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**The Quantum Birkhoff Normal Form and Spectral Asymptotics**

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# The Quantum Birkhoff Normal Form and Spectral Asymptotics

San Vũ Ngọc

## Abstract

In this talk we explain a simple treatment of the quantum Birkhoff normal form for semiclassical pseudo-differential operators with smooth coefficients. The normal form is applied to describe the discrete spectrum in a generalised non-degenerate potential well, yielding uniform estimates in the energy  $E$ . This permits a detailed study of the spectrum in various asymptotic regions of the parameters  $(E, \hbar)$ , and gives improvements and new proofs for many of the results in the field. In the completely resonant case we show that the pseudo-differential operator can be reduced to a Toeplitz operator on a reduced symplectic orbifold. Using this quantum reduction, new spectral asymptotics concerning the fine structure of eigenvalue clusters are proved.

## 1. Introduction: spectral asymptotics for potential wells

The motivation for this work is to understand spectral asymptotics for the Schrödinger operator  $\hat{H} = -\frac{\hbar^2}{2}\Delta + V(x)$ , acting on  $\mathbb{R}^n$ , or on a compact riemannian manifold, where  $V$  is a smooth confining potential. In this talk we restrict for simplicity to the case of  $\mathbb{R}^n$ , and  $V \in C^\infty(\mathbb{R}^n)$ . Of course, it is one of the fundamental questions of (non-relativistic) quantum mechanics. The point where the potential reaches a global minimum will be called a global potential well. This is where the ground state lives, at least in the semiclassical limit  $\hbar \rightarrow 0$ . One could think for instance of a molecule oscillating in its fundamental state, subject to small excitations which may put it in various neighbouring states, corresponding to eigenvalues of  $\hat{H}$  that remain close to the smallest one. The idea of semiclassical analysis is to understand the behaviour of the lowest eigenvalues in terms of the shape of the potential near its minimum. This will naturally require a *détour* by classical mechanics.

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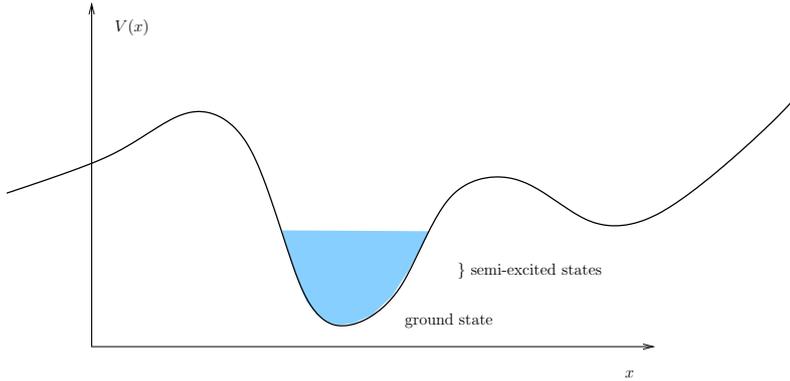


Figure 1: A potential with a global well.

Exactly how many eigenvalues above the ground state will be accessible by investigating the shape of the well is a key issue to address. The use of a Birkhoff normal form — initially introduced for classical mechanics in Birkhoff’s book [1] — will greatly improve the naive number of such eigenvalues that could be hoped for by a simple perturbation analysis.

The idea to use a Birkhoff normal form in semiclassical analysis is not new. After Gustavson popularised Birkhoff’s construction, many theoretical physicists have used it in different fields of quantum physics, with great success in particular in spectroscopy. A mathematical formulation in terms of pseudo-differential operators appears in [11], [9], [7] and certainly many other articles. Please refer to our original article [3] for a slightly more complete bibliography.

## 2. Hamiltonian classical mechanics

Our phase space is here  $\mathbb{R}^{2n}$  with canonical variables  $(x, \xi)$ . The Hamiltonian function  $H(x, \xi)$  into consideration is the energy of the classical system.

For instance, the Hamiltonian function corresponding to the Schrödinger operator is  $H(x, \xi) = \frac{1}{2} \|\xi\|^2 + V(x)$ .

Any function  $H$  in  $C^\infty(\mathbb{R}^{2n})$  gives rise to a dynamical system

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial \xi} \\ \dot{\xi} = -\frac{\partial H}{\partial x} \end{cases}$$

More generally at this point  $(M, \omega)$  could be any symplectic manifold. A function  $H \in C^\infty(M)$  defines a vector field  $\mathcal{X}_H$  on  $M$  by duality:

$$\omega(\mathcal{X}_H, \cdot) = -dH.$$

The corresponding evolution of a point  $m \in M$  is then the dynamical system  $\frac{dm}{dt} = \mathcal{X}_H(m)$ .

If  $M = \mathbb{R}^{2n}$ , the symplectic form is the canonical 2-form  $\omega = \sum_j d\xi_j \wedge dx_j$ .

In any case, the basic fact for such a Hamiltonian system is that the function  $H$  is constant along trajectories.

### 3. Classical wells

We are interested in *non-degenerate global wells* (although many constructions remain valid for local wells), which precisely means the following:

There is a unique global minimum  $z_0$  for  $H$ , and it is non-degenerate:

(H1).  $H(z_0) = 0, dH(z_0) = 0, H''(z_0)$  is invertible.  
 Moreover  $\exists E_\infty > 0$  such that  $\{H \leq E_\infty\}$  is compact.

An immediate consequence of this hypothesis is that  $z_0$  is a fixed point, and moreover for any  $E \leq E_\infty$ , trajectories in  $H^{-1}(E)$  are complete. Of course, all trajectories starting close to the bottom of the well will stay forever in a small neighbourhood of it. But this would still hold for *local* minima. The global assumption will ensure that, for the corresponding quantum system, no “tunneling effect” can occur, and all eigenfunctions contributing to the low eigenvalues are indeed localised near  $z_0$ .

### 4. The classical Birkhoff normal form

The Birkhoff normal form is a well known refinement of the averaging method : under a suitable canonical transformation, a perturbation of a harmonic oscillator  $H_2$  can be replaced by its average along the classical Hamiltonian flow generated by  $H_2$ . With the averaging method, this remains valid as long as one restricts the dynamics to times bounded by  $\mathcal{O}(1/\epsilon)$ , where  $\epsilon$  is the size of the perturbation. Using the Birkhoff normal form, this time can be extended to  $\mathcal{O}(1/\epsilon^N)$  for arbitrary  $N$ , provided one takes into account higher order terms which are also averaged, but in a more intricate sense.

Let  $H(x, \xi)$  be a Hamiltonian function in  $C^\infty(\mathbb{R}^{2n})$  with a global non-degenerate well at  $z_0 = 0$ .

Then by a simple argument of symplectic linear algebra, there exist canonical coordinates  $(x, \xi)$  in which  $H = H_2 + \mathcal{O}((x, \xi)^3)$  where  $H_2$  is a *harmonic oscillator*:

$$H_2(x, \xi) = \sum_{j=1}^n \nu_j (x_j^2 + \xi_j^2)/2, \quad \nu_j > 0.$$

The classical Birkhoff normal form is a formal result, in terms of Taylor series, which can be stated as follows.

**Theorem 4.1.** *there exists a formal Taylor series*

$$K = K_3 + K_4 + \dots, \quad \text{with } K_j \text{ homogenous of degree } j \text{ in } (x, \xi)$$

and new formal canonical coordinates  $(x, \xi)$  such that

$$H(x, \xi) = H_2 + K$$

where  $K$  is invariant under the flow of  $H_2$ :  $\{H_2, K\} = 0$ .

### 5. Classical dynamics

What can we get from the Birkhoff normal form ? Of course, one can truncate the Taylor series at an arbitrary order, for the normal form  $K$  itself but also for the

normalising canonical transformation as well. This shows that, for points starting close to the origin,  $K$  is an approximate integral of motion for  $H$ .

Hence the motion (approximately) takes place in a submanifold of dimension  $2n - 2$ , instead of the usual codimension-1 hypersurface given by fixing the energy  $H$ .

In general one can do much better.

- if the frequencies  $\nu_j$  are  $\mathbb{Q}$ -independent then, as a formal series,  $K = f(I_1, \dots, I_n)$  with  $I_j = (x_j^2 + \xi_j^2)/2$ . So we have actually  $n$  independent commuting integral of motion !

The dynamics is (formally) completely integrable and the flow of  $H_2$  winds densely on a Lagrangian  $n$ -torus.

- if the frequencies are completely resonant:  $\nu_j = \nu_c \mathbf{p}_j$  with  $\mathbf{p}_j \in \mathbb{N}$ ,  $\nu_c \in \mathbb{R}^+$ . Then there is no (obvious) integral for  $H$ , apart from  $K^{(1)}$ . But the flow of  $H_2$  is *periodic*, and we can at least use this new symmetry to gain a better understanding of the dynamics.
- Of course there are intermediate cases, where the dimension of the  $\mathbb{Q}$ -vector space generated by  $\nu_1, \dots, \nu_n$  may range from 2 to  $n - 1, \dots$ , and which in principle could be handled by a combination of both techniques. We do not deal with them here.

## 6. Extensions

This well-known Birkhoff normal form still raises fascinating questions.

- *Natural question for a mathematician.* What about *convergence of the series* (for the normal form  $K$  and/or the canonical transformation) ? This question is known to be very difficult, and was already raised by Poincaré. Amongst recent results, one can cite :
  - The canonical transformation is convergent if  $H$  is analytic and integrable [14].
  - The convergence of the normal form itself is more difficult. A partial answer was given in [8] and states that it is “either always convergent or generically divergent”.

We won't be interested here in these convergence issues. For us,  $H$  is any  $C^\infty$  function, and its Birkhoff normal form can safely be considered divergent.

- *Natural question for a physicist.* What about *quantum mechanics* ? Many formal results by physicists and quantum chemists since at least 1960 attest the interest to perform such a normal form in quantum mechanics. It may lead to very accurate numerics, much more accurate than could be achieved by a simple perturbation analysis. Many works show that *resonances* (integer

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<sup>(1)</sup>For instance when  $\nu = (1, 1, 2)$  Duistermaat proved that in general  $H$  cannot be completely integrable [5].

valued relations between the frequencies  $\nu_j$ ) are very important. Even though, from a purist viewpoint, one could consider that in nature no such resonance can occur and that the generic case of non-resonant frequencies is the only relevant one, this would lead to huge problems. When frequencies are *close* to being resonant, then *small denominators* appear in the normal form, making it very rapidly divergent. In many cases, it is much wiser to deal with such systems as perturbations of resonant cases. This phenomenon shows up, for instance, in the so-called *Fermi resonance* for the  $CO_2$  molecule.

The goal of this talk is to explain what kind of rigorous statements one can obtain by using a quantum mechanical Birkhoff normal form. Of course we shall have to give sense to divergent series. This is another reason why the semiclassical limit is so useful : divergent series can still produce asymptotic expansions when  $\hbar \rightarrow 0$ .

## 7. Schrödinger operators with a potential well

Let us return to our Schrödinger operator  $P = -\frac{\hbar^2}{2}\Delta + V(x)$ . Assume  $V$  has a global, non-degenerate minimum at  $x_0 = 0$  with  $V(0) = 0$ .

By a linear, unitary change of variables in local position coordinates  $x = (x_1, \dots, x_n)$  near 0,  $V''(0)$  is diagonal; let  $(\nu_1^2, \dots, \nu_n^2)$  be its eigenvalues, with  $\nu_j > 0$ .

Now, the rescaling  $x_j \mapsto \sqrt{\nu_j}x_j$  transforms  $P$  into a perturbation of the harmonic oscillator  $\hat{H}_2$ :

$$P = \hat{H}_2 + W(x), \quad \text{with } \hat{H}_2 = \sum_{j=1}^n \frac{\nu_j}{2} \left( -\hbar^2 \frac{\partial^2}{\partial x_j^2} + x_j^2 \right),$$

where  $W(x)$  is a smooth potential of order  $\mathcal{O}(|x|^3)$  at the origin. We are thus in position to discuss Birkhoff normal forms.

*In fact, in the article [3] we just assume that  $P$  is a self-adjoint pseudo-differential operator in a standard class with symbol  $p(x, \xi)$  having a global non-degenerate well in the sense of hypothesis **(H1)**. Since we don't use any specific property of the Schrödinger operator, everything still goes through.*

Then the spectrum of  $P$  is *discrete* below  $E_\infty$ : let us denote its eigenvalues by  $\lambda_1^P(\hbar) \leq \lambda_2^P(\hbar) \leq \dots$ .

In 1983, Simon and Helffer-Sjöstrand proved independently that for any fixed  $j$ ,  $\lambda_j^P(\hbar)$  admits an asymptotic expansion in *half-integer* powers of  $\hbar$ . The appearance of these non-integer exponents was a bit of a surprise. They do not appear in dimension 1, and in general they do not appear for non-resonant systems. However they do show up for the Fermi resonance 1 : 2.

## 8. Known results for semi-excited states

The Helffer-Sjöstrand result was not completely satisfactory in the sense that, if one fixes the index  $j$  (in  $\lambda_j^P(\hbar)$ ), then this means that we are looking at a very small energy:  $E = \mathcal{O}(\hbar)$ .

So how to reach higher energies is a natural question if we have physical applications in mind.

The answer is known for *non-resonant frequencies* and is due to Sjöstrand:

**Theorem 8.1** ([11]). *If the  $\nu_i$ 's are independent over  $\mathbb{Q}$ , then*

$$\text{Spec}(P) \cap [0, C\hbar^\delta] = \{f(\hbar(k_1 + \frac{1}{2}), \dots, \hbar(k_n + \frac{1}{2}); \hbar), \quad k_i \in \mathbb{N}\} + O(\hbar^\infty)$$

where  $f = f(I_1, \dots, I_n; \hbar)$  admits an asymptotic expansion in integral powers of  $\hbar$ .

We can thus describe a growing to infinity number of eigenvalues in the limit  $\hbar \rightarrow 0$ , provided the energy does not exceed  $C\hbar^\delta$ . The corresponding states were called *semi-excited states*, understanding that true excited states should be those corresponding to a fixed spectral interval, not shrinking with  $\hbar$ .

## 9. Heuristics

Let us explain heuristically how the Birkhoff normal form is likely to produce such a result as Sjöstrand's theorem, and why the important resonant case was more difficult to handle.

- *The non-resonant case* (where  $K$  is formally completely integrable).

It is fairly easy to understand. The original Hamiltonian  $H$  become  $f(I_1, \dots, I_n)$  under some canonical transformation, up to an error term  $R$ .

Using pseudo-differential operators and the Fourier integral calculus, one can quantise this transformation: then  $P = \hat{H}$  becomes microlocally unitarily equivalent to  $f(\hat{I}_1, \dots, \hat{I}_n)$ . Since the spectrum of  $I_j$  is merely  $\hbar(N + \frac{1}{2})$ ,  $N \in \mathbb{N}$ , this give the expected quasi-eigenvalues, with corresponding quasi-modes.

Then this gives the whole spectrum because  $H$  is confining, which ensures not only that no other eigenvalue might show up, but also that the error term  $R$  must be very small. Techniques for this step are now quite standard in the pseudo-differential setting.

- *The resonant case*. The possibilities for the normal form  $K$  are vast. Essentially, the only thing we have is an  $S^1$  symmetry. Hence one should perform a *reduction*. But this is not easy for several reasons:
  - the  $S^1$  action is in general not free: the orbit space is not a manifold... (in general it has orbifold singularities);
  - unfortunately, pseudo-differential operators are not stable by reduction...

Only these reasons were enough to make the problem look quite unpleasant. The fact that several unsuccessful attempts were made doesn't add any encouragement<sup>(2)</sup>. Nevertheless, related results exist in the literature, which could serve as guides, as the well-known clustering phenomenon for operators with periodic bicharacteristics.

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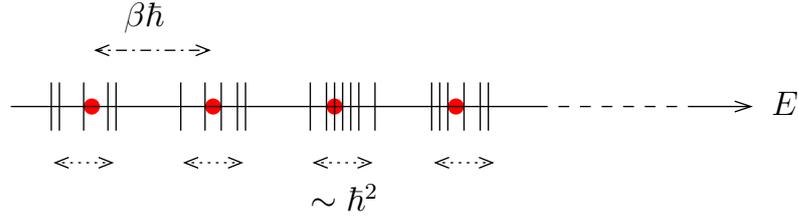
<sup>(2)</sup>For instance, in [12], I could only deal with the case  $\nu_1 = \nu_2 = \dots = \nu_n$ .

## 10. Asymptotics for eigenvalue clusters

There is a long tradition of results for pseudo-differential operators with periodic bicharacteristics, in the hands of Weinstein, Colin de Verdière, Guillemin, Helffer, Dozias, and certainly many others.

The main message is the following:

**Theorem 10.1** ([13], [4], etc.). *Consider a pseudo-differential operator  $P$ . If the Hamiltonian flow of the principal symbol is simply periodic (in particular without fixed points) around a smooth energy hypersurface  $P = E$ , and the subprincipal symbol vanishes there, then the spectrum around  $E$  is clustered on an arithmetic progression  $\alpha + \beta\hbar(N + \gamma)$ ,  $N \in \mathbb{Z}$ , where the width of the cluster is of order  $\mathcal{O}(\hbar^2)$ .*



**Proof.** Using the pseudo-differential calculus, one can perturb  $P$  by adding a smaller order term in such a way that the new operator  $\tilde{P}$  satisfies

$$\exp(-i\tilde{P}/\beta\hbar) = C.Id. \quad \text{See [4].}$$

□

In view of our initial problem, this theorem raises the following questions:

- Does it hold for fixed points (bottom of well) ?
- Can we describe the internal structure of each cluster ?

For the sake of simplicity, let us assume here that  $\nu_c = 1$ , so that all frequencies  $\nu_i$  are positive integers.

Let  $E_N = \hbar(\frac{|\nu|}{2} + N)$ ,  $N \in \mathbb{N}$ , be the eigenvalues of  $\hat{H}_2$ .

Then one of the main results of [3] is the following:

**Theorem 10.2** ([3]). 1. *There exists  $\hbar_0 > 0$  and  $C > 0$  such that for every  $\hbar \in (0, \hbar_0]$*

$$\text{Spec}(P) \cap (-\infty, C\hbar^{\frac{2}{3}}) \subset \bigcup_{E_N \in \text{Spec}(\hat{H}_2)} \left[ E_N - \frac{\hbar}{3}, E_N + \frac{\hbar}{3} \right].$$

2. *When  $E_N \leq C\hbar^{\frac{2}{3}}$ , let  $m(E_N, \hbar) = \#\text{Spec}(P) \cap \left[ E_N - \frac{\hbar}{3}, E_N + \frac{\hbar}{3} \right]$ . Then for  $\hbar < \hbar_0$ ,  $m(E_N, \hbar)$  is precisely the dimension of  $\ker(\hat{H}_2 - E_N)$ .*

3. *Let  $k = k(x, \xi)$  be the average of  $W$  along the flow of  $H_2$ .*

*Let  $S_N \subset \mathbb{R}^{2n}$  be the sphere:  $S_N = \{(x, \xi) \in \mathbb{R}^{2n}, H_2(x, \xi) = E_N\}$ .*

Let  $E_N + \lambda_1(E_N, \hbar), \dots, E_N + \lambda_{m(E_N, \hbar)}(E_N, \hbar)$  be the eigenvalues of  $P$  in this  $N$ -th band in increasing order. Then, uniformly for  $\hbar < \hbar_0$  and  $N$  such that  $E_N \leq C\hbar^{\frac{2}{3}}$ ,

$$\lambda_1(E_N, \hbar) = \inf_{(x, \xi) \in S_N} |k(x, \xi)| + (E_N)^{\frac{3}{2}} \mathcal{O}(N^{-1}),$$

$$\lambda_{m(E_N, \hbar)}(E_N, \hbar) = \sup_{(x, \xi) \in S_N} |k(x, \xi)| + (E_N)^{\frac{3}{2}} \mathcal{O}(N^{-1})$$

and for any function  $g \in C^\infty(\mathbb{R})$ ,

$$\sum_{i=1}^{m(E_N, \hbar)} g\left(\frac{\lambda_i(E_N, \hbar)}{(E_N)^{\frac{3}{2}}}\right) = \left(\frac{1}{2\pi\hbar}\right)^{n-1} \int_{S_N} g\left(\frac{k(x, \xi)}{(E_N)^{\frac{3}{2}}}\right) \mu_{E_N}(x, \xi) + \mathcal{O}(N^{2-n})$$

where  $\mu_{E_N}$  is the Liouville measure of  $S_N$ .

The first point establishes the existence of clusters. The second one relates these clusters to the corresponding eigenspaces of  $\hat{H}_2$ . The last point gives the asymptotics for the spectral density in each cluster. One can remark that these asymptotics behave as if we had a new semiclassical problem restricted to each cluster, whose principal symbol is the averaged perturbation  $k$ .

*Remark:* In our article [3] we explain how to improve the exponent  $2/3$  in the spectral upper bound  $\hbar^{2/3}$  — and the corresponding inverse exponent  $3/2$  in  $(E_N)^{3/2}$  — in some situations.

## 11. The formal Birkhoff normal form

Let us mention now some of the most important steps leading to the above theorem. As usual, the Birkhoff normal form is primarily a formal result, and this can be formulated directly in a quantum setting.

We work with the space

$$\mathcal{E} = \mathbb{C} \llbracket x_1, \dots, x_n, \xi_1, \dots, \xi_n, \hbar \rrbracket,$$

and define the weight of the monomial  $x^\alpha \xi^\beta \hbar^\ell$  to be  $|\alpha| + |\beta| + 2\ell$ . The finite dimensional vector space spanned by monomials of weight  $N$  is denoted by  $\mathcal{D}_N$ . Let  $\mathcal{O}_N$  be the subspace consisting of formal series whose coefficients of weight  $< N$  vanish.  $(\mathcal{O}_N)_{N \in \mathbb{N}}$  is a filtration :

$$\mathcal{E} = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \dots, \quad \bigcap_N \mathcal{O}_N = \{0\}.$$

The space  $\mathcal{E}$  becomes an associative algebra under the Weyl product (the usual Weyl star-product of  $\mathbb{R}^{2n}$ , for which  $\hbar$  is a central element). We consider it as a Lie algebra with the corresponding associative bracket.

The formal quantum Birkhoff normal form can be expressed as follows.

**Theorem 11.1.** *Let  $H_2 \in \mathcal{D}_2$  be the harmonic oscillator (as before) and  $L \in \mathcal{O}_3$ . Then there exists  $A \in \mathcal{O}_3$  and  $K \in \mathcal{O}_3$  such that*

- $e^{i\hbar^{-1} \text{ad}_A}(H_2 + L) = H_2 + K$  ;
- $[K, H_2] = 0$  .

Moreover if  $H_2$  and  $L$  have real coefficients then  $A$  and  $K$  can be chosen to have real coefficients as well.

Here  $\text{ad}_A$  is the endomorphism:  $L \mapsto [A, L]$ , and the series  $e^{i\hbar^{-1}\text{ad}_A}(H_2 + L)$ , which is convergent in the filtration  $(\mathcal{O}_N)$ , can be seen as the formal conjugation of  $H_2 + L$  by  $e^{i\hbar^{-1}A}$ .

The proof is very easy. When one expands the wanted relation in the  $\mathcal{O}_N$ -filtration, one remarks that the formula can be verified inductively if the operator  $i\hbar^{-1}\text{ad}_{H_2}$  restricts to a semisimple endomorphism of each  $\mathcal{D}_N$ . This fact is indeed true, and can be explicitly verified on the basis consisting of the monomials  $\hbar^\ell z^\beta \bar{z}^\gamma$ , with  $z_j = x_j + i\xi_j$ ,  $\bar{z}_j = x_j - i\xi_j$ , thanks to the formula

$$\hbar^{-1}\text{ad}_{H_2}(z^\beta \bar{z}^\gamma) = \langle \beta - \gamma, \nu \rangle z^\beta \bar{z}^\gamma.$$

## 12. The quantum Birkhoff normal form

We are now looking for a concrete, usable version of the Birkhoff normal form. The following version is a refinement of Sjöstrand's statement in [11]; it has the advantage of controlling independently asymptotics both in the energy  $E$  and the semiclassical parameter  $\hbar$ .

**Theorem 12.1** ([3]). *Let  $P$  be a semiclassical self-adjoint pseudo-differential operator of order zero such that the principal symbol  $p$  admits a non-degenerate global well in the sense of (H1), with arbitrary frequencies  $\nu_i > 0$ .*

*Then for any compact domain  $D \subset \mathbb{R}^{2n}$  containing the origin in its interior there exists a pseudo-differential operator  $K$  of order zero such that*

- $[K, \hat{H}_2] = 0$ ;
- $K$  vanishes microlocally outside of  $D$ ;
- the total Weyl symbol  $\sigma_W(K) \in \mathcal{O}_3$ ,

and for each  $\eta > 0$  there exists  $E_0 > 0$ ,  $\hbar_0 > 0$  and for each  $N$  a constant  $C_N > 0$  such that for all  $(\hbar, E) \in [0, \hbar_0] \times [0, E_0]$ ,

$$\left( \lambda_j^P \leq E \text{ or } \lambda_j^Q \leq E \right) \Rightarrow \left| \lambda_j^P - \lambda_j^Q \right| \leq C_N (E^N + \hbar^N),$$

where  $Q = Q((1 + \eta)E) := (\hat{H}_2 + K)_{|\Pi_{(-\infty, (1+\eta)E]}^{\hat{H}_2}(L^2(\mathbb{R}^n))}$ .

Here  $\Pi_J^{\hat{H}_2}$  is the spectral projector of  $\hat{H}_2$  on the interval  $J$ , and  $\lambda_1^Q \leq \lambda_2^Q \leq \dots$  are the eigenvalues of  $Q$ .

Actually, we have assumed in this statement that the subprincipal symbol of  $P$  vanishes, as in the case of the Schrödinger operator. But here it is not a problem to bring it back, since it only affects the result by a shift of the spectrum by the amount  $\hbar H_0$ , where  $H_0$  is the value of the subprincipal symbol at the origin. See [3].

## 13. Applications

Besides being the first step for the proof of theorem 10.2, this quantum Birkhoff normal form has many applications. Let us mention some of them here.

1. One can recover the semi-excited states of Sjöstrand [11] (in the non-resonant case). Indeed, it essentially amounts to apply the theorem with  $E = C\hbar^\gamma$ , for some  $\gamma \in (0, 1)$ .

2. One can find a good approximation of the spectrum using only *polynomial differential operators* (this, in particular, gives a mathematical justification for the accurate numerics of spectroscopists.)

3. One can prove a “semi-excited Weyl law” for the spectral counting function:

$$\mathcal{N}^P(E, \hbar) \sim \frac{1}{(2\pi\hbar)^n} \int_{p \leq E} |dx d\xi|.$$

4. One can recover the Low-Lying Eigenvalues of Helffer-Sjöstrand [6] and Simon [10]. For this purpose, just use the theorem with  $E = C\hbar$ . Then  $Q$  is a finite dimensional matrix (of size independent of  $\hbar$ ), and standard perturbations results for eigenvalues apply. The appearance of non-integer exponents of  $\hbar$  is guaranteed if the normal form  $K$  has a non-vanishing part in  $\mathcal{D}_3$ .

## 14. The Bargmann side

In order to deal with the difficulties mentioned earlier in the case of completely resonant frequencies  $\nu_j$ , we find it appropriate to work in the Bargmann representation, not only for the harmonic oscillator  $\hat{H}_2$ , but also for all pseudo-differential operators, which then become Toeplitz operators.

The *Bargmann space*  $\mathcal{B}_\hbar$  is the space of entire holomorphic functions on  $\mathbb{C}^n$  with finite  $L^2$  norm, with a weighted scalar product:

$$\langle \psi, \psi' \rangle_{\mathbb{C}^n} = \int_{\mathbb{C}^n} (\psi, \psi')(z) \mu(z), \quad \text{with } (\psi, \psi')(z) = \psi(z) \overline{\psi'(z)} e^{-|z|^2/\hbar}$$

where  $|z|^2 = \sum z_i \bar{z}_i$  and  $\mu$  is the Lebesgue measure on  $\mathbb{C}^n = \mathbb{R}^{2n}$ .

Operators on  $L^2(\mathbb{R}^n)$  can be transported on  $\mathcal{B}_\hbar$  via the *Bargmann transform* which is the unitary map  $U_{\mathcal{B}} : L^2(\mathbb{R}^n) \rightarrow \mathcal{B}_\hbar$  given by

$$U_{\mathcal{B}}(\varphi)(z) = \frac{2^{n/4}}{(2\pi\hbar)^{3n/4}} \int_{\mathbb{R}^n} e^{\hbar^{-1}(z \cdot x \sqrt{2} - (z^2 + x^2)/2)} \varphi(x) dx, \quad (1)$$

where  $z \cdot x = \sum z_i x_i$ ,  $z^2 = z \cdot z$ ,  $x^2 = x \cdot x$ .

The harmonic oscillator becomes

$$\hat{H}_2^{\mathcal{B}}(\hbar) := U_{\mathcal{B}} \tilde{H}_2(\hbar) U_{\mathcal{B}}^* = \hbar \sum_{j=1}^n \nu_j \left( z_j \frac{\partial}{\partial z_j} + \frac{1}{2} \right).$$

and  $H_2 = \sum_j \nu_j |z_j|^2$ .

## 15. Toeplitz operators

Using the Bargmann transform, *pseudo-differential operators*  $P$  are transformed into *Toeplitz operators*  $T_g : \mathcal{B}_\hbar \rightarrow \mathcal{B}_\hbar$ ,  $\psi \mapsto \mathbf{\Pi}^{\mathcal{B}}(g\psi)$ , where  $\mathbf{\Pi}^{\mathcal{B}}$  is the orthogonal projector of  $L^2(\mathbb{C}^n, e^{-|z|^2/\hbar})$  onto  $\mathcal{B}_\hbar$ , and  $g = g(\hbar)$  is a function on  $\mathbb{C}^n$  with an asymptotic expansion in powers of  $h$ . The principal term  $g_0$  is just the principal symbol of  $P$ , when we identify  $\mathbb{C}^n$  with  $T^*\mathbb{R}$  by  $z_j = (x_j - i\xi_j)/\sqrt{2}$ .

Let us go back to the resonant case. We have our  $S^1$  action, which is given by the flow of  $H_2$ . Then the essential ingredient of the proof is, roughly, the following:

Let  $K_N : \mathbf{H}^N \rightarrow \mathbf{H}^N$  be the restriction of  $K$  to  $\mathbf{H}^N = \ker(\hat{H}_2 - E_N)$ , the  $N$ -th eigenspace of  $\hat{H}_2$ .

Let  $M = M_{E_N}$  be the symplectic reduced space at energy  $E_N$ :  $M = H_2^{-1}(E_N)/S^1$ . It would be awkward to try and use Toeplitz quantisation directly on  $M_{E_N}$ , since this orbifold shrinks to a point as  $E \rightarrow 0$ . The trick here is to use the homogeneity of the harmonic oscillator : the unitary map  $f(x) \mapsto E^{\frac{n}{4}} f(E^{\frac{1}{2}}x)$  allows us to identify  $\ker(\hat{H}_2(\hbar) - E)$  with  $\ker(\hat{H}_2(\hbar/E) - 1)$ . Thus we may choose  $h = \hbar/E_N \sim 1/N$  as a new semiclassical parameter, and then only consider the reduced space  $M = M_1$ . We can now state the result.

**Theorem 15.1** ([3]).  $K_N$  can be identified to a semiclassical Toeplitz operator on  $M$ :

$$K_N = \mathbf{\Pi}_N g + \mathcal{O}(N^{-\infty})$$

where  $g$  is a smooth function on  $M$  admitting an asymptotic expansion in powers of  $N^{-1}$ .

The technical details needed for a precise formulation of this result are not so important for this talk. The main idea to have in mind is that this gives a strong version of the “quantisation commutes with reduction” principle. Here the symmetry is the  $S^1$  action, the reduced symplectic orbifold is  $M$ , and its quantising Hilbert spaces are simply the  $N$ -th eigenspaces of  $\hat{H}_2$  for  $N \in \mathbb{N}$ . This version is strong in the sense that our whole algebra of pseudo-differential operators commuting with  $\hat{H}_2$  is now reduced to an algebra of Toeplitz operators on  $M$ . The proof of theorem 10.2 can now be seen as an application of “standard” spectral asymptotics for Toeplitz operators, developed in the case of orbifolds by Charles [2].

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