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Résumé

On présente ici une série de travaux dont le but est de décrire les flots géophysiques dans la zone équatoriale, en tenant compte de l'influence prédominante de la rotation de la terre. Pour cela, on procède par approximations successives, en calculant pour chaque modèle la réponse du fluide à la pénalisation par la force de Coriolis. La principale difficulté provient des variations spatiales de l'accélération de Coriolis : en particulier, comme elle s'annule à l'équateur, les oscillations rapides sont piégées dans une fine bande de latitudes.

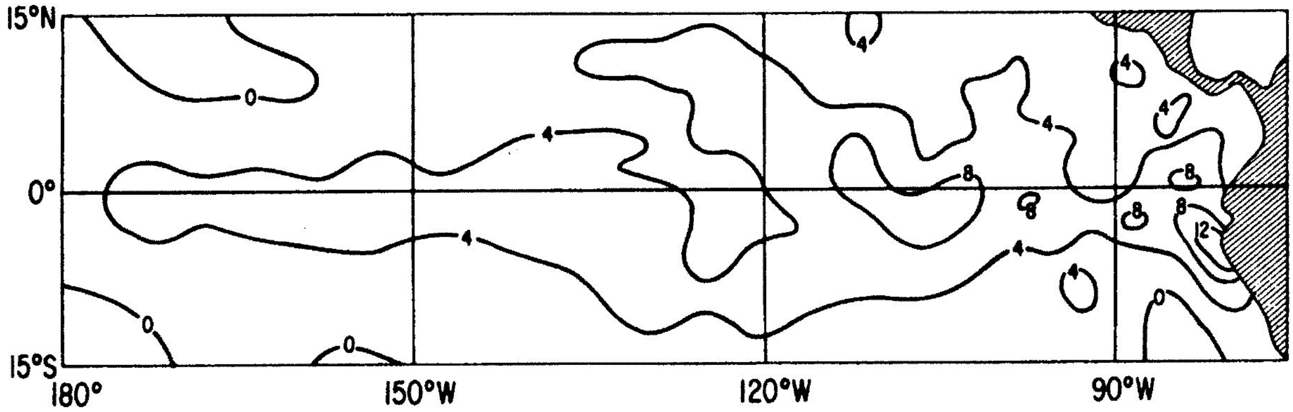
Abstract

We present here a series of works which aims at describing geophysical flows in the equatorial zone, taking into account the dominating influence of the earth rotation. We actually proceed by successive approximations computing for each model the response of the fluid to the strong Coriolis penalisation. The main difficulty is due to the spatial variations of the Coriolis acceleration : in particular, as it vanishes at the equator, fast oscillations are trapped in a thin strip of latitudes.

The present paper is devoted to some recent results about the modelling of geophysical flows. These results have been obtained in a series of joint works with Isabelle Gallagher [3, 4, 5, 6], which aims at describing the motions of the ocean and of the atmosphere in the equatorial zone, taking into account the dominating influence of the earth rotation.

In order to get an overview of the various physical phenomena occuring in such complex dynamics, we proceed by successive approximations computing for each model the response of the fluid to the strong Coriolis penalisation. Actually we will focus here on a very simplified model of oceanography, which we will describe precisely in the next section pointing out for instance the underlying physical and geometrical approximations. We will just discuss in the last part the different corrections to the dynamics we expect to find when these assumptions are relaxed.

The main difficulty in this whole study is due to the spatial variations of the Coriolis acceleration. Indeed it is well-known that a rotating fluid does not adjust to a state of a rest, but rather to a geostrophic equilibrium which is a balance between the Coriolis acceleration and the pressure gradient divided by density [2]. But this adjustment process is somewhat special when the Coriolis acceleration (or more precisely its horizontal component) vanishes. Because of this singularity, equatorial waves are expected to show a decay with respect to latitude [7, 9], which corresponds to the physical observation that the equatorial zone behaves as a waveguide.



The sea surface temperature in the Pacific ocean in december 1972 relative to december 1971.
Contours are in degrees Farenheit.
From Fishing Information, December 1972, No. 12, U.S. Dept. of Commerce,
National Marine Fisheries Service, La Jolla, California.

Equatorial waves are trapped in a thin strip of some hundreds of kilometers around the equator

1. The β-plane approximation

In order to get a suitable description of the oceanic motion, it is crucial to have a precise idea of the general features of the ocean. In view of the typical length scales occuring in oceanography, it is relevant to consider the ocean as an incompressible viscous fluid with free surface, submitted to gravitation. From a mathematical point of view this means that the motion is governed by a system of partial differential equations of Navier-Stokes type, set on a variable domain, and supplemented with some condition on the interface taking into account the capillarity effects. The structure of such a system is thus very complex and requires a sharp analysis. Then, as our goal here is to understand the influence of the Coriolis force in the equatorial zone, we will restrict our attention to a simplified model.

The main simplification is an invariance assumption, leading to the so-called shallow water approximation. We indeed suppose that the flow is essentially bidimensionnal and that it depends only on the horizontal coordinates. The fluid is then completely characterized by its local depth and bulk velocity. Its density is assumed to be homogeneous, and its pressure is given by the hydrostatic law.

The other simplification comes from geometrical approximations, leading to the so-called β-plane approximation. On the one hand, we consider the latitude and longitude (or more precisely the distance northward from the equator and the eastward distance) as cartesian coordinates (x_1, x_2), neglecting the curvature of the earth. On the other hand, we use a linearization of the sinus of the latitude occuring in the Coriolis force.

1.1. A viscous Saint-Venant model

Under these various assumptions, we come down to the study of the following viscous Saint-Venant model :

$$\begin{aligned} \partial_t h + \nabla \cdot (hu) = 0, \\ \partial_t(hu) + \nabla \cdot (hu \otimes u) + \frac{1}{Fr^2} h\nabla h + \beta x_1 (hu)^\perp - \nu \nabla \cdot (h\nabla u) + h\nabla K(h) = 0, \end{aligned} \tag{1.1}$$

where h and u denote as usual the depth and bulk velocity of the fluid, Fr stands for the Froude number, and the coefficient β in the Coriolis term is defined as twice the ratio between the earth angular velocity Ω and the earth radius R.

The first equation expresses the local conservation of mass, whereas the second one gives the local conservation of momentum. The pressure is ruled as agreed by the hydrostatic law, and thus after integration with respect to the vertical coordinate it is given by

$$p = \rho g \frac{h^2}{2}.$$

The viscous effects are supposed to increase with the depth, and in particular to cancel when h vanishes. The last term occuring in the conservation of momentum generates also some dissipation, it comes from the capillarity, modelled by a differential operator K to be specified later.

For such a model, weak solutions are defined globally provided that cavitation is controlled [1].

Theorem 1 (Bresch-Desjardins)**.** *Let $H > 0$ be the reference depth. Consider $(h^0, u^0) \in H^{2\alpha} \times L^2(\mathbf{R} \times \mathbf{T})$ with $\alpha > \frac{1}{2}$ such that*

$$\begin{aligned} \mathcal{E}^0 &= \int \left(\frac{(h^0 - H)^2}{2Fr^2} + \frac{\kappa}{2} |\Delta^\alpha h^0|^2 + \frac{1}{2} h^0 |u^0|^2 \right)(x)dx \\ &< C_0 H^2, \end{aligned} \tag{1.2}$$

where $C_0 = \frac{\kappa}{2} C_{2\alpha}^2$ and $C_{2\alpha}$ is the embedding constant from $H^{2\alpha}$ in L^∞. This ensures in particular that h^0 is bounded from below.

Then there exists $(h,u) \in L^\infty(\mathbf{R}^+, H^{2\alpha} \times L^2(\mathbf{R} \times \mathbf{T}))$ satisfying

- *the viscous Saint-Venant system (1.1) with initial condition (h^0, u^0), and with capillarity*

$$K(h) = \kappa \Delta^{2\alpha} h$$

(preventing the formation of singularities due to cavitation);

- *the energy estimate*

$$\mathcal{E}(t) + \nu \int_0^t \int h |\nabla u|^2 (s,x) dx ds \leq \mathcal{E}^0,$$

where the energy is defined by

$$\mathcal{E}(t) = \int \left(\frac{(h-H)^2}{2Fr^2} + \frac{\kappa}{2} |\Delta^\alpha h|^2 + \frac{1}{2} h |u|^2 \right) (t,x) dx.$$

This existence result is actually a little different from that in [1] since x_1 describes the whole real axis. In particular, getting a uniform bound from below on h requires a better control on its derivatives, which is the reason why we take $\alpha > \frac{1}{2}$ instead of the more physical $\alpha = \frac{1}{2}$.

1.2. Orders of magnitude in the equatorial zone

In the previous paragraph, we have fixed the mathematical framework for our study. Before starting with the asymptotic analysis when the Coriolis force has a dominating influence, let us now consider the physical framework, giving for instance the orders of magnitude of the parameters occuring in System (1.1).

The range of validity of the β-plane approximation is determined by geometrical considerations. We have

$$\left| \frac{\sin \phi}{\phi} - 1 \right| \leq 0.14 \quad \text{if} \quad |\phi| \leq \frac{\pi}{6},$$

meaning that it is relevant to approximate the Coriolis force by a linear function of x_1 provided that x_1 take its values in the strip

$$|x_1| = R|\phi| \leq 3000\, km.$$

In this approximation, the coefficient β measuring the strength of the Coriolis force is given by

$$\beta = \frac{2\Omega}{R} \sim 2 \times 10^{-11} m^{-1} s^{-1},$$

so that the Coriolis radius of deformation at the equator is typically

$$a_e = \left(\frac{\sqrt{gH}}{2\beta} \right)^{1/2} \sim 100\, km.$$

In other words, the range of decay of the equatorial waves is of the order of some hundred of kilometers, which is very small compared with the range of validity of the β-plane approximation. We thus expect the northern and southern boundary conditions not to play an important role, and for the sake of simplicity we consider, as said in the previous paragraph, that x_1 describes the whole real axis.

It remains then to give an estimate of the Froude number. Of course the depth variation is expected to be very small compared with the reference depth. More precisely we will consider depth variations

$$h = H(1 + \varepsilon\eta)$$

where ε stands for the order of magnitude of the Froude number.

In order for gravity waves to be notably modified by rotation effects, the Rossby radius of deformation has to be comparable to the typical horizontal length scales. In order to derive the quasi-geostrophic equations with free-surface term used in oceanography, we will therefore assume that ε is also the order of magnitude of the Rossby number.

In non-dimensional variables, the viscous Saint-Venant system (1.1) can then be rewritten

$$\begin{aligned} \partial_t \eta + \frac{1}{\varepsilon}\nabla \cdot u + \nabla \cdot (\eta u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{1}{\varepsilon}\nabla\eta + \frac{\beta}{\varepsilon}x_1 u^\perp - \nu\Delta u - \frac{\nu}{1+\varepsilon\eta}(\nabla\eta \cdot \nabla)u + \varepsilon\nabla\Delta^{2\alpha}\eta = 0, \end{aligned} \tag{1.3}$$

where β denotes from now on the ratio between the Froude and Rossby numbers.

In order to investigate the influence of the Coriolis force, we are then interested in describing the asymptotic behaviour of this system as ε tends to zero. Of course it is not clear that the use of the shallow water approximation is relevant in this context since the Coriolis force is known to generate vertical oscillations. Nevertheless this very simplified model is commonly used by physicists [7, 10] since it allows to obtain the qualitative features of the horizontal motion, in particular the trapping of equatorial waves.

1.3. Description of the equatorial waves

Before determining properly the asymptotic motion using a multi-scale analysis, let us just exhibit this trapping property on the linear system.

The wave equations have the following structure :

$$\partial_t(\eta, u) + \frac{1}{\varepsilon} L(\eta, u) = 0,$$

with

$$L(\eta, u) = (\nabla \cdot u, \beta x_1 u^\perp + \nabla \eta). \tag{1.4}$$

Note for instance that L is a skew-symmetric operator and that its resolvent would be compact if the coefficient of the rotation did not vanish. We thus expect L to have only purely imaginary discrete spectrum, but its eigenmodes should have a particular behaviour because of the singularity at $x_1 = 0$.

In order to characterize the eigenmode corresponding to some eigenvalue τ, we use a rather standard method. Assume that $u_1 \neq 0$. Then, by Fourier transform with respect to x_2, if $k^2 \neq \tau^2$, we get the following ordinary differential equation

$$-\partial_{11}\hat{u}_1 + \left(k^2 - \tau^2 - \beta\frac{k}{\tau} + \beta^2 x_1^2\right)\hat{u}_1 = 0, \tag{1.5}$$

where $\hat{u}_1$ denotes the coefficient of $\exp(ikx_2)$ in the decomposition of u_1.

Such an equation has a solution in $L^2(\mathbf{R} \times \mathbf{T})$ if and only if the following dispersion relation

$$\begin{gathered} k^2 - \tau^2 - \beta\frac{k}{\tau} = -(2n+1)\beta \text{ holds for some } n \in \mathbf{N}^*, \\ \text{or } \tau^2 + k\tau = 1 \text{ (corresponding to } n = 0). \end{gathered} \tag{1.6}$$

Furthermore the solution is proportionnal to

$$\psi_n(x_1) = \exp\left(-\frac{\beta x_1^2}{2}\right) P_n(x_1\sqrt{\beta})$$

where P_n is the Hermite polynomial of degree n.

For $n \neq 0$ we get in this way three possible values of τ :

$$\tau^3 - (k^2 + \beta(2n+1))\tau + \beta k = (\tau - \tau(k,n,1))(\tau - \tau(k,n,2))(\tau - \tau(k,n,3)), \tag{1.7}$$

with

$$|\tau(k,n,1)|, |\tau(k,n,2)| \to \infty \text{ as } |k| \to \infty,$$

corresponding to the so-called Poincaré waves, and

$$|\tau(k,n,3)| \to 0 \text{ as } |k| \to \infty,$$

corresponding to the so-called Rossby wave.

For $n = 0$ we get in this way two possible values of τ

$$\tau^2 + k\tau - \beta = (\tau - \tau(k, 0, 1))(\tau - \tau(k, 0, 2)) \tag{1.8}$$

corresponding to the mixed Poincaré-Rossby waves, to be supplemented by

$$\tau(k, 0, 3) = -k \tag{1.9}$$

which is associated to the Kelvin wave, i.e. to the wave satisfying

$$u_1 = 0, \quad u_2 = \eta. \tag{1.10}$$

By decomposition on Hermite functions with respect to x_1, we then get a Hilbertian basis of $L^2(\mathbf{R} \times \mathbf{T}, \mathbf{R}^3)$ consisting of eigenvectors of L indexed by

$$n \in \mathbf{N}, \quad k \in \mathbf{Z} \quad \text{and} \quad i \in \{1, 2, 3\}.$$

The rigorous proof of such a result is however technical because of the particularity of the case $n = 0$. Note that the eigenvectors of L show the expected exponential decay with respect to x_1.

2. The fast rotation limit

Consider for all $\varepsilon > 0$ a solution $(\eta_\varepsilon, u_\varepsilon)$ of the scaled viscous Saint-Venant system (1.3). Because of the uniform bounds coming from the energy estimate

$$\int \left(\frac{1}{2}\eta_\varepsilon^2 + \frac{\kappa}{2}\varepsilon^2|\Delta^\alpha \eta_\varepsilon|^2 + \frac{1 + \varepsilon\eta_\varepsilon}{2}|u_\varepsilon|^2\right)(t, x)dx + \nu \int_0^t \int (1+\varepsilon\eta_\varepsilon)|\nabla u_\varepsilon|^2(s, x)dxds < C_0, \tag{2.1}$$

there exists $(\eta, u) \in L^2_{loc}(\mathbf{R}^+, L^2 \times H^1(\mathbf{R} \times \mathbf{T}))$ such that, up to extraction of a subsequence,

$$(\eta_\varepsilon, u_\varepsilon) \rightharpoonup (\eta, u) \text{ in w-}L^2_{loc}(\mathbf{R}^+, L^2 \times H^1(\mathbf{R} \times \mathbf{T})). \tag{2.2}$$

More precisely, considering the structure of the penalized system (1.3)

$$\partial_t(\eta_\varepsilon, u_\varepsilon) + \frac{1}{\varepsilon}L(\eta_\varepsilon, u_\varepsilon) + Q_\varepsilon(\eta_\varepsilon, u_\varepsilon) = 0$$

where Q_ε is some bounded non linear term, we expect $(\eta_\varepsilon, u_\varepsilon)$ to behave asymptotically as

$$(\eta_\varepsilon, u_\varepsilon)(t, x) = (\eta, u)(t, x) + (\eta_{osc}, u_{osc})(\frac{t}{\varepsilon}, t, x) + O(\varepsilon), \tag{2.3}$$

where $(\eta, u)(t, .) \in KerL$ describes the mean motion, and (η_{osc}, u_{osc}) oscillates essentially according to the eigenmodes of L.

2.1. Description of the mean motion

In order to characterize the mean motion, the weak compactness stated previously is sufficient : a compensated compactness argument allows indeed to take limits in the non linear terms (and actually to prove that they do not contribute to the limiting equation), using only the structure of the wave equations [5].

Theorem 2. *Let $(\eta_\varepsilon^0, u_\varepsilon^0)$ be a family of $H^{2\alpha} \times L^2(\mathbf{R}\times\mathbf{T})$ such that*

$$\mathcal{E}_\varepsilon^0 = \int \left(\frac{1}{2}(\eta_\varepsilon^0)^2 + \frac{\kappa}{2}\varepsilon^2 |\Delta^\alpha \eta_\varepsilon^0|^2 + \frac{1+\varepsilon\eta_\varepsilon^0}{2}|u_\varepsilon^0|^2 \right)(x)dx < C_0. \tag{2.4}$$

Consider any family of weak solutions to the scaled β-plane model :

$$\begin{aligned} \partial_t \eta_\varepsilon + \frac{1}{\varepsilon}\nabla\cdot u_\varepsilon + \nabla\cdot(\eta_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \frac{1}{\varepsilon}\nabla\eta_\varepsilon + \frac{\beta}{\varepsilon}x_1 u_\varepsilon^\perp - \nu\Delta u_\varepsilon - \frac{\nu\nabla\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}\cdot u_\varepsilon + \varepsilon\nabla\Delta^{2\alpha}\eta_\varepsilon = 0, \\ (\eta_\varepsilon, u_\varepsilon)_{|t=0} = (\eta_\varepsilon^0, u_\varepsilon^0). \end{aligned} \tag{2.5}$$

Then the following results hold for the asymptotics $\varepsilon \to 0$:

- *The family $(\eta_\varepsilon, u_\varepsilon)$ is weakly compact in $L^2_{loc}(\mathbf{R}^+, L^2 \times H^1(\mathbf{R}\times\mathbf{T}))$.*
- *Any limit point (η, u) satisfies the constraints*

$$u_1 = 0, \quad \partial_2 u_2 = \partial_2\eta = 0, \quad \beta x_1 u_2 + \partial_1\eta = 0, \tag{2.6}$$

in the sense of distributions.

- *The mean motion (η, u) is governed by the heat type equation given in weak formulation :*

for all $(\eta^*, u^*) \in L^2 \times H^1(\mathbf{R})$ such that $u_1^* = 0$ and $x_1 u_2^* + \partial_1\eta^* = 0$,

$$\int (\eta\eta^* + u\cdot u^*)(t,x)dx + \nu\int_0^t\int \nabla u_2\cdot\nabla u_2^*(s,x)dxds = \int(\eta^0\eta^* + u^0 u^*)(x)dx, \tag{2.7}$$

where (η^0, u^0) is the weak limit of $(\eta_\varepsilon^0, u_\varepsilon^0)$ in $L^2 \times L^2(\mathbf{R}\times\mathbf{T})$.

Note that (η^0, u^0) does not necessarily satisfy the constraints (2.6), thus in general

$$(\eta, u)_{|t=0} \neq (\eta^0, u^0).$$

In order to get a strong formulation of the limiting system, we would then have to introduce the L^2 projection Π_0 on the kernel of L (considered as a subspace of $L^2 \times L^2(\mathbf{R}\times\mathbf{T})$), which is a pseudo-differential operator with singularity at $x_1 = 0$, and then to extend it to some space of distributions containing for instance $(0, \Delta u)$. The mean motion would then satisfy

$$\begin{aligned} \partial_t(\eta, u) - \nu\Pi_0(0,\Delta u) = 0 \text{ with the constraint } (\eta, u) = \Pi_0(\eta, u), \\ (\eta, u)_{|t=0} = \Pi_0(\eta^0, u^0). \end{aligned}$$

Sketch of the proof. By the uniform bounds on $(\eta_\varepsilon, u_\varepsilon)$ in $L^2_{loc}(\mathbf{R}^+, L^2 \times H^1(\mathbf{R}\times\mathbf{T}))$ and the upper and lower bounds on $1+\varepsilon\eta_\varepsilon$ in $L^\infty(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$, we get the weak compactness as detailed in the introduction of this section, as well as the convergence

$$L(\eta_\varepsilon, u_\varepsilon) = -\varepsilon\partial_t(\eta_\varepsilon, u_\varepsilon) - \varepsilon Q_\varepsilon(\eta_\varepsilon, u_\varepsilon) \to 0 \tag{2.8}$$

in the sense of distributions.

Thus any limit point (η, u) of $(\eta_\varepsilon, u_\varepsilon)$ belongs to $KerL$, meaning that

$$\nabla \cdot u = 0, \quad \beta x_1 u^\perp + \nabla\eta = 0,$$

which is equivalent to the constraints (2.6) given in the Theorem.

As L is a skew-symmetric operator, for all $(\eta^*, u^*) \in (L^2 \times H^1(\mathbf{R}\times\mathbf{T})) \cap KerL$,

$$\begin{aligned}
&\int (\eta_\varepsilon\eta^* + u_\varepsilon \cdot u^*)(t,x)dx + \int_0^t \int (-\eta_\varepsilon u_\varepsilon \cdot \nabla\eta^* + (u_\varepsilon\cdot\nabla)u_\varepsilon \cdot u^*)(s,x)dxds \\
&+\nu \int_0^t \int \nabla u_\varepsilon : \nabla u^*(s,x)dxds - \nu \int_0^t \int u^* \cdot \frac{1}{1+\varepsilon\eta_\varepsilon}(\nabla(\varepsilon\eta_\varepsilon)\cdot\nabla)u_\varepsilon(s,x)dxds \\
&= \int (\eta^0_\varepsilon\eta^* + u^0_\varepsilon u^*)(x)dx
\end{aligned} \tag{2.9}$$

The weak compactness allows to take limits in the first and third terms of the left-hand side. The last term converges to 0 since, by interpolation,

$$\varepsilon\eta_\varepsilon \to 0 \text{ in } L^\infty_{loc}(\mathbf{R}^+, H^s(\mathbf{R}\times\mathbf{T})) \text{ for } s < 2\alpha.$$

It remains therefore to take limits in the coupling terms, using both the structure of the nonlinearity and the structure of the wave equations.

Using the identity

$$(u\cdot\nabla)u = \nabla\frac{|u|^2}{2} + u^\perp\omega,$$

where $\omega = \nabla^\perp \cdot u$ denotes the vorticity, and the characterization of $KerL$

$$\nabla\eta^* = -\beta x_1 (u^*)^\perp, \quad u_1^* = 0 \text{ and } \partial_2 u_2^* = 0,$$

we rewrite the coupling terms

$$\int_0^t \int (-\eta_\varepsilon u_\varepsilon \cdot \nabla\eta^* + (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot u^*)(s,x)dxds = \int_0^t \int (\beta x_1 \eta_\varepsilon u_{\varepsilon 1} - u_{\varepsilon 1}\omega_\varepsilon)u_2^*(s,x)dxds.$$

The fast oscillations are governed by

$$\begin{aligned}
\varepsilon\partial_t\eta_\varepsilon + \nabla\cdot u_\varepsilon &= O(\varepsilon), \\
\varepsilon\partial_t\omega_\varepsilon + \nabla^\perp\cdot(\beta x_1 u_\varepsilon^\perp + \nabla\eta_\varepsilon) &= O(\varepsilon),
\end{aligned}$$

from which we deduce that

$$\varepsilon\partial_t(\omega_\varepsilon - \beta x_1\eta_\varepsilon) + \beta u_{\varepsilon 1} = O(\varepsilon). \tag{2.10}$$

Formally the coupling terms are then equal to

$$\begin{aligned}
&\int_0^t \int (-\eta_\varepsilon u_\varepsilon \cdot \nabla\eta^* + (u_\varepsilon\cdot\nabla)u_\varepsilon \cdot u^*)(s,x)dxds \\
&= \int_0^t \int (\beta x_1\eta_\varepsilon - \omega_\varepsilon)\Big(\frac{\varepsilon}{\beta}\partial_t(\beta x_1\eta_\varepsilon - \omega_\varepsilon) + O(\varepsilon)\Big)u_2^*(s,x)dxds \\
&= \frac{\varepsilon}{2\beta}\int (\beta x_1\eta_\varepsilon - \omega_\varepsilon)^2 u_2^*(t,x)dx - \frac{\varepsilon}{2\beta}\int(\beta x_1\eta_\varepsilon^0 - \omega_\varepsilon^0)^2 u_2^*(x)dx + O(\varepsilon),
\end{aligned}$$

and thus converge to 0 as $\varepsilon \to 0$. Taking limits in (2.9) leads then to the expected heat equation (in weak formulation).

In order to make the previous compensated compactness argument rigorous (especially to deal with the remainders), we have actually to introduce some regularization $(\eta_\varepsilon^\delta, u_\varepsilon^\delta)$ of $(\eta_\varepsilon, u_\varepsilon)$ such that

$$\begin{gathered}
u_\varepsilon^\delta - u_\varepsilon \to 0 \text{ in } L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{R}\times\mathbf{T})) \\
\omega_\varepsilon^\delta - \omega_\varepsilon \to 0 \text{ and } \eta_\varepsilon^\delta - \eta_\varepsilon \to 0 \text{ in } L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))
\end{gathered} \tag{2.11}$$

uniformly in $\varepsilon > 0$.

The regularized version of (2.10) states then

$$\varepsilon\partial_t(\omega_\varepsilon^\delta - \beta x_1\eta_\varepsilon^\delta) + \beta u_{\varepsilon 1}^\delta = O\left(\frac{\varepsilon}{\delta^s}\right)_{L^2_{loc}(\mathbf{R}^+,L^2(\mathbf{R}\times\mathbf{T}))} + O(\delta)_{L^2_{loc}(\mathbf{R}^+,L^2(\mathbf{R}\times\mathbf{T}))} \tag{2.12}$$

Thus approximating the coupling terms by

$$\int_0^t\int (\beta x_1\eta_\varepsilon^\delta u_{\varepsilon 1}^\delta - u_{\varepsilon 1}^\delta\omega_\varepsilon^\delta)u_2^*(s,x)dxds,$$

and taking limits first as $\varepsilon \to 0$, then as $\delta \to 0$ leads to the expected convergence. □

2.2. Description of the fast oscillations

Describing the corrections to the mean motion requires more sophisticated tools since these corrections involve many time scales. As the system oscillates according to the eigenmodes of L, a filtering method allows actually to get rid of the fast time scale and to get the following strong convergence result [5] :

Theorem 3. *Let $(\eta_\varepsilon^0, u_\varepsilon^0)$ be a family of $H^{2\alpha} \times L^2(\mathbf{R} \times \mathbf{T})$ satisfying the uniform energy estimate (3.22) as well as the strong convergence*

$$(\eta_\varepsilon^0, u_\varepsilon^0) \to (\eta^0, u^0) \text{ strongly in } L^2 \times L^2(\mathbf{R} \times \mathbf{T}). \tag{2.13}$$

Consider any family of solutions to the scaled β-plane model :

$$\begin{aligned} \partial_t \eta_\varepsilon + \frac{1}{\varepsilon}\nabla \cdot u_\varepsilon + \nabla \cdot (\eta_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \frac{1}{\varepsilon}\nabla \eta_\varepsilon + \frac{\beta}{\varepsilon} x_1 u_\varepsilon^\perp - \nu \Delta u_\varepsilon - \frac{\nu}{1+\varepsilon\eta_\varepsilon}(\nabla \varepsilon \eta_\varepsilon \cdot \nabla) u_\varepsilon + \varepsilon \nabla \Delta^{2\alpha} \eta_\varepsilon = 0, \\ (\eta_\varepsilon, u_\varepsilon)_{|t=0} = (\eta_\varepsilon^0, u_\varepsilon^0). \end{aligned} \tag{2.14}$$

Assume that the condition of non resonance (2.15) is satisfied for all $k, k^* \in \mathbf{Z}$, $n, n^*, m \in \mathbf{N}$ and $i, i^*, j \in \{1,2,3\}$ (which is expected to occur except for a countable number of β) :

$$\begin{aligned} &\tau(k,n,i) + \tau(k^*,n^*,i^*) = \tau(k+k^*,m,j) \neq 0 \\ \text{if and only if } \ &\text{either } \tau(k,n,i) = 0 \text{ or } \tau(k^*,n^*,i^*) = 0 \\ &\text{or } n = n^* = m = 0 \text{ and } i = i^* = j = 3, \end{aligned} \tag{2.15}$$

where the eigenvalues $(\tau(k,n,i))_{k,n,i}$ are defined by (1.7)(1.8)(1.9), which means that fast oscillations other than Kelvin waves should not interfer at leading order.

Then the following results hold :

- *The asymptotic behaviour of $(\eta_\varepsilon, u_\varepsilon)$ as $\varepsilon \to 0$ is given by*

$$\left\| (\eta_\varepsilon, u_\varepsilon) - \exp\left(\frac{tL}{\varepsilon}\right)\Psi \right\|_{L^2[0,T], L^2 \times L^2(\mathbf{R} \times \mathbf{T}))} \to 0 \text{ for all } T > 0,$$

where Ψ does not depend on ε, meaning that the system oscillates really according to the eigenmodes of L.

- *The filtered motion $\Psi \in L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{R} \times \mathbf{T}))$ is governed by the heat type equation (which makes sense in weak formulation as in Theorem 2)*

$$\begin{aligned} \partial_t \Psi - \nu \sum_\tau \Pi_\tau \Delta' \Pi_\tau \Psi + \sum_\tau \Pi_\tau Q(\Pi_\tau \Psi, \Pi_0 \Psi) + \Pi_K Q(\Pi_K \Psi, \Pi_K \Psi) = 0, \\ \Psi^0 = (\eta^0, u^0), \end{aligned} \tag{2.16}$$

where Π_τ denotes the projection on the eigenspace associated with the eigenvalue τ, Π_K is the projection on the Kelvin modes, Δ' and Q are respectively the linear and symmetric bilinear operators defined by

$$\Delta'(\Psi_1, \Psi') = (0, \Delta \Psi') \text{ and } Q(\Psi, \Psi) = \Big(\nabla \cdot (\Psi_1 \Psi'), (\Psi' \cdot \nabla)\Psi'\Big).$$

Note that the strong convergence result is expected to hold independently from β when the spatial domain under consideration is $\mathbf{R}^2$. Indeed, the discrete Fourier transform with respect to x_2 is replaced by a continuous Fourier transform, and for $|\xi|$ in a strip bounded from up and below, waves generate dispersion and should vanish in the limit by a Strichartz argument.

Sketch of proof. The method of proof is rather standard once the condition of non resonance is established [8, 11].

The basic idea is to introduce the semi-group $\exp\left(\frac{tL}{\varepsilon}\right)$ generated by the linear penalization L, and to establish the strong convergence of the field $\Psi_\varepsilon = \exp\left(\frac{tL}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$.

Conjugating formally equation (2.14) by the semi-group leads to

$$\partial_t \Psi_\varepsilon + \exp\left(\frac{tL}{\varepsilon}\right)\left(Q\left(\exp\left(-\frac{tL}{\varepsilon}\right)\Psi_\varepsilon, \exp\left(-\frac{tL}{\varepsilon}\right)\Psi_\varepsilon\right) - \nu\Delta' \exp\left(\frac{-tL}{\varepsilon}\right)\Psi_\varepsilon\right) = O(\varepsilon), \tag{2.17}$$

(which makes sense provided to be tested against a finite combination of eigenmodes of L). We therefore get a bound on the time derivative of Ψ_ε in some space of distributions, and, because of the bounds on the spatial derivatives of u_ε coming from the energy estimate, we expect (Ψ_ε) to be strongly compact in $L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{R}))$.

Furthermore a formal passage to the limit in (2.17) (in weak formulation) leads to (2.16) (in weak formulation). Indeed, $\Pi_0\Psi$ is expected to satisfy an autonomous equation because of the compensated compactness argument detailed in the previous paragraph

$$\partial_t \Pi_0 \Psi - \nu \Pi_0 \Delta' \Pi_0 \Psi = 0,$$

the Kelvin modes ae expected to interact together, as well as with $\Pi_0\Psi$ because of the condition of resonance (2.15)

$$\partial_t \Pi_K - \nu \sum_\tau \Pi_K \Delta' \Pi_K \Psi + \sum_\tau \Pi_K Q(\Pi_K\Psi, \Pi_0\Psi) + \Pi_K Q(\Pi_K\Psi, \Pi_K\Psi) = 0,$$

and the other components $\Pi_\tau\Psi$ of Ψ are expected to be only transported by $\Pi_0\Psi$ and diffused.

The rigorous proof of convergence is actually based on a stability property of the limiting equation (2.16). We introduce an approximation $\Psi^{(N)}$ having a finite number N of eigenmodes of the limiting field Ψ (which also satisfies (2.16)). Considering (2.16) as a linear equation with coefficients depending on $\Pi_0\Psi = \Pi_0\Psi^{(N)}$, we get

$$\|\Psi^{(N)} - \Psi\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \to 0 \text{ as } N \to \infty.$$

On the other hand, by integration by parts with respect to time, using the condition of non-resonance for fixed N, we prove that $\Psi^{(N)}$ satisfies (2.14) modulo a small remainder, and therefore by the energy estimate

$$\|\Psi^{(N)} - \Psi^\varepsilon\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \to 0 \text{ as } \varepsilon \to 0, \text{ then } N \to \infty.$$

Combining both convergences leads then to the expected result. $\square$

Remark. *The difficulty here is therefore to study the condition of non resonance, and more precisely to establish that (2.15) holds except for a countable number of β. Let us first recall that the eigenvalues of L are obtained as the roots of the following algebraic equation :*

$$\tau^3 - (k^2 + (2n+1)\beta)\tau + \beta k = 0 \text{ for } n \in \mathbf{N}^*,$$
$$\tau^3 + 2k\tau^2 + (k^2 - \beta)\tau - \beta k = 0 \text{ for } n = 0.$$

In particular, the quantity

$$\mathcal{P}_{n,n^*,m,k,k^*}(\beta) = \Pi_{i,i^*,j\in\{1,2,3\}}(\tau(k,n,i) + \tau(k^*,n^*,i^*) - \tau(k+k^*,m,j))$$

depends only on the symmetric functions respectively of $(\tau(k,n,i))_{i\in\{1,2,3\}}$, $(\tau(k^*,n^*,i^*))_{i^*\in\{1,2,3\}}$ and $(\tau(k+k^*,m,j))_{j\in\{1,2,3\}}$, and is therefore a polynomial with respect to β. Thus it is identically zero or it admits a finite number of roots.

Cases involving $n = k = 0$, or $n^* = k^* = 0$ or $n = n^* = m = 0$ ask for a special care since Kelvin waves are known to provide resonant triads, meaning that $\mathcal{P}_{n,n^*,m,k,k^*} \equiv 0$. We should then introduce polynomials involving less factors to conclude.

In other cases we expect $\mathcal{P}_{n,n^*,m,k,k^*}$ not to be identically zero, which should be proved by considering the asymptotics $\beta \to \infty$. The main technical point is actually to get an equivalent of the factors involving three Rossby waves.

3. The vertical motion

In view of the orders of magnitude given in paragraph 1.2, the geometrical approximations used in the β-plane model seems relevant. On the contrary, the invariance assumption leading to the shallow water approximation seems not to be realistic since experimental observations show the existence of vertical currents in the ocean, and of course in the atmosphere. The Coriolis force is indeed expected to generate vertical oscillations which are completely neglected in the previous study.

To get a more realistic description of the equatorial flows, we should study the fast rotation limit for the 3-dimensional incompressible Navier-Stokes equations with free-surface.

The difficulty is therefore to understand the coupling between the following phenomena

(i) vertical oscillations due to the incompressibility constraint,

(ii) wave trapping due to the singularity at $x_1 = 0$,

(iii) structural changes on the asymptotic system due to the free-surface.

3.1. A preliminary study

In order to separate the problems, we first study the role of vertical oscillations for the 3-dimensional rotating Navier-Stokes system, set in a fixed domain :

$$\partial_t u + (u \cdot \nabla) u + \frac{1}{\varepsilon} x_1 u \wedge e_3 + \nabla p = 0 \text{ with the constraint } \nabla \cdot u = 0. \tag{3.18}$$

As previously, let us start with the description of the corresponding equatorial waves. In this framework wave equations state

$$\partial_t u + \frac{1}{\varepsilon} \mathcal{L} u = 0,$$

with

$$\mathcal{L} u = P(x_1 u \wedge e_3) \tag{3.19}$$

where P denotes as usual the Leray projection on divergence-free vector fields. The penalisation $\mathcal{L}$ is therefore a skew-symmetric operator with a singularity at $x_1 = 0$.

Oscillations are actually of two types, namely purely 2D oscillations corresponding to $k_3 = 0$, and vertical Fourier oscillations ($k_3 \neq 0$). In order to get a suitable formulation of the fast oscillating waves which allows for instance to study their coupling in the nonlinear term, we split systematically the velocity field u as follows : $\bar{u} + \tilde{u}$ where $\bar{u} = \int u dx_3$.

The 2D oscillations are characterized by means of the horizontal vorticity $\bar{\omega} = \nabla_h^\perp \cdot \bar{u}$:

$$\begin{aligned} \partial_t \bar{\omega} + \frac{1}{\varepsilon} \bar{u}_1 = 0 \text{ with } \nabla_h \cdot \bar{u}_h = 0, \\ \partial_t \bar{u}_3 = 0 \end{aligned} \tag{3.20}$$

from which we deduce that

$$\partial_t \bar{\omega} + \frac{1}{\varepsilon} \partial_2 \Delta_h^{-1} \bar{\omega} = 0.$$

In particular the penalisation is skew-symmetric and compact. We therefore get

$$\omega(t, x) = \sum_{k_2 \in \mathbf{Z}^*} \int \exp\left(ik_2 x_2 + i\xi_1 x_1\right) \exp\left(\frac{ik_2}{\xi_1^2 + k_2^2} \frac{t}{\varepsilon}\right).$$

In order to describe vertical oscillations, we have to consider all the components of the rotational $\tilde{\Omega} = \nabla \wedge \tilde{u}$:

$$\partial_t \tilde{\Omega} - \frac{1}{\varepsilon} \tilde{u}_1 e_3 + \frac{1}{\varepsilon} \partial_3 (x_1 \tilde{u}) = 0 \text{ with } \nabla \cdot \tilde{u} = 0. \tag{3.21}$$

By Fourier transform with respect to t/ε, x_2 and x_3, we then get the following ordinary differential equation for $v = \hat{u}_1$:

$$-\partial_{11} v_{k_2,k_3,\tau} + \left(k_2^2 + k_3^2 - \frac{k_2}{\tau} - \frac{k_3^2}{\tau^2} x_1^2 \right) v_{k_2,k_3,\tau} = 0 \text{ if } \tau \neq 0.$$

In order to describe completely the asymptotic behaviour of the Navier-Stokes system penalized by the singular Coriolis force, it should be necessary to first understand the structure of the vertical oscillations, for instance the nature of the spectrum of the operator $-\partial_{11} - (k_3^2/\tau^2)x_1^2$, and the behaviour of the eigenmodes (or generalized eigenmodes) as $|x_1| \to \infty$.

Nevertheless the particular structure of the wave equations, coupled with the structure of the coupling term, allows to already get a very simple description of the mean motion [3] :

Theorem 4. *Let u_ε^0 be a family of $L^2(\mathbf{R} \times \mathbf{T})$ such that*

$$\mathcal{E}_\varepsilon^0 = \frac{1}{2} \int |u_\varepsilon^0|^2(x) dx \leq \mathcal{C}_0 \tag{3.22}$$

for some nonnegative constant $\mathcal{C}_0$.

Consider any family of weak solutions to the scaled Navier-Stokes model :

$$\begin{aligned} \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + + \frac{1}{\varepsilon} x_1 u_\varepsilon^\perp - \nu \Delta u_\varepsilon + \nabla p_\varepsilon = 0 \textit{ with } \nabla \cdot u_\varepsilon = 0, \\ u_{\varepsilon|t=0} = u_\varepsilon^0. \end{aligned} \tag{3.23}$$

Then the following results hold for the asymptotics $\varepsilon \to 0$:

- *Because of the uniform bounds coming from the energy estimate*

$$\frac{1}{2} \int |u_\varepsilon|^2(t,x) dx + \nu \int_0^t \int |\nabla u_\varepsilon|^2(s,x) dx ds \leq \mathcal{C}_0, \tag{3.24}$$

the family (u_ε) is weakly compact in $L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{R} \times \mathbf{T}))$.

- *Any limit point u satisfies the constraints*

$$u_1 = 0, \quad \partial_2 u_2 = \partial_3 u_2 = 0, \quad \partial_3 u_3 = 0, \tag{3.25}$$

in the sense of distributions.

- *The mean motion u is governed by the heat type equation given in weak formulation :*

for all $(u_2^*, u_3^*) \in L^2(\mathbf{R}) \times L^2(\mathbf{R} \times \mathbf{T})$,

$$\int u \cdot u^*(t,x) dx - \int_0^t \int u_3 u_2 \partial_2 u_3^*(s,x) dx ds + \nu \int_0^t \int \nabla u : \nabla u^*(s,x) dx ds = \int u^0 u^*(x) dx, \tag{3.26}$$

where u^0 is the weak limit of u_ε^0 in $L^2(\mathbf{R} \times \mathbf{T}^2)$.

As previously most of the nonlinear terms disappear in the limiting process. This can be understood as some sort of turbulent behaviour, where all scales are mixed due to the variation of the Coriolis coefficient. Technically the result is due to the fact that the kernel of $\mathcal{L}$ (and also of L) is very small as soon as the Coriolis coefficient is not a constant, which induces a lot of rigidity in the limit equation.

Sketch of proof. The method of proof is very similar to that used for Theorem 1. From the energy estimate we deduce the weak compactness statement as well as the convergence $\mathcal{L}u_\varepsilon \to 0$ in the sense of distributions. The constraints come then just from the characterization of Ker$\mathcal{L}$. In order to derive the evolution equation for the mean velocity, we proceed by duality, multiplying (3.23) by some element of $H^1(\mathbf{R} \times \mathbf{T}^2) \cap \mathrm{Ker}\mathcal{L}$ and integrating with respect to x and t. The only point is to check that the (nonlinear) convection term vanishes in the limit.

Using the characterization of the kernel of $\mathcal{L}$ and the identity $(u \cdot \nabla)u = \frac{1}{2}\nabla|u|^2 - u \wedge \Omega$, we rewrite the coupling terms

$$\begin{aligned} &\int_0^t \int (u_\varepsilon \cdot \nabla)u_\varepsilon \cdot u^*(s,x)dxds \\ &= -\int_0^t \int \bar{u}_\varepsilon \wedge \bar{\Omega}_\varepsilon \cdot u^*(s,x)dxds - \int_0^t \int \tilde{u}_\varepsilon \wedge \tilde{\Omega}_\varepsilon \cdot u^*(s,x)dxds \\ &= \int_0^t \bar{u}_{\varepsilon 3}\bar{u}_{\varepsilon h} \cdot \nabla u_3^*(s,x)dxds + \int_0^t \int \bar{\omega}_\varepsilon \bar{u}_\varepsilon^\perp \cdot u^*(s,x)dxds + \int_0^t \int \partial_3 \tilde{u}_\varepsilon \wedge \tilde{\mathcal{O}}_\varepsilon \cdot u^*(s,x)dxds, \end{aligned} \tag{3.27}$$

where $\tilde{\mathcal{O}}_\varepsilon$ is defined by $\partial_3 \tilde{\mathcal{O}}_\varepsilon = \tilde{\Omega}_\varepsilon$ and $\int \tilde{\mathcal{O}}_\varepsilon dx_3 = 0$.

From the energy bound and the control on $\partial_t \bar{u}_{\varepsilon 3}$, we get some strong compactness on $\bar{u}_{\varepsilon 3}$ and obtain the convergence of the first term in (3.27). The second one is proved to converge to 0 using the wave equation

$$\varepsilon \partial_t \bar{\omega}_\varepsilon + \bar{u}_{\varepsilon 1} = O(\varepsilon),$$

and making rigorous the following formal computation :

$$\begin{aligned} \int_0^t \int \bar{\omega}_\varepsilon \bar{u}_\varepsilon^\perp \cdot u^*(s,x)dxds &= \int_0^t \int \bar{\omega}_\varepsilon (\varepsilon \partial_t \bar{\omega}_\varepsilon + O(\varepsilon)) u_2^*(s,x)dxds \\ &= \frac{\varepsilon}{2} \int \bar{\omega}_\varepsilon^2(t,x)dx - \frac{\varepsilon}{2} \int (\bar{\omega}_\varepsilon^0)^2(x)dx + O(\varepsilon). \end{aligned}$$

The proof of convergence for the last term in (3.27) is based on the following formulation of the wave equations

$$\varepsilon \partial_t (x_1 \tilde{\Omega}_\varepsilon + \tilde{\mathcal{O}}_{\varepsilon 1} e_3) + x_1^2 \partial_3 \tilde{u}_\varepsilon = O(\varepsilon).$$

We have indeed

$$\int_0^t\int \partial_3\tilde u_\varepsilon \wedge \tilde{\mathcal O}_\varepsilon \cdot u^*(s,x)dxds = \int_0^t\int (-\frac{\varepsilon}{x_1}\partial_t\partial_3\tilde{\mathcal O}_\varepsilon - \frac{\varepsilon}{x_1^2}\partial_t\tilde{\mathcal O}_{\varepsilon 1}e_3 + O(\varepsilon))\wedge \tilde{\mathcal O}_\varepsilon \cdot u^*(s,x)dxds$$
$$= -\frac{\varepsilon}{2}\int \frac{1}{x_1}\partial_3\tilde{\mathcal O}_\varepsilon \wedge \mathcal O_\varepsilon \cdot u^*(t,x)dx + \frac{\varepsilon}{2}\int \frac{1}{x_1}\partial_3\tilde{\mathcal O}^0_\varepsilon \wedge \mathcal O^0_\varepsilon \cdot u^*(x)dx$$
$$+\frac{\varepsilon}{2}\int \frac{1}{x_1^2}(\tilde O_{\varepsilon 1})^2 u_2^*(t,x)dx - \frac{\varepsilon}{2}\int \frac{1}{x_1^2}(\tilde O^0_{\varepsilon 1})^2 u_2^*(x)dx + O(\varepsilon).$$

The only limiting coupling term corresponds therefore to the convection of the vertical velocity by the horizontal velocity. □

3.2. Some perspectives

Boundary conditions are known to possibly modify the mean motion in the fast rotation limi. This phenomenon is referred to as the Ekman pumping [2]. Free boundaries modify furthermore the wave equations and the global structure of the asymptotic system. In particular the mean motion satisfies different constraints.

A natural question is to understand the role of the free surface in the 3-dimensional fast rotation limit, first in the case of a constant Coriolis coefficient which arises at mid-latitudes considering a small geographical zone. This will have then to be extended to the case of a non singular inhomogeneous Coriolis coefficient, in order to modell portions of the sea located at mid-latitudes but expanding on a large range of latitudes. Our ultimate goal in this direction is to deal with the case of a singular Coriolis coefficient which is relevant for equatorial geophysical flows, and in particular to see if the trapping property observed in two dimensions is still satisfied when vertical oscillations are taken into account.

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Laboratoire J.-L. Lions UMR 7598
Université Paris VI
175, rue du Chevaleret
75013 Paris
FRANCE
saintray@ann.jussieu.fr