*h(x, v) = \int_{\mathbb{R}} (-\Delta_x)^{\frac{\alpha}{2}} h(x - tv)\tilde{\rho}(t)dt\varphi(v)$$

$$= \frac{1}{|v|^\alpha} \int_{\mathbb{R}} (-\Delta_t)^{\frac{\alpha}{2}} (h(x - tv))\tilde{\rho}(t)dt\varphi(v)$$

$$= \frac{1}{|v|^\alpha} \int_{\mathbb{R}} h(x - tv)(-\Delta)^{\frac{\alpha}{2}} \tilde{\rho}(t)dt\varphi(v),$$

whence, for any $1 \le p \le \infty$,

$$\left\| (-\Delta_x)^{\frac{\alpha}{2}} T^*h \right\|_{L_x^p L_v^1} \le \left\| (-\Delta_x)^{\frac{\alpha}{2}} T^*h \right\|_{L_v^1 L_x^p} \le C \|h\|_{L_x^p},$$

which, when combined with (5.1), yields

$$\left\| (1 - \Delta_x)^{\frac{\alpha}{2}} T^*h \right\|_{L_x^p L_v^1} \le C \|h\|_{L_x^p}. \tag{5.2}$$

Finally, interpolating (we use here complex interpolation; see [2] for full details) the bound (5.1), where we set $q = \infty$, with (5.2), we obtain the estimate

$$\left\| (1 - \Delta_x)^{\frac{\alpha}{2q}} T^*h \right\|_{L_x^p L_v^q} \le C \|h\|_{L_x^p}.$$

for any $1 < p < \infty$, $1 \le q < \infty$ and $0 < \alpha < 1$, which concludes the proof of the theorem. $\qquad \square$

Remark. Note from the proof above that it is possible to improve the gain of regularity in the preceding theorem by assuming that the support of the velocity weight $\varphi(v)$ does not contain the origin. However, this is a rather unnatural setting which we prefer to avoid here.

6. The two-dimensional case

Our study of the one-dimensional case in the previous section showed that it is possible to largely improve the classical velocity averaging results in that setting. In particular, we showed therein that the gain of regularity of velocity averages is, in some cases, substantially improved beyond the value $\frac{1}{2}$.

While such a general improvement is not achievable in higher dimensions ($n \geq 2$), in view of the counterexamples from [9, Theorem 1.3] discussed earlier, it is nevertheless possible, as shown below, to obtain new cases displaying a gain of regularity of velocity averages of almost half a derivative.

In two dimensions ($n = 2$), this was already strongly suggested in [17, Theorem 1.2], where it is shown that velocity averages of f belong to $W_x^{s,\frac{4}{3}}$, for any $0 \leq s < \frac{1}{2}$, provided f and g belong to $L_x^{\frac{4}{3}} L_v^2$ and under the peculiar assumption that $g(x,v)\varphi(v)$ is an even function in v (see [17, Theorem 1.2]). The latter assumption is rather unnatural and it remained unclear whether this evenness condition could be removed or not.

By building upon the work from [17], the results from [2] brought a definitive answer to this two-dimensional problem, which is precisely the content of the following result. In the next section, these methods are generalized to higher dimensions ($n \geq 3$), without achieving a gain of half a derivative, though.

Theorem 6.1 ([2, 17]). *In dimension $n = 2$, let $f, g \in L^{\frac{4}{3}}\left(\mathbb{R}_x^2; L^2\left(\mathbb{R}_v^2\right)\right)$ be such that (0.2) holds true. Then,*

$$\int_{\mathbb{R}^2} f(x,v)\varphi(v)dv \in W^{s,\frac{4}{3}}\left(\mathbb{R}_x^2\right),$$

for all $0 \leq s < \frac{1}{2}$ and any $\varphi \in L_c^\infty\left(\mathbb{R}^2\right)$.

Remark. Notice that interpolating (at least formally) the above result with the Hilbertian case from Theorem 1.1 establishes the validity of the estimates predicted in (4.6) for the range of parameters $\frac{4}{3} \leq r \leq 2$. Recall that the range $r > 2$ in (4.6) has already been excluded by counterexamples. As for the restriction $r \geq \frac{4}{3}$, a counterexample from [17, Section 4.3] can be adapted to the present situation to suggest that this condition is optimal.

Sketch of proof. As in the proofs of Theorems 1.1 and 3.1, this result follows from the boundedness of the velocity averaging operator $T : L_x^{\frac{4}{3}} L_v^2 \to W_x^{s,\frac{4}{3}}$ defined by (1.2), where $\rho(r)$ is a smooth cutoff function such that $\rho(0) = 1$, and of a similar operator where $\rho(r)$ is replaced by $\tau(r) = \frac{1-\rho(r)}{ir}$. For simplicity, we only deal with the cutoff $\rho(r)$ and refer to [2] for full details.

Then, by duality, the boundedness of $T : L_x^{\frac{4}{3}} L_v^2 \to W_x^{s,\frac{4}{3}}$, for all $0 \leq s < \frac{1}{2}$, is a consequence of the boundedness of the adjoint operator $T^* : L_x^4 \to W_x^{s,4} L_v^2$ defined in (1.4), for all $0 \leq s < \frac{1}{2}$.

We define now, in any dimension $n \geq 1$, the velocity averaging operator S and its adjoint S^* on the sphere $\mathbb{S}^{n-1}$ by

$$Sf(x) = \int_{\mathbb{S}^{n-1}} \mathcal{F}_x^{-1} \rho(v \cdot \eta) \mathcal{F}_x f(x,v) dv = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(x - tv, v)\tilde{\rho}(t)\varphi(v) dt dv,$$

$$S^* h(x,v) = \mathcal{F}_x^{-1} \rho(v \cdot \eta) \mathcal{F}_x h(x) = \int_{\mathbb{R}} h(x - tv)\tilde{\rho}(t) dt,$$

(6.1)

where $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$. These operator correspond to the kinetic transport equation (0.2) with velocities restricted to the sphere $v \in \mathbb{S}^{n-1}$ and are introduced here for mere convenience and simplicity of analysis.

The operators S and S^* enjoy boundedness properties which are similar to those of the original operators T and T^*, respectively. In particular, employing the classical Hilbertian methods of velocity averaging, one finds that $S : L_{x,v}^2 \to H_x^{\frac{1}{2}}$ and $S^* : L_x^2 \to H_x^{\frac{1}{2}} L_v^2$. Furthermore, representing

$\mathbb{R}_v^n$ in polar coordinates, it is possible to show that any boundedness property of S^* results in a corresponding boundedness property for T^*. In particular, setting $n = 2$, it is seen that the present theorem follows from the boundedness of $S^* : L_x^4 \to W_x^{s,4} L_v^2$, for every $0 \leq s < \frac{1}{2}$.

From now on, we assume that the cutoff function $\rho(r)$ may be decomposed as a product $\rho(r) = \rho_1(r)\rho_2(r)$, so that $\tilde{\rho}(s) = \tilde{\rho}_1 * \tilde{\rho}_2(s)$. Moreover, we suppose that both $\tilde{\rho}_1(s)$ and $\tilde{\rho}_2(s)$ are compactly supported. Naturally, we denote by S^{i*}, where $i = 1, 2$, the operators S^* where we replace the cutoff ρ by ρ_i. It is then readily seen that

$$S^* h(x, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} h\left(x - (s+t)v\right) \tilde{\rho}_1(s)\tilde{\rho}_2(t) ds dt = \int_{\mathbb{R}} S^{2*} h(x - sv, v)\tilde{\rho}_1(s) ds.$$

A key idea here is to use the above trick to gain integration in one dimension along v through the straightforward estimate

$$
\begin{aligned}
|S^* h(x, v)|^2 &\leq \|\tilde{\rho}_1\|_{L^1} \int_{\mathbb{R}} \left|S^{2*} h(x + sv, v)\right|^2 |\tilde{\rho}_1(s)|\, ds \\
&\leq \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{[-r_1, r_1]} \left|S^{2*} h(x + sv, v)\right|^2 ds,
\end{aligned}
\tag{6.2}
$$

where $\mathrm{supp}\tilde{\rho}_1 \subset [-r_1, r_1]$, for some $r_1 > 0$.

We use now the trivial yet crucial fact that the exponent 4 is an even integer to control the square of the adjoint transport operator $|S^* h|^2$ in L_x^2 rather than the operator $S^* h$ itself in L_x^4. Thus, employing (6.2), we have the bilinear estimate

$$
\begin{aligned}
\|S^* h\|_{L_x^4 L_v^2}^4 &= \int_{\mathbb{R}^2 \times \mathbb{S}^1} |S^* h(x, v_2)|^2 \left(\int_{\mathbb{S}^1} |S^* h(x, v_1)|^2\, dv_1\right) dx dv_2 \\
&\leq C \int_{\mathbb{R}^2 \times \mathbb{S}^1} \left(\int_{[-r_1, r_1]} \left|S^{2*} h(x + s_2 v_2, v_2)\right|^2 ds_2\right)\left(\int_{\mathbb{S}^1} |S^* h(x, v_1)|^2\, dv_1\right) dx dv_2 \\
&= C \int_{\mathbb{R}^2 \times \mathbb{S}^1} \left|S^{2*} h(x, v_2)\right|^2 \left(\int_{[-r_1, r_1]} \int_{\mathbb{S}^1} \left|S^* h(x + s_2 v_2, v_1)\right|^2 dv_1 ds_2\right) dx dv_2 \\
&\leq C \int_{\mathbb{R}^2 \times \mathbb{S}^1} \left|S^{2*} h(x, v_2)\right|^2 \\
&\qquad \times \left(\int_{[-r_1, r_1]^2} \int_{\mathbb{S}^1} \left|S^{2*} h(x + s_1 v_1 + s_2 v_2, v_1)\right|^2 dv_1 ds_1 ds_2\right) dx dv_2 \\
&\leq C \left\|S^{2*} h\right\|_{L_{x,v}^2}^2 \sup_{\substack{x \in \mathbb{R}^2 \\ v_2 \in \mathbb{S}^1}} I(x, v_2),
\end{aligned}
$$

where we define the nonlinear operator

$$I(x, v_2) = \int_{[-r_1, r_1]^2} \int_{\mathbb{S}^1} \left|S^{2*} h(x + s_1 v_1 + s_2 v_2, v_1)\right|^2 dv_1 ds_1 ds_2. \tag{6.3}$$

Further using the classical Hilbertian boundedness of $S^{2*} : L_x^2 \to H_x^{\frac{1}{2}} L_v^2$, which is analog to the boundedness of $T^* : L_x^2 \to H_x^{\frac{1}{2}} L_v^2$, and assuming for simplicity that $h(x)$ has frequencies localized inside an annulus of inner and outer radii comparable to 2^k, with $k \in \mathbb{N}$, we deduce that

$$\|S^* h\|_{L_x^4 L_v^2}^4 \leq \frac{C}{2^k} \|h\|_{L_x^2}^2 \sup_{\substack{x \in \mathbb{R}^2 \\ v_2 \in \mathbb{S}^1}} I(x, v_2), \tag{6.4}$$

where $C > 0$ is independent of k.

The next key idea consists in exploiting the two-dimensional integration in $ds_1 ds_2$ in the above definition of the non-linear operator to express $I(x, v_2)$ as the $L_{x,v}^2$ norm of $S^{2*} h$. To this end, we employ the change of variables $(s_1, s_2) \mapsto z = s_1 v_1 + s_2 v_2$, whenever v_1 and v_2 form a basis, which holds almost everywhere. Of course, this transformation becomes degenerate when v_1 and v_2 are close to being colinear, which requires technical care.

It is readily seen that the Jacobian determinant of this transformation is given by $\sin\theta$, where $\theta \in [0, \pi]$ is the angle between v_1 and v_2. The degeneracy at $\sin\theta = 0$ is then handled by noticing that the domain of integration in z becomes localized in the direction of v_2 when $\sin\theta$ is small.

Thus, decomposing the domain of integration $v_1 \in \mathbb{S}^1$ into small dyadic subsets

$$S_i = \left\{ 2^{-(i+1)} < |\sin\theta| \le 2^{-i} \right\},$$

with $i \in \mathbb{N}$, one notices, whenever $v_1 \in S_i$, since S^{2*} only acts in the direction of v_1, that it is possible to replace the function $h(x + z)$ in the definition (6.3) by $h_{x,i}(z) = h(x + z)\mathbb{1}_{K_i}(z)$, where $K_i \subset \mathbb{R}^2$ is a cylinder of fixed length pointing in the direction of v_2 whose width is comparable to 2^{-i}.

Therefore, sweeping important technical difficulties under the carpet (see [2] for details) and using the classical Hilbertian boundedness of $S^{2*} : L_x^2 \to H_x^{\frac{1}{2}} L_v^2$ again, we arrive at the estimate

$$
\begin{aligned}
I(x, v_2) &\le \sum_{i=0}^{\infty} \int_{\mathbb{R}^2 \times S_i} \left| \int_{\mathbb{R}} h\left(x + z + sv_1\right) \mathbb{1}_{K_i}(z + sv_1)\tilde{\rho}_2(s)ds \right|^2 \frac{1}{\sin\theta} dz dv_1 \\
&\le C \sum_{i \le k} 2^i \int_{\mathbb{R}^2 \times \mathbb{S}^1} \left| \int_{\mathbb{R}} h\left(x + z + sv_1\right) \mathbb{1}_{K_i}(z + sv_1)\tilde{\rho}_2(s)ds \right|^2 dz dv_1 \\
&\quad + C \sum_{i > k} 2^i \int_{\mathbb{R}^2 \times S_i} \left| \int_{\mathbb{R}} h\left(x + z + sv_1\right) \mathbb{1}_{K_i}(z + sv_1)\tilde{\rho}_2(s)ds \right|^2 dz dv_1 \\
&\le C \sum_{i \le k} 2^i \left\| S^{2*} h_{x,i}(z, v_1) \right\|_{L_{z,v_1}^2}^2 + C \sum_{i > k} 2^i |S_i| \left\| h_{x,i}(z) \right\|_{L_z^2}^2 \\
&\le C \sum_{i \le k} 2^{i-k} \left\| h_{x,i}(z) \right\|_{L_z^2}^2 + C \sum_{i > k} \left\| h_{x,i}(z) \right\|_{L_z^2}^2 \\
&\le C \left(\sum_{i \le k} 2^{i-k} |K_i| + \sum_{i > k} |K_i| \right) \left\| h \right\|_{L_x^\infty}^2 \le \frac{C}{2^{\lambda k}} \left\| h \right\|_{L_x^\infty}^2 ,
\end{aligned}
\tag{6.5}
$$

for any $0 < \lambda < 1$ and where $C > 0$ is independent of k.

Hence, combining estimates (6.4) and (6.5), we infer, for any $0 < s < \frac{1}{2}$,

$$\|S^* h\|_{L_x^4 L_v^2}^4 \le \frac{C}{2^{k4s}} \|h\|_{L_x^2}^2 \|h\|_{L_x^\infty}^2 , \tag{6.6}$$

where $C > 0$ is independent of k. It follows that, for any $0 \le s < \frac{1}{2}$,

$$\left\| (1 - \Delta_x)^{\frac{s}{2}} S^* h \right\|_{L_x^4 L_v^2} \le C \|h\|_{L_x^2}^{\frac{1}{2}} \|h\|_{L_x^\infty}^{\frac{1}{2}} ,$$

which almost establishes the boundedness of $S^* : L_x^4 \to W_x^{s,4} L_v^2$.

In order to conclude, we write $|h(x)| = \int_0^\infty \mathbb{1}_{\{|h(x)| \ge s\}} ds$ to deduce from the preceding estimate, assuming h is non-negative, that

$$
\begin{aligned}
\left\| (1 - \Delta_x)^{\frac{s}{2}} S^* h \right\|_{L_x^4 L_v^2} &\le \int_0^\infty \left\| (1 - \Delta_x)^{\frac{s}{2}} S^* \mathbb{1}_{\{|h(x)| \ge s\}} \right\|_{L_x^4 L_v^2} ds \\
&\le C \int_0^\infty |\{|h(x)| \ge s\}|^{\frac{1}{4}} ds = C \|h\|_{L_x^{4,1}} ,
\end{aligned}
$$

where $L_x^{4,1}$ denotes a standard Lorentz space (see [4, Section 1.3] or [14, Section 1.4] for definitions and properties of Lorentz spaces). When, h is signed, we arrive at the same estimate simply by decomposing h into its positive and negative parts. Finally, by allowing an arbitrarily small loss of regularity, that is by replacing $0 < s < \frac{1}{2}$ by a slightly smaller value, it is possible to replace the Lorentz space $L_x^{4,1}$ by the standard Lebesgue space L_x^4 in the right-hand side of the above estimate, which concludes the justification of the theorem. $\qquad\square$

Remark. Some features of the preceding proof are strikingly reminiscent of the proofs of boundedness of Bochner-Riesz multipliers and Fourier restriction operators in two dimensions. We refer to [15] and the references therein for more on these subjects from harmonic analysis.

7. The higher-dimensional case

We address now the higher-dimensional setting. In [2], the methods leading to Theorem 6.1 were extended to establish an analog result valid in any dimension. Unfortunately, the ensuing result does not reach a maximal gain of regularity of half a derivative on the velocity averages, but only a gain of $\frac{n}{4(n-1)}$ derivatives, where $n \geq 3$ is the dimension. This drawback mainly stems from the fact that the methods of proof of Theorem 6.1 are restricted to the L_x^4 setting, by exploiting the trivial fact that the exponent 4 is an even integer in order to control the square of some transport operator in L_x^2 rather than the operator itself in L_x^4.

The main result in this setting is the following.

Theorem 7.1 ([2]). *In any dimension $n \geq 3$, let $f, g \in L^{\frac{4}{3}}\left(\mathbb{R}_x^n; L^2\left(\mathbb{R}_v^n\right)\right)$ be such that (0.2) holds true. Then,*

$$\int_{\mathbb{R}^n} f(x,v)\varphi(v)dv \in W^{s,\frac{4}{3}}\left(\mathbb{R}_x^n\right),$$

for all $0 \leq s < \frac{n}{4(n-1)}$ and any $\varphi \in L_c^\infty\left(\mathbb{R}^n\right)$.

Remark. The above result does not cover any of the estimates predicted by (4.6) or (4.7), for it never gets close to a gain of half a derivative. However, it can be interpreted, using interpolation, as a weaker version of (4.6), at least for some restricted values of $1 \leq r \leq 2$. Indeed, note that a formal interpolation yields

$$\left(L_x^1 L_v^2, L_x^{\frac{2n}{n+1}} L_v^2\right)_{\frac{n}{2(n-1)}} = L_x^{\frac{4}{3}} L_v^2 \quad \text{and} \quad \left(L_x^1, W_x^{\frac{1}{2}, \frac{2n}{n+1}}\right)_{\frac{n}{2(n-1)}} = W_x^{\frac{n}{4(n-1)}, \frac{4}{3}}.$$

In particular, formally extrapolating the above regularity result has us believe that estimate (4.6) must hold for the value $r = \frac{2n}{n+1}$. In other words, Theorem 7.1 would follow from a formal interpolation of (4.6), where we set $r = \frac{2n}{n+1}$, with the degenerate L^1 case. Thus, this formal argument suggests that (4.6) may only be valid in the range $\frac{2n}{n+1} \leq r \leq 2$, which also agrees with the restriction $\frac{4}{3} \leq r \leq 2$ for the two-dimensional case discussed in the previous section.

Sketch of proof. As in the proofs of the preceding theorems, this result will follow from the study of the boundedness of the transport operator $T : L_x^{\frac{4}{3}} L_v^2 \to W_x^{s,\frac{4}{3}}$, for all $0 \leq s < \frac{n}{4(n-1)}$ defined by (1.2), where $\rho(r)$ is a smooth cutoff function such that $\rho(0) = 1$, and of a similar operator where $\rho(r)$ is replaced by $\tau(r) = \frac{1-\rho(r)}{ir}$. For simplicity, we only deal with the cutoff $\rho(r)$ and refer to [2] for full details.

As explained in the proof of Theorem 6.1, the continuity of the operator $T : L_x^{\frac{4}{3}} L_v^2 \to W_x^{s,\frac{4}{3}}$ will then follow from the boundedness of the adjoint operator $S^* : L_x^4 \to W_x^{s,4} L_v^2$ with velocities restricted to the sphere $\mathbb{S}^{n-1}$, for every $0 \leq s < \frac{n}{4(n-1)}$, defined by (6.1).

Here, we are also assuming that the cutoff function $\rho(r)$ may be decomposed as a product $\rho(r) = \rho_1(r)\rho_2(r)$, so that $\tilde{\rho}(s) = \tilde{\rho}_1 * \tilde{\rho}_2(s)$, where both $\tilde{\rho}_1(s)$ and $\tilde{\rho}_2(s)$ are compactly supported. In particular, the crucial trick (6.2) used to gain a one-dimensional integration along v is also valid here.

We introduce now the following non-linear operator, for any integer $N \geq 2$, denoting the variables $S = (s_2, \ldots, s_{N-1}) \in \mathbb{R}^{N-2}$ and $V = (v_1, \ldots, v_{N-1}) \in \left(\mathbb{S}^{n-1}\right)^{N-1}$:

$$I_N h = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S^* h(x, v_N)|^2$$

$$\times \left(\int_{[-r_1, r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S^* h\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right) \right|^2 dV dS \right) dv_N dx,$$

where $[-r_1, r_1]$ contains the support of $\tilde{\rho}_1$. Recall that, employing (6.2), it is possible to extract a one-dimensional integration from $S^* h(x, v_N)$ and $S^* h\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right)$ along v_N and v_1, respectively. Therefore, it is possible, at least formally, to gain an N-dimensional spatial integration in the above integrand by exploiting the integration along the variables s_j. Thus, loosely speaking,

the number N represents the expected gain of spatial dimension on the domain of integration in I_N.

As shown below, the boundedness of $S^* : L_x^4 \to W_x^{s,4} L_v^2$ will follow from the following four properties of the non-linear operator I_N:

- For $N = 2$,
$$I_2 h = \|S^* h\|_{L_x^4 L_v^2}^4 . \tag{7.1}$$

- For $N = n$ and any $0 < s < \frac{1}{2}$, assuming for simplicity that $h(x)$ has frequencies localized inside an annulus of inner and outer radii comparable to 2^k, with $k \in \mathbb{N}$,
$$I_n h \le \frac{C}{2^{k4s}} \|h\|_{L_x^2}^2 \|h\|_{L_x^\infty}^2 , \tag{7.2}$$

where $C > 0$ is independent of k. This estimate is rather involved and is based on the methods employed to reach (6.6) in the proof of Theorem 6.1.

- For any $N \ge 2$,
$$(I_N h)^2 \le \|S^* h\|_{L_x^4 L_v^2}^4 I_{2N-2} h, \tag{7.3}$$
which is a simple consequence of the Cauchy-Schwarz inequality (in x) followed by a careful change of variable.

- For any $N \ge 2$,
$$(I_N h)^2 \le C \left\|S^{2*} h\right\|_{L_{x,v}^4}^4 I_{2N-1} h, \tag{7.4}$$

which is a direct consequence of an application of (6.2) followed by a careful use of the Cauchy-Schwarz inequality with a change of variable.

Now, in order to establish the boundedness of $S^* : L_x^4 \to W_x^{s,4} L_v^2$ from the above four properties of I_N, we further introduce the mappings $\Lambda_0, \Lambda_1 : \mathbb{N} \setminus \{0,1\} \to \mathbb{N} \setminus \{0,1\}$ defined by

$$\Lambda_0 k = 2k - 2 \quad \text{and} \quad \Lambda_1 k = 2k - 1.$$

It is possible to show that, for any integer $n \ge 3$, there exists $L \in \mathbb{N}$ and $a_0, a_1, \ldots, a_L \in \{0,1\}$ such that

$$n = \Lambda_{a_0} \Lambda_{a_1} \ldots \Lambda_{a_L} 2 \quad \text{and} \quad n - 2 = \sum_{k=0}^{L} a_k 2^k.$$

Moreover, the above decomposition is unique provided $a_L = 1$. We will only consider this decomposition applied to the dimension n.

Notice that one may now unify (7.3) and (7.4) into the single estimate, for any $N \ge 2$,

$$(I_N h)^2 \le C \|S^* h\|_{L_x^4 L_v^2}^{4(1-a)} \left\|S^{2*} h\right\|_{L_{x,v}^4}^{4a} I_{\Lambda_a N} h, \tag{7.5}$$

where $a \in \{0,1\}$.

At last, combining the above properties by using (7.1) first and then applying successively ($L+1$ times) estimate (7.5), we deduce that

$$\|S^* h\|_{L_x^4 L_v^2}^{2^{L+1}} = (I_2 h)^{2^{L-1}}$$

$$\le C \|S^* h\|_{L_x^4 L_v^2}^{(1-a_L)2^L} \left\|S^{2*} h\right\|_{L_{x,v}^4}^{a_L 2^L} \left(I_{\Lambda_{a_L}} 2 h\right)^{2^{L-2}}$$

$$\le C \|S^* h\|_{L_x^4 L_v^2}^{(1-a_{L-1})2^{L-1} + (1-a_L)2^L} \left\|S^{2*} h\right\|_{L_{x,v}^4}^{a_{L-1} 2^{L-1} + a_L 2^L} \left(I_{\Lambda_{a_{L-1}} \Lambda_{a_L}} 2 h\right)^{2^{L-3}}$$

$$\le \ldots$$

$$\le C \|S^* h\|_{L_x^4 L_v^2}^{\sum_{j=0}^{L}(1-a_j)2^j} \left\|S^{2*} h\right\|_{L_{x,v}^4}^{\sum_{j=0}^{L} a_j 2^j} \left(I_{\Lambda_{a_0} \Lambda_{a_1} \ldots \Lambda_{a_L}} 2 h\right)^{\frac{1}{4}}$$

$$= C \|S^* h\|_{L_x^4 L_v^2}^{2^{L+1} - (n-1)} \left\|S^{2*} h\right\|_{L_{x,v}^4}^{n-2} (I_n h)^{\frac{1}{4}} .$$

Hence, in view of the classical Hilbertian boundedness of $S^{2*} : L_x^2 \to H_x^{\frac{1}{2}} L_v^2$ and (7.2), recalling that we are assuming for simplicity that $h(x)$ has localized frequencies of order 2^k, it follows that, for any $0 < s < \frac{n}{4(n-1)}$,

$$\|S^*h\|_{L_x^4 L_v^2}^{n-1} \le C \left\|S^{2*}h\right\|_{L_{x,v}^4}^{n-2} (I_n h)^{\frac{1}{4}}$$

$$\le C \left\|S^{2*}h\right\|_{L_{x,v}^2}^{\frac{n-2}{2}} \left\|S^{2*}h\right\|_{L_{x,v}^\infty}^{\frac{n-2}{2}} (I_n h)^{\frac{1}{4}}$$

$$\le \frac{C}{2^{k(n-1)s}} \|h\|_{L_x^2}^{\frac{n-1}{2}} \|h\|_{L_x^\infty}^{\frac{n-1}{2}} ,$$

where $C > 0$ does not depend on k. It follows that, for any $0 \le s < \frac{n}{4(n-1)}$,

$$\left\| (1 - \Delta_x)^{\frac{s}{2}} S^* h \right\|_{L_x^4 L_v^2} \le C \|h\|_{L_x^2}^{\frac{1}{2}} \|h\|_{L_x^\infty}^{\frac{1}{2}} .$$

Finally, the remainder of the demonstration follows the arguments from the end of the proof of Theorem 6.1, which show that the preceding estimate is sufficient to conclude the boundedness of $S^* : L_x^4 \to W_x^{s,4} L_v^2$.

The justification of the theorem is now complete. $\square$

8. Dispersion and velocity averaging

Apart from the one-dimensional setting treated in Section 5, where (4.6) and (4.7) are both shown to hold for every admissible value of r, we have not yet discussed the possible validity of (4.7) in higher dimensions $n \ge 2$. In fact, it was shown in [2] that (4.7) holds for the value $r = 2$ as a consequence of dispersion of the kinetic transport flow combined with the classical Hilbertian result from Theorem 1.1, which is precisely the content of the following result.

Theorem 8.1 ([2]). *In any dimension $n \ge 1$, let $f, g \in L^{\frac{2n}{n+1}} \left(\mathbb{R}_x^n ; L^{\frac{2n}{n-1}} (\mathbb{R}_v^n) \right)$ such that (0.2) holds true. Then,*

$$\int_{\mathbb{R}^n} f(x, v) \varphi(v) dv \in H^s (\mathbb{R}_x^n) ,$$

for all $0 \le s < \frac{1}{2}$ and any $\varphi \in L_c^\infty (\mathbb{R}^n)$.

Remark. Such a result had already been hinted at in [16, 17], where it had been established that, in two dimensions only ($n = 2$), velocity averages of f belong to H_x^s, for any $0 \le s < \frac{1}{2}$, provided f and g belong to $L_x^{\frac{4}{3}} L_v^\infty$ (see [17, Theorem 1.3]).

Sketch of proof. As usual, this result follows from the boundedness of the velocity averaging operator $T : L_x^{\frac{2n}{n+1}} L_v^{\frac{2n}{n-1}} \to H_x^s$ defined by (1.2), where $\rho(r)$ is a smooth cutoff function such that $\rho(0) = 1$, and of a similar operator where $\rho(r)$ is replaced by $\tau(r) = \frac{1 - \rho(r)}{ir}$. For simplicity, we only deal with the cutoff $\rho(r)$ and refer to [2] for full details.

The following basic dispersive estimate on the kinetic transport flow was established in [7], for all $1 \le q \le p \le \infty$ and any $t \ne 0$:

$$\|f(x - tv, v)\|_{L_x^p L_v^q} = \frac{1}{|t|^{\frac{n}{q}}} \left\| f \left(y, \frac{x - y}{t} \right) \right\|_{L_x^p L_y^q}$$

$$\le \frac{1}{|t|^{\frac{n}{q}}} \left\| f \left(y, \frac{x - y}{t} \right) \right\|_{L_y^q L_x^p} = \frac{1}{|t|^{n\left(\frac{1}{q} - \frac{1}{p}\right)}} \|f(x, v)\|_{L_x^q L_v^p} .$$

Combining the above kinetic dispersion with a TT^*-argument, it is then readily seen that, for all $p \geq 2$,

$$\left\| \int_{\mathbb{R}} f(x-tv,v)\tilde{\rho}(t)dt \right\|_{L^2_{x,v}}^2 = \left\| \int_{\mathbb{R}\times\mathbb{R}} f(x-tv,v)\,\tilde{\rho}(t)f(x-sv,v)\,\tilde{\rho}(s)dsdt \right\|_{L^1_{x,v}}$$

$$\leq \int_{\mathbb{R}\times\mathbb{R}} \|f(x,v)f(x-(s-t)v,v)\|_{L^1_{x,v}} |\tilde{\rho}(s)\tilde{\rho}(t)|\,dsdt$$

$$\leq \int_{\mathbb{R}\times\mathbb{R}} \|f(x,v)\|_{L^{p'}_x L^p_v} \|f(x-(s-t)v,v)\|_{L^p_x L^{p'}_v} |\tilde{\rho}(s)\tilde{\rho}(t)|\,dsdt$$

$$\leq \|f\|_{L^{p'}_x L^p_v}^2 \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{|s-t|^{n\left(1-\frac{2}{p}\right)}} |\tilde{\rho}(s)\tilde{\rho}(t)|\,dsdt.$$

Therefore, further requiring that $2 \leq p < \frac{2n}{n-1}$ so that the last integral above is finite by virtue of the Hardy-Littlewood-Sobolev inequality, we deduce that

$$\left\| \int_{\mathbb{R}} f(x-tv,v)\tilde{\rho}(t)dt \right\|_{L^2_{x,v}} \leq C \|f\|_{L^{p'}_x L^p_v}, \tag{8.1}$$

where $C > 0$ is a finite independent constant.

We employ now a trick which has already proved itself extremely useful in the proofs of Theorems 6.1 and 7.1. Namely, we assume that the cutoff function $\rho(r)$ may be decomposed as a product $\rho(r) = \rho_1(r)\rho_2(r)$, so that $\tilde{\rho}(s) = \tilde{\rho}_1 * \tilde{\rho}_2(s)$. Naturally, we denote by T^i, where $i = 1, 2$, the operators T where we replace the cutoff ρ by ρ_i. According to the representation formula (3.1), it then holds that

$$Tf(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-(s+t)v,v)\,\tilde{\rho}_1(s)\tilde{\rho}_2(t)\varphi(v)dsdtdv$$

$$= T^2 \left(\int_{\mathbb{R}} f(x-sv,v)\tilde{\rho}_1(s)ds \right)(x).$$

Thus, combining the classical Hilbertian boundedness of $T^2 : L^2_{x,v} \to H^{\frac{1}{2}}_x$ (from the proof of Theorem 1.1) with the dispersive bound (8.1), we conclude that

$$\|Tf\|_{H^{\frac{1}{2}}_x} \leq C \left\| \int_{\mathbb{R}} f(x-sv,v)\tilde{\rho}_1(s)ds \right\|_{L^2_{x,v}} \leq C \|f\|_{L^{p'}_x L^p_v},$$

for any $2 \leq p < \frac{2n}{n-1}$, which establishes the boundedness of $T : L^{p'}_x L^p_v \to H^{\frac{1}{2}}_x$. Since this result is local in v, we infer the boundedness of $T : L^{p'}_x L^{\frac{2n}{n-1}}_v \to H^{\frac{1}{2}}_x$, as well. Finally, by allowing an arbitrarily small loss of regularity, that is by replacing $s = \frac{1}{2}$ by a slightly smaller value $0 < s < \frac{1}{2}$, it is also possible to replace, using a Sobolev embedding, the space $L^{p'}_x$ by $L^{\frac{2n}{n+1}}_x$. This establishes the boundedness of $T : L^{\frac{2n}{n+1}}_x L^{\frac{2n}{n-1}}_v \to H^s_x$, for any $0 < s < \frac{1}{2}$, and thus concludes the proof of the theorem. $\qquad\square$

Remark. As previously mentioned, the preceding theorem establishes the validity of (4.7) for the value $r = 2$ in every dimension. However, nothing is claimed regarding other values $r \neq 2$, which remains open.

As shown by the proof above, the case $r = 2$ in (4.7) is a consequence of the action of dispersion on the estimate provided by the case $r = 2$ in (4.6). In fact, it seems natural to believe that this is the same for every admissible value of r, i.e. that every valid estimate in (4.7) is the consequence of the action of dispersion on the estimate provided by the same value r in (4.6). In particular, notice that the pair of exponents $\left(\frac{2nr}{2n+r}, \frac{2n}{n-1}\right)$ in (4.7) has the same harmonic mean $\frac{4r}{2+r}$ as the pair of exponents $(r, 2)$ in (4.6), which is a requirement for the application of kinetic dispersive estimates such as (8.1) (see also the more general dispersive estimates from [2]). This viewpoint suggests that estimates (4.6) and (4.7) are valid on the exact same range of parameters. Unfortunately, we do not have any counterexample to substantiate this claim.

9. Open problems and final thoughts

The search for maximal regularity in velocity averaging lemmas first motivated by [17] and then conducted in [2] has proved so far a very rich endeavor requiring diverse and original methods (extending beyond the classical settings of velocity averaging), producing interesting new results and leading to exciting research perspectives.

Thus, a straightforward dimensional analysis performed in Section 4 first revealed that this problem can be reduced to establishing estimates of the types (4.6) and (4.7), whose exact range of validity remains yet to be determined in general.

This task turns out to be heavily dependent on the dimension where it is set. The one-dimensional setting presented in Section 5 has been fully solved by showing that both estimates (4.6) and (4.7) are valid in the largest possible range of admissible parameters, i.e. $1 < r < \infty$ for (4.6) and $2 < r < \infty$ for (4.7).

The higher-dimensional setting tells a whole different story.

In two dimensions, the results described in Section 6 establish that (4.6) holds for values $\frac{4}{3} \leq r \leq 2$. Moreover, as explained therein, a counterexample from [17] to the boundedness of certain operators strongly suggests that this range of parameters is optimal. In higher dimensions $n \geq 3$, a formal extrapolation argument of the results from Section 7 further implies that (4.6) should only hold in the range $\frac{2n}{n+1} \leq r \leq 2$. We summarize these considerations in the following conjecture.

Conjecture 9.1. *In any dimension $n \geq 2$, let $\frac{2n}{n+1} \leq r \leq 2$ and $f, g \in L^r \left(\mathbb{R}_x^n; L^2 \left(\mathbb{R}_v^n \right) \right)$ be such that (0.2) holds true. Then,*

$$\int_{\mathbb{R}^n} f(x,v)\varphi(v)dv \in W^{s,r} \left(\mathbb{R}_x^n \right),$$

for all $0 \leq s < \frac{1}{2}$ and any $\varphi \in L_c^\infty \left(\mathbb{R}^n \right)$. Moreover, this result does not hold for values $1 \leq r < \frac{2n}{n+1}$.

The behavior of the range of admissible parameters conjectured above is strikingly reminiscent of the famous problems of boundedness of Bochner-Riesz multipliers and Fourier restriction operators (see [15] and the references therein). From this viewpoint, by further analogy with the unboundedness results from [11] for the ball multiplier, we can only hypothesize the following.

Conjecture 9.2. *In any dimension $n \geq 2$, let $\frac{2n}{n+1} \leq r \leq 2$ satisfy that, for any $f, g \in L^r \left(\mathbb{R}_x^n; L^2 \left(\mathbb{R}_v^n \right) \right)$ such that (0.2) holds true, one has*

$$\int_{\mathbb{R}^n} f(x,v)\varphi(v)dv \in W^{\frac{1}{2},r} \left(\mathbb{R}_x^n \right),$$

for any $\varphi \in L_c^\infty \left(\mathbb{R}^n \right)$. Then, necessarily $r = 2$.

As for the validity of estimate (4.7), it was addressed by the results from Section 8, where it was identified that dispersion of the kinetic flow could produce bounds such as the ones predicted in (4.7). Theorem 8.1 is very satisfying in that it holds in every dimension. However, it only handles the case $r = 2$ in (4.7). One could further argue that any estimate of the type (4.7) is produced by the effects of dispersion on an estimate of the type (4.6) for the same parameter r. From this perspective and in view of the above conjectures, it is then only natural to speculate that the following should hold.

Conjecture 9.3. *In any dimension $n \geq 2$, let $\frac{2n}{n+1} \leq r \leq 2$ and $f, g \in L^{\frac{2nr}{2n+r}} \left(\mathbb{R}_x^n; L^{\frac{2n}{n-1}} \left(\mathbb{R}_v^n \right) \right)$ be such that (0.2) holds true. Then,*

$$\int_{\mathbb{R}^n} f(x,v)\varphi(v)dv \in W^{s,r} \left(\mathbb{R}_x^n \right),$$

for all $0 \leq s < \frac{1}{2}$ and any $\varphi \in L_c^\infty \left(\mathbb{R}^n \right)$. Moreover, this result does not hold for values $1 \leq r < \frac{2n}{n+1}$.

The above set of problems provides many interesting research perspectives. We believe that any progress (positive or negative) in this area will require new and original ideas. It would be judicious though to search first for definitive counterexamples, directly applicable to the equation (0.2), capable of probing the limits of validity of such conjectures, which would also greatly strenghten their plausibility.

Finally, as mentioned before, the results presented here and their proofs bear a strong resemblance with the famous and notoriously difficult problems of boundedness of Bochner-Riesz multipliers and Fourier restriction operators. One may also debate about their similitude with smoothing conjectures for Schrödinger and wave equations. These considerations thus suggest the possibility of interesting links between kinetic theory, dispersive equations and harmonic analysis, which should be investigated.

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