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**ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

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# Scattering amplitude for the Schrödinger equation with strong magnetic field

Laurent Michel

**Abstract**

In this note, we study the scattering amplitude for the Schrödinger equation with constant magnetic field. We consider the case where the strengh of the magnetic field goes to infinity and we discuss the competition between the magnetic and the electrostatic effects.

## 1. Introduction

We consider the Schrödinger operator with constant magnetic field

$$\mathrm{H}(b) = \mathrm{H}_0(b) + b^\gamma V$$

with $\gamma \in [0, 1]$ and

$$\mathrm{H}_0(b) = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2 - \Delta_z.$$

Here $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-2}$, $\Delta_z$ denotes the Laplacian on $\mathbb{R}^{n-2}_z$ with $n \geq 3$ and the potential $V$ satisfies the following hypothesis

**Assumption 1.** *There exists* $V^\infty \in C_0^\infty(\mathbb{R}^{n-2})$ *and* $W \in C_0^\infty(\mathbb{R}^n)$ *such that*

$$V(x, y, z) = V^\infty(z) + W(x, y, z)$$

*and* $V, V^\infty \geq 0$.

Under this assumption, it is well-known (see [2]) that the scattering operator $\mathbf{S} = \mathbf{S}(b)$ associated to the pair $(\mathrm{H}_0, \mathrm{H})$ is well defined. The first aim of this note is to show that the scattering matrix $S(E, b)$ can be defined via the spectral representation of $\mathrm{H}_0$ and admits a convenient representation formula away from the point

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spectrum and the Landau levels. The term "matrix" has to be understood differently according to the dimension $n$. In the case $n = 3$, $S(E, b)$ will be a two by two matrix with coefficient in $\mathcal{L}(L^2(\mathbb{R}^2_{x,y}))$, whereas for $n \geq 4$ it will be a unitary operator on $L^2(S^{n-3} \times \mathbb{R}^2_{x,y})$. More precisely, let us denote

$$\widehat{\mathrm{H}}_0 = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2 \tag{1.1}$$

the Schrödinger operator with constant magnetic field on $L^2(\mathbb{R}^2)$. It is well-known that the spectrum of $\widehat{\mathrm{H}}_0$ is pure point [2], [13] and given by the sequence of Landau levels $\sigma(\widehat{\mathrm{H}}_0) = \sigma_{pp}(\widehat{\mathrm{H}}_0) = \{b(2q-1),\ q \in \mathbb{N}^*\}$. Let us denote $\widetilde{\Pi}_q :\ L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the projector on the eigenspace associated to $b(2q-1)$ and define $\mathcal{H}$ by $\mathcal{H} = \mathbb{R}^2$ if $n = 3$ and $\mathcal{H} = L^2(S^{n-3})$ if $n \geq 4$. For $E > 0$, we define

$$\widetilde{\mathcal{F}}_0(E) :\ L^2(\mathbb{R}^{n-2}_z, (1+|z|^2)^{\alpha/2}dz) \to \mathcal{H}$$

by

$$\widetilde{\mathcal{F}}_0(E)\varphi = \frac{E^{-1/4}}{\sqrt{4\pi}} \int_{\mathbb{R}} (e^{-i\sqrt{E}\langle z,\xi\rangle}\varphi(z), e^{i\sqrt{E}\langle z,\xi\rangle}\varphi(z))dz$$

if $n = 3$ and

$$\widetilde{\mathcal{F}}_0(E)\varphi(\xi) = \frac{E^{\frac{n-4}{4}}}{\sqrt{2}(2\pi)^{\frac{n-2}{2}}} \int_{\mathbb{R}^{n-2}} e^{-i\sqrt{E}\langle z,\xi\rangle}\varphi(z)dz,$$

if $n \geq 4$. For $E > b$, we introduce

$$\begin{aligned}\mathcal{F}_0(E) :\ L^2(\mathbb{R}^n,&(1+|z|^2)^{\alpha/2}dxdydz) \to L^2(\mathbb{R}^2_{x,y}, \mathcal{H})\\ &\varphi \mapsto \sum_{b(2q-1)\leq E} \widetilde{\Pi}_q \otimes \widetilde{\mathcal{F}}_0(E - b(2q-1))\varphi\end{aligned}$$

and in all dimensions we define

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^*_+, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$$

by $\mathcal{F}\varphi(E) = \mathcal{F}_0(E)\varphi$. Remark that for $\varphi \in L^2(\mathbb{R}^n)$, $\mathcal{F}_0(E)\varphi$ makes sense only as an $L^2$ function with respect to $E$. The operator $\mathcal{F}$ is a unitary isomorphism and

- $\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*$ is the multiplication by $E$ on $L^2(\mathbb{R}^*_+, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$
- for all $t > 0$, $\mathcal{F}S(b)\mathcal{F}^*$ and $e^{it\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*}$ commute.

Hence (cf. [14], Theorem XIII.84), there exists a function $E \mapsto S(E, b)$ in the space $L^\infty(\mathbb{R}^*_+, \mathcal{L}(L^2(\mathbb{R}^2 \times S^{n-3})))$ such that

$$\forall \varphi \in L^2(\mathbb{R}^n),\ (\mathcal{F}S(b)\varphi)(E) = S(E, b)(\mathcal{F}\varphi)(E).$$

For $E > 0$, $S(E, b)$ is called the scattering matrix (it is a matrix with operator-valued coefficients only in the case $n = 3$).
Our first result consists in a representation formula for the scattering matrix in terms of generalized eigenfunctions. For $B_1, B_2$ two Banach spaces, we denote by $\mathcal{L}(B_1, B_2)$ the space of linear bounded operators from $B_1$ into $B_2$. When $B_1 = B_2$ we simply denote $\mathcal{L}(B_1)$. Under our assumptions on the potential $V$, the scattering matrix can be represented in a convenient way.

**Theorem 1.** *Suppose that Assumption 1 is satisfied and denote by* $\sigma_{pp}(\mathrm{H})$ *the point spectrum of* H*. Then, for all* $E \in ]b, +\infty[\setminus((2\mathbb{N}^* - 1) \cup \sigma_{pp}(\mathrm{H}))$*, one has*

$$\begin{aligned} S(E,b) - Id = &-2i\pi b^{\gamma}\mathcal{F}_0(E)V(x)\mathcal{F}_0(E)^* \\ &+ 2i\pi b^{2\gamma}\mathcal{F}_0(E)V(x)R(E+i0)V(x)\mathcal{F}_0(E)^*, \end{aligned} \tag{1.2}$$

*where*

$$R(E+i0) = \lim_{\mu\to 0^+}(\mathrm{H} - E - i\mu)^{-1}$$

*exists in the space* $\mathcal{B}_\alpha = \mathcal{L}(L^2(\mathbb{R}^n, (1+|z|^2)^{\alpha/2}dxdydz), L^2(\mathbb{R}^n, (1+|z|^2)^{-\alpha/2}dxdydz))$ *for* $\alpha > 1/2$*.*

This representation formula is analogous to the formula arising in absence of magnetic field (see for instance [1] for a proof in the short range case). Some generalisations to less regular and non compactly supported potentials can be founded in [10]. From this representation formula we deduce by a straightforward computation the following

**Corollary 1.1.** *For* $E \in ]b, +\infty[\setminus((2\mathbb{N}^* - 1) \cup \sigma_{pp}(\mathrm{H}))$*,* $T(E,b) := S(E,b) - Id$ *has a kernel*

$$(\omega, \omega') \in S^{n-3} \times S^{n-3} \to T(\omega, \omega', E, b) \in \mathcal{L}(L^2(\mathbb{R}^2)$$

*which is smooth on* $S^{n-3} \times S^{n-3}$*.*

**Remark 1.2.** *Thanks to the form of the potential* $V$*, we can find* $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}_z^{n-2})$ *such that* $\chi_2 V = V$ *and* $\chi_1 = 1$ *on* $\operatorname{supp}\chi_2$*. Then, it was shown in* [10] *that for all* $E \in ]b, +\infty[\setminus(\mathbb{L} \cup \sigma_{pp}(\mathrm{H}))$

$$T(E,b) = -2i\pi\mathcal{F}_0(E)[\Delta_z, \chi_1]R(E+i0)[\Delta_z, \chi_2]\mathcal{F}_0(E)^*. \tag{1.3}$$

An important difference between the Schrödinger operator without magnetic field and the case we study in this article is the following. In the case $A = 0$, the scattering amplitude $T(.,.,E,b)$ is a complex-valued function whereas in our case it takes its values in the space of bounded operators on $L^2(\mathbb{R}^2)$. The aim of this paper is to study this kernel for energies far from the Landau levels in the following sense. We take $\lambda \in [1, +\infty[\setminus(2\mathbb{N}+1)$ and we study the behavior of $T(\omega, \omega', \lambda b, b)$ when $b \to +\infty$. It is important to remark that the distance between the energy $\lambda b$ and the set of Landau levels is proportional to $b$ and so goes to infinity.
Throughout this paper, we will use the semi-classical pseudo-differential calculus in a standard way (see [4]). Let us recall briefly the notations. For $m : \mathbb{R}^d \to [0, +\infty[$ an order function (see Definition 7.5 in [4]) and $\delta \in [0,1]$, we say that $a(x,h) \in C^\infty(\mathbb{R}^d \times ]0,1])$ belongs to the class $S^\delta(\mathbb{R}^d, m, h)$ if

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0,\ |\partial_x^\alpha a(x,h)| \le C_\alpha h^{-\delta|\alpha|}m(x).$$

In the special case where $m = 1$, we write $S^\delta(\mathbb{R}^d, h)$ instead of $S^\delta(\mathbb{R}^d, 1, h)$ and if the semi-classical parameter $h$ is unambiguous we write $S^\delta(\mathbb{R}^d)$. When the symbol $a$ is an operator in $\mathcal{L}(F,G)$ ($F, G$ being subspaces of a Hilbert space), the notation $|.|$ above has to be understood as a norm in $\mathcal{L}(F,G)$ and the associated class of symbols

will be denoted by $S^\delta(\mathbb{R}^d, m, \mathcal{L}(F,G), h)$. We refer to the appendix of [3] for a a nice review of elementary properties of pseudo-differential operators with operator valued symbols. As in [3], for $p$ in a suitable class of symbol, we will denote by $p^w(x, hD_x)$ the standard Weyl-quantization of $p$.
First we state our results in the case $\gamma \in ]0, 1/2[$. In all the paper, we denote $\lambda_q = \lambda - (2q-1)$.

**Theorem 2.** *Suppose that Assumption 1 is satisfied. Let $\gamma \in [0, 1/2[$ and $\lambda \in ]2q_0 - 1, 2q_0 + 1[$ for some $q_0 \in \mathbb{N}^*$. When $b$ tends to infinity, $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$ and*

$$T(\omega, \omega', \lambda b, b) = \frac{ib^{\gamma + \frac{n-4}{2}}}{2(2\pi)^{n-1}} \sum_{q=1}^{q_0} \lambda_q^{\frac{n-4}{2}} \widehat{V}^z(x, y, b^{1/2}\lambda_q^{1/2}(\omega - \omega'))\Pi_q + \mathcal{O}(b^{\frac{n-5}{2}+2\gamma})$$

*in $\mathcal{L}(L^2(\mathbb{R}^2)$.*

**Remark 1.3.** *This result is similar to the high-energy asymptotic for the Schrödinger equation without magnetic field. In particular, it permits to obtain easily some inverse scattering results. We refer to* [10] *for details.*

The proofs of Theorems 1 and 2 can be found in [10]. The remainder of the paper is devoted to the case $\gamma = 1$.
Before setting our theorems, we may discuss the elementary but instructive example where the potential $V$ depends only on $z$. Then $V(z)$ and the projectors $\widetilde{\Pi}_q$ commute, so that

$$R(\lambda b + i0) = \sum_{q \in \mathbb{N}} (-\Delta_z + bV(z) - b(\lambda - (2q-1)) - i0)^{-1} \otimes \widetilde{\Pi}_q.$$

Setting $h = b^{-1/2}$, it follows from (1.3) and the definition of $\mathcal{F}_0(E)$ that

$$T(\omega, \omega', \lambda b, b) = \sum_{2q-1 \leq \lambda} T_L(\omega, \omega', \lambda - (2q-1), h)\widetilde{\Pi}_q$$

where $T_L(\omega, \omega', \mathcal{E}, h)$ denotes the scattering amplitude for the pair $(-h^2\Delta_z, -h^2\Delta_z + V(z))$ at the energy $\mathcal{E}$. This shows why our problem becomes semi-classical. Moreover, it is well-known since the works of Vainberg [16] and Robert-Tamura [15] that the asymptotic behavior of the scattering amplitude when $h$ goes to zero at the energy $\mathcal{E}$ depends on the nature of the classical trajectories on this energy surface.
From now, we assume that $\gamma = 1$ and $\lambda \in ]2q_0 - 1, 2q_0 + 1[$, $q_0 \in \mathbb{N}^*$. For $(x, y) \in \mathbb{R}^2$, we introduce the symbol $p_{x,y}(z, \xi) = \xi^2 + V(x, y, z)$, $\forall (z, \xi) \in T^*\mathbb{R}^{n-2}$. We denote by $H_{p_{x,y}} = \partial_\xi p_{x,y}\partial_z - \partial_z p_{x,y}\partial_\xi$ the associated vector field and $t \mapsto \exp(tH_{p_{x,y}})(z, \xi)$ the solution of the Hamiltonian system

$$\dot{Z} = 2Z^*, \ \dot{Z}^* = -\nabla_z V(x, y, Z) \tag{1.4}$$

with initial condition $(z, \xi)$ at $t = 0$. We make the following non-trapping assumption

**Assumption 2.** *For all $q = 1, \ldots, q_0$ and all $(x, y) \in \mathbb{R}^2$,*

$$\lim_{|t|\to\infty} |\exp(tH_{p_{x,y}})(z, \xi)| = +\infty$$

*for all $(z, \xi) \in T^*\mathbb{R}^{n-2}$ such that $\xi^2 + V(x, y, z) = \lambda_q$.*

Remark that if the $\lambda_q$, $q = 1, \ldots, q_0$, are non-trapping for the symbol $\xi^2 + V^\infty(z)$ in the sense of [15] and $W, \nabla_z W$ are small enough in $C^0$ norm, then Assumption 2 is satisfied.
In dimension $n = 3$, the structure of the classical scattered trajectories is quiet simple and the preceding assumption is sufficient to state a theorem. Let us denote

$$S(E, b) = \begin{pmatrix} S_{11}(E, b) & S_{12}(E, b) \\ S_{21}(E, b) & S_{22}(E, b) \end{pmatrix}$$

the scattering matrix in dimension 3, with $S_{ij}(E, b) \in \mathcal{L}(L^2(\mathbb{R}))$. It is well known that $S_{11}(E, b) = S_{22}(E, b)$ so that we only have to study $S_{11}$, $S_{12}$ and $S_{21}$. From Assumption 2, we deduce easily that there exists $q_1 \in \{1, \ldots, q_0 + 1\}$ such that:
• for all $q \in \{1, \ldots, q_1 - 1\}$ and all $(x, y) \in \mathbb{R}^2$, $\lambda_q > \sup_{z\in\mathbb{R}} V(x, y, z)$
• for all $q \in \{q_1, \ldots, q_0\}$ and all $(x, y) \in \mathbb{R}^2$, the equation $V(x, y, z) = \lambda_q$ has exactly two solutions $\alpha_q(x, y) < \beta_q(x, y)$ and these solutions are non-critical points.
Remark that $\alpha_q$ and $\beta_q$ are smooth functions of the variable $(x, y)$ and are independent on $(x, y)$ when $|(x, y)| >> 1$.

**Theorem 3.** *Suppose that $n = 3$ and that Assumption 2 is fulfilled, then the diagonal coefficients are given by*

$$S_{11}(\lambda b, b) = \sum_{q=1}^{q_1-1} s^w_{d,q}(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\widetilde{\Pi}_q + \sum_{q=q_0+1}^{+\infty} \widetilde{\Pi}_q + \mathcal{O}(b^{-\infty})$$

*in $\mathcal{L}(L^2(\mathbb{R}^2))$, with*

$$s_{d,q}(\lambda, y, \eta) = \exp(ib^{1/2} \int_{-\infty}^{+\infty} \sqrt{\lambda_q - V(\eta, y, z)} - \sqrt{\lambda_q}dz) + b^{-1/2}r_{d,q}(\lambda, y, \eta),$$

*where $r_{d,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$.*
*The off-diagonal coefficients satisfy*

$$S_{21}(\lambda b, b) = \sum_{q=q_1}^{q_0} s^w_{a,q}(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\widetilde{\Pi}_q + \mathcal{O}(b^{-\infty})$$

*in $\mathcal{L}(L^2(\mathbb{R}^2))$, with*

$$s_{a,q}(\lambda, y, \eta) = i\exp(2ib^{1/2}(\sqrt{\lambda_q}\alpha_q(\eta, y) + \int_{-\infty}^{\alpha_q(\eta,y)} \sqrt{\lambda_q - V(\eta, y, z)} - \sqrt{\lambda_q}dz)) \\ + b^{-1/2}r_{a,q}(\lambda, y, \eta),$$

*where $r_{a,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$.*

There is a similar formula for $S_{12}(\lambda b, b)$ that can be found in [11]. On the other hand, notice that thanks to the assumption on the support of $V$ and the properties of $\alpha_q$, $\beta_q$, the symbols $s_{d,q}$ and $s_{a,q}$ belong to the class $S^{1/2}(\mathbb{R}^2, b^{-1})$.

When $n \geq 4$, we need an additional assumption on the classical flow. Now, let us fix $\omega$ and $\omega'$ in $S^{n-3}$ such that $\omega \neq \omega'$. We denote by $\Upsilon_\omega$ the hyper-plane orthogonal to $\omega$, by $\tilde{z} = (z_1, \ldots, z_{n-3})$ the variable in $\Upsilon_\omega$ and we set $\hat{z} = (\tilde{z}, 0) \in \mathbb{R}^{n-2}$. As $V$ is compactly supported in the variable $z$, out of a compact set the solutions of (1.4) are straight lines and it is easy to see that for all $q = 1, \ldots, q_0$ and $\tilde{z} \in \Upsilon_\omega$, there exists a unique solution $(Z_{q,\infty}(t, x, y, \tilde{z}, \omega), Z^*_{q,\infty}(t, x, y, \tilde{z}, \omega))$ of (1.4) such that for $-t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\omega t + \hat{z}$$

Under Assumption 2, we can precise the behavior of these particles when $t$ goes to $+\infty$. Indeed, $V$ being compactly supported with respect to $z$, there exists $\theta_{q,\infty}(x, y, \tilde{z}, \omega) \in S^{n-3}$ and $r_{q,\infty}(x, y, \tilde{z}, \omega) \in \mathbb{R}^{n-2}$ such that for $t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega)t + r_{q,\infty}(x, y, \tilde{z}, \omega) \tag{1.5}$$

$$Z^*_{q,\infty}(t, x, y, \tilde{z}, \omega) = \sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega). \tag{1.6}$$

For $\tilde{z} \in \Upsilon_\omega$, let us define the angular densities by

$$\widehat{\sigma}_q(x, y, \tilde{z}) = |\det(\theta_{q,\infty}, \partial_{z_1}\theta_{q,\infty}, \ldots, \partial_{z_{n-3}}\theta_{q,\infty})| \tag{1.7}$$

**Assumption 3.** *We suppose that for all $q \in \{1, \ldots, q_0\}$, $(x, y) \in \mathbb{R}^2$ and all $\tilde{z} \in \Upsilon_\omega$ with $\theta_{q,\infty}(x, y, \tilde{z}) = \omega'$, we have $\widehat{\sigma}_q(x, y, \tilde{z}) \neq 0$.*

If Assumption 3 is satisfied, we say that the final direction $\omega'$ is regular with respect to the initial direction $\omega$. Once again, let us remark that if we suppose that $W$ and $\nabla_z W$ are small enough, Assumption 3 will be satisfied as soon as for all $q \in \{1, \ldots, q_0\}$, $\omega'$ is regular with respect to $\omega$ at the energy level $\lambda_q$ for the symbol $\xi^2 + V^\infty(z)$ in the sense of [15]. This permits in particular to build potentials satisfying Assumptions 2 and 3.

Assume that the direction $\omega'$ is regular with respect to $\omega$. For $q \in \{1, \ldots, q_0\}$ and $(x, y) \in \mathbb{R}^2$ fixed, we deduce from the implicit function theorem that there exist $l_q = l_q(x, y) \in \mathbb{N}$ and $\tilde{z}_{q,1}(x, y), \ldots, \tilde{z}_{q,l_q}(x, y) \in \Upsilon_\omega$ such that

$$\theta_{q,\infty}(x, y, z) = \omega' \Longleftrightarrow z \in \{\tilde{z}_{q,1}, \ldots, \tilde{z}_{q,l_q}\}.$$

Moreover, we can deduce from the local inverse theorem that the function $(x, y) \mapsto l_q$(x,y) is constant on $\mathbb{R}^2$. We show also that the functions $(x, y) \mapsto \tilde{z}_{q,j}(x, y)$ are $C^\infty$ with respect to $(x, y)$. On the other hand, $W$ being compactly supported they are constant at infinity and therefore belongs to the class $S^0(\mathbb{R}^2)$.

**Theorem 4.** *Suppose that $n \geq 4$ and that $(\omega, \omega') \in S^{n-3} \times S^{n-3}$ with $\omega \neq \omega'$ satisfies Assumptions 2 and 3. Then $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$ and there exists a sequence $T_{q,j}(\omega, \omega', \lambda, b, ., .), q = 1, \ldots, q_0,\ j \in \mathbb{N}$ of symbols in $S^{1/2}(\mathbb{R}^2, b^{-1})$ such that for all $N \in \mathbb{N}$,*

$$T(\omega, \omega', \lambda b, b) = b^{\frac{n-3}{4}} \sum_{q=1}^{q_0} \lambda_q^{\frac{n-3}{4}} T_q(\omega, \omega', \lambda b, b)\widetilde{\Pi}_q + \mathcal{O}(b^{\frac{n-3}{4}-N})$$

*in* $\mathcal{L}(L^2(\mathbb{R}^2))$*, with*

$$T_q(\omega, \omega', \lambda b, b) = \sum_{j=0}^{N} b^{-j/2} T^w_{q,j}(\omega, \omega', \lambda, b, y/2 - b^{-1}D_x, x/2 + b^{-1}D_y).$$

*Moreover,*

$$T_{q,0}(\omega, \omega', \lambda, b, y, \eta) = c(n) \sum_{l=1}^{l_q} \widehat{\sigma}_q(\eta, y, \tilde{z}_{q,l}(\eta, y))^{-1/2} e^{ib^{1/2}\mathbf{S_{q,l}}(y,\eta) - i\mu_{q,l}\pi/2}$$

*where* $c(n) = \frac{1}{2} e^{i(n-3)\pi/4}(2\pi)^{-(n-3)/2}$,

$$\begin{aligned}\mathbf{S_{q,l}}(y, \eta) = \int_{-\infty}^{+\infty} (|Z^*_{q,\infty}(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)|^2 \\ - V(\eta, y, Z_{q,\infty}(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)) - \lambda + 2q - 1)dt \\ - r_{q,\infty}(\eta, y, \tilde{z}_{q,l}(\eta, y), \omega)\end{aligned}$$

*and* $\mu_{q,l}$ *is the Maslov index of the trajectory* $(Z_{q,\infty}, Z^*_{q,\infty})(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)$ *on the Lagrangian manifold*

$$\{(z, \xi) \in T^*\mathbb{R}^{n-2} \mid z = Z_{q,\infty}(t, \eta, y, \tilde{z}, \omega), \xi = Z^*_{q,\infty}(t, \eta, y, \tilde{z}, \omega)), \tilde{z} \in \Upsilon_q,\ t \in \mathbb{R}\}$$

*and* $\mu_{q,l}$ *is independent on* $(y, \eta)$*.*

This result should be compared with the results obtained by Vainberg [16], Robert-Tamura [15] for the Schrödinger equation without magnetic field in the semi-classical setting. In their case, the only variable is $z$ and the scattering amplitude is a function whose principal term is described by the underlying Hamiltonian system. Here we show that the scattering amplitude is a pseudo-differential operator in the variable $(x, y)$ and roughly speaking, the dominating term of its symbol is given by the quantization (with respect to $(x, y)$) of the term of Vainberg, Robert-Tamura for the potential $z \mapsto V(x, y, z)$ where $(x, y)$ is considered as a parameter.

## 2. Reduction by mean of an effective Hamiltonian

The aim of this section is to introduce the effective Hamiltonian associate to the operator $\mathrm{H}(b)$ and the parameter $\lambda$. As usual, this is done by considering a suitable Grushin problem. We refer to [5] and [17] for connected problems involving similar techniques. First we introduce a well-known unitary operator that transforms $\widehat{\mathrm{H}}_0$ into the harmonic oscillator. For $f \in L^2(\mathbb{R}^n)$, let us set

$$\mathrm{U}\, f(x, y, z) = \frac{b^{3/4}}{2\pi} \int_{\mathbb{R}^2} e^{i\varphi_b(x,y,x',y')} f(x', y', z) dx'dy',$$

with $\varphi_b(x, y, x', y') = \frac{b}{2}xy + b^{1/2}(y' - y)x' - bxy'$. Then $\mathrm{U}$ is unitary from $L^2(\mathbb{R}^n)$ onto itself and it is well-known that

$$\mathrm{U}\, \mathrm{H}(b)\, \mathrm{U}^* = -\Delta_z + bN_x + bV^w(b^{-1}D_y + b^{-1/2}D_x, y - b^{-1/2}x, z), \qquad (2.1)$$

where $N_x = -\frac{d^2}{dx^2} + x^2$ is the harmonic oscillator and we use the notation $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. Moreover the right-hand side of (2.1) is self-adjoint with domain $\mathcal{D} = D(N_x) \otimes L^2(\mathbb{R}_y) \otimes H^2(\mathbb{R}_z^{n-2})$, where $D(N_x)$ is the domain of the harmonic oscillator. We refer to [3] and [13] for more details. Now, we introduce the semi-classical parameter $h = b^{-1/2} > 0$, which will tend to 0 as $b$ goes to infinity. Hence, (2.1) takes the form

$$\mathrm{U}\,\mathrm{H}(b)\,\mathrm{U}^* = b(-h^2\Delta_z + N_x + V^w(h^2D_y + hD_x, y - hx, z)). \tag{2.2}$$

Let us introduce the eigenfunctions of the harmonic oscillator. For $j \in \mathbb{N}^*$, we denote by $\phi_j \in L^2(\mathbb{R}_x)$ the function such that $N_x\phi_j = (2j-1)\phi_j$ and $\|\phi_j\|_{L^2(\mathbb{R})} = 1$. We define $R_-$ from $L^2(\mathbb{R}_{y,z}^{n-1})^{q_0}$ into $L^2(\mathbb{R}^n)$ by

$$R_-(\varphi_1, \dots, \varphi_{q_0})(x, y, z) = \sum_{q=1}^{q_0} \varphi_q(y, z)\phi_q(x),$$

and $R_+$ from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}_{y,z}^{n-1})^{q_0}$ by

$$R_+\phi = (\langle\phi, \phi_1\rangle_{L^2(\mathbb{R}_x)}, \dots, \langle\phi, \phi_{q_0}\rangle_{L^2(\mathbb{R}_x)}).$$

Let us denote by $\Pi_q = \mathrm{U}^*\,\widetilde{\Pi}_q \otimes \mathrm{Id}_z\,\mathrm{U}$ the projection onto $Span(\phi_q) \otimes L^2(\mathbb{R}_{y,z}^{n-1})$, $\Pi = \sum_{q=1}^{q_0} \Pi_q$ and $\widehat{\Pi} = \mathrm{Id} - \Pi$. Then we have

$$R_+R_- = \mathrm{Id} \text{ and } R_-R_+ = \Pi \tag{2.3}$$

Let us introduce $F_{q_0} = \{s \in \mathbb{C}, \operatorname{Im} s \geq 0 \text{ and } \operatorname{Re} s = \lambda\}$. For $s \in F_{q_0}$ and $(v, v_-) \in L^2(\mathbb{R}^n) \times \widetilde{\mathcal{D}}^{q_0}$ we can find $(u, u_-) \in \mathcal{D} \times L^2(\mathbb{R}_{y,z}^{n-1})^{q_0}$ which solves the Grushin problem

$$\begin{cases} (\widetilde{P}(h) - s)u + R_-u_- = v \\ R_+u = v_-. \end{cases} \tag{2.4}$$

This permits to obtain the following

**Proposition 2.1.** *Assume that $s \in F_{q_0} \cap \{\operatorname{Im} s > 0\}$, then we have*

$$R_+(\widetilde{P}(h) - s)^{-1}R_- = E_\pm(s)^{-1}. \tag{2.5}$$

*with*

$$E_\pm(s) = R_+(\widetilde{P}(h) - s)R_- - R_+\widetilde{P}(h)\widehat{\Pi}(\widehat{P}(h) - s)^{-1}\widehat{\Pi}\widetilde{P}(h)R_-. \tag{2.6}$$

**Proof.** See [11]. □

The following proposition precises on the structure of the effective Hamiltonian $E_\pm(s)$.

**Proposition 2.2.** *Suppose that $s \in F_{q_0}$. Then, there exists a sequence of matrix valued symbols $(E_j(y, \eta, z, \xi, s))_{j\in\mathbb{N}}$ such that $E_0 \in S^0(\mathbb{R}^{2n-2}, \langle\xi\rangle^2, \mathcal{L}(\mathbb{R}^{q_0}))$, $E_j \in S^0(\mathbb{R}^{2n-2}, 1, \mathcal{L}(\mathbb{R}^{q_0})), \forall j \geq 1$ and for all $N \in \mathbb{N}^*$,*

$$E_\pm(s) = \sum_{j=0}^{N} h^j E_j^w(y, h^2D_y, z, hD_z, s) + R_N(h, s),$$

*with* $\|R_N(h,s)\|_{L^2(\mathbb{R}^{n-1}_{y,z}),L^2(\mathbb{R}^{n-1}_{y,z})} = \mathcal{O}(h^N)$ *uniformly with respect to* $s \in F_{q_0}$*. Moreover, we have*

$$E_0(y,\eta,z,\xi,s) = \mathrm{diag}((\xi^2 + V(\eta,y,z) + 2q - 1 - s)_{q=1,\dots,q_0}),$$

$E_1$ *is off-diagonal and for all* $j \geq 1$ *the semi-norms of* $E_j$ *are bounded uniformly with respect to* $s \in F_{q_0}$*.*

**Proof.** It is based on a Taylor expansion of $\langle W^w(h^2D_y + hD_x, y - hx, z)\phi_p, \phi_q\rangle_{L^2(\mathbb{R}_x)}$ and the construction of a parametrix for $(\widehat{P}(h) - s)^{-1}$.We refer to [11] for details. □

## 3. Resolvent estimates

In this section, we prove that under the non-trapping condition on $\lambda$, $\lambda b$ is not an eigenvalue of $\mathrm{H}(b)$ and we establish a resolvent estimate. The main ingredient is the existence of a global escape function.

**Proposition 3.1.** *Suppose that* $\lambda$ *satisfies Assumption 2 and let* $q \in \{1,\dots,q_0\}$*. Let* $K$ *be a compact subset of* $\cup_{(x,y)\in\mathbb{R}^2} p_{x,y}^{-1}([\lambda_q - \epsilon, \lambda_q + \epsilon])$*. Then, one can find a function* $G_q(x,y,z,\xi) \in C_b^\infty(\mathbb{R}^2_{x,y} \times T^*\mathbb{R}^{n-2}_z)$ *such that* $H_{p_{x,y}}G_q \geq 0$ *and*

$$\forall (x,y) \in \mathbb{R}^2, \forall (z,\xi) \in K,\ H_{p_{x,y}}G_q \geq 1$$

**Proof.** This is a consequence of the existence of an escape function for the symbol $p_{x,y}$ at non-trapping energies [6]. As $V$ depends only on $z$ at infinity, the parameters $(x,y)$ no not pose any problem. □

**Corollary 3.2.** *Suppose that* $\lambda$ *satisfies Assumption 2, then for* $b$ *large enough,* $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$*.*

From the preceding corollary, we know that $\lambda b$ is not an eigenvalue of $\mathrm{H}(b)$. Therefore, the scattering matrix at the energy $\lambda b$ is well defined and satisfies

$$S(\lambda b, b) - Id = -2i\pi\mathcal{F}_0(\lambda b)[\Delta_z, \chi_1]R(\lambda b + i0)[\Delta_z, \chi_2]\mathcal{F}_0(\lambda b)^*, \tag{3.1}$$

where for all $\alpha > 1/2$, $R(\lambda b + i0) = \lim_{\mu\to 0^+}(\mathrm{H}(b) - \lambda b - i\mu)^{-1}$ exists in the space $\mathcal{B}_\alpha$ defined in Theorem 1 (cf. Proposition 2.1 of [10]). Hence, it follows from (2.2) that

$$R(\lambda b + i0) = h^2\,\mathrm{U}^*(\widetilde{P}(h) - \lambda - i0)^{-1}\,\mathrm{U}$$

where $(\widetilde{P}(h) - \lambda - i0)^{-1} = \lim_{\mu\to 0^+}(\widetilde{P}(h) - \lambda - i\mu)^{-1}$ exists in the space $\mathcal{B}_\alpha$. Plugging this expression into (3.1) and using the fact that U acts only on the variables $(x,y)$, we obtain

$$T(\lambda b, b) = -2i\pi h^2\,\mathrm{U}^*\,\mathcal{F}_h(\lambda)[\Delta_z, \chi_1]E_\pm(\lambda)^{-1}[\Delta_z, \chi_2]\mathcal{F}_h(\lambda)^*\,\mathrm{U}, \tag{3.2}$$

where $E_\pm(\lambda) = \lim_{\mu\to 0^+} E_\pm(\lambda + i\mu)$ exists in $\mathcal{L}((L^2_\alpha)^{q_0}, (L^2_{-\alpha})^{q_0})$ for $\alpha > 1/2$ and $\mathcal{F}_h(\lambda):\ L^2_\alpha(\mathbb{R}^{n-1}_{y,z})^{q_0} \to L^2(\mathbb{R}^2_{x,y})$ is defined by $\mathcal{F}_h(\lambda) = \mathcal{F}_0(h^{-2}\lambda)R_-$.

**Proposition 3.3.** *Suppose that Assumption 1 is satisfied, then*

$$\|E_\pm(\lambda)^{-1}\|_{L^2_\alpha(\mathbb{R}^{n-1}_{y,z})^{q_0},L^2_{-\alpha}(\mathbb{R}^{n-1}_{y,z})^{q_0}} = \mathcal{O}(h^{-1})$$

*for all* $\alpha > 1/2$.

**Sketch of proof.** We apply Mourre's method (see [12] and [7] for a semi-classical version) and search a conjugate operator for $\mathcal{P}_{N_0}(\lambda)$ that is an operator $\mathcal{G}$ such that

$$i[\mathcal{P}_{N_0}, \mathcal{G}] \geq Ch$$

when it is localized in energy. It suffices to build a conjugate operator for each $p^w_{q,0}(y, z, h^2D_y, hD_z)$ at the energy $\lambda_q$.
Let $G$ be the escape function built in Proposition 3.1. For $\chi \in C_0^\infty(\mathbb{R})$ localizing near $\{\lambda_q\}$ and all $(y, \xi)$, we have

$$i\chi(p^w_{\eta,y}(z, hD_z))[p^w_{\eta,y}(z, hD_z), G^w(y, \eta, z, hD_z)]\chi(p^w_{\eta,y}(z, hD_z)) \\ \geq h\chi^2(p_{\eta,y}(z, hD_z))$$

From the symbolic calculus (in the variable $y$) with operator-valued symbols, it follows

$$i[p^w_{q,0}(y, z,h^2D_y, hD_z), G^w(y, h^2D_y, z, hD_z)] \\ = i[p^w_{\eta,y}(z, hD_z), G^w(y, \eta, z, hD_z)](y, h^2D_y) + \mathcal{O}(h^2)$$

Combining the two last equations, we obtain the Mourre estimate. □

## 4. Diagonalization of $E_\pm(\lambda)$

From Proposition 2.2, we know that $E_\pm(\lambda)$ is equal to a matrix-diagonal Schrödinger operator plus terms of lower order. The following proposition says that it can be diagonalized at all order.

**Proposition 4.1.** *For all* $N_0 \in \mathbb{N}^*$ *there exists a unitary transformation* $U_{N_0}$ *on* $L^2(\mathbb{R}^{n-1}_{y,z})$ *such that*

$$T(\lambda b, b) = -2i\pi h^{-2}U^*U^*_{N_0}\mathcal{F}_h(\lambda)[h^2\Delta_z, \chi_1]\mathcal{P}_{N_0}(\lambda)^{-1}[h^2\Delta_z, \chi_2]\mathcal{F}_h(\lambda)^*U_{N_0}U + \mathcal{O}(h^{N_0}) \tag{4.1}$$

*with*

$$\mathcal{P}_{N_0}(\lambda) = \mathrm{diag}((p^w_q(y, z, h^2D_y, hD_z, N_0) - \lambda_q)_{q=1,\dots,q_0})$$

*and* $p_q(., N_0) \in S^0(T^*\mathbb{R}^{n-1}, \langle\xi\rangle^2)$. *Moreover,* $p_q(y, z, \eta, \xi, N_0) = \sum_{m=0}^{N_0} h^m p_{q,m}(y, z, \eta, \xi)$ *with* $p_{q,0} = \xi^2+V(\eta, y, z)$, $p_{q,1} = 0$, $p_{q,m} \in S^0(T^*\mathbb{R}^{n-1})$ *for* $m \geq 1$ *and* $p_{q,m}$ *compactly supported with respect to* $z$.

**Sketch of proof.** We just give the main ideas and forget technical points. We restrict also ourselves to prove that we can eliminate the off-diagonal terms of order $h$. We start from formula (3.2) and we search $U_2$ such that

$$U_2 E_\pm(\lambda) U_2^* = \mathcal{P}_2(\lambda) + h^2 R_2(h).$$

We look $U_2$ for under the form

$$U_2 = exp(hu^w(y, z, h^2 D_y, hD_z))$$

with $u = (u_{i,j})_{i,j=1,\dots,q_0}$ such that $u_{i,j} \in S^0(T^*\mathbb{R}^{n-1})$ is real valued and $u_{i,j} = -u_{j,i}$ for all $i, j$. Hence, $U_2$ is clearly unitary, $U_2^* = \exp(-hu^w(y, z, h^2 D_y, hD_z))$ and we get

$$U_2 E_\pm(\lambda) U_2^* = E_0^w(\dots) + +h\left(E_1^w(\dots) + u^w(\dots)E_0^w(\dots) - E_0^w(\dots)u^w(\dots)\right) + \mathcal{O}(h^2),$$

where $(\dots) = (y, z, h^2 D_y, hD_z)$. Hence, we have to find $u$ such that

$$E_1^w(\dots) = E_0^w(\dots)u^w(\dots) - u^w(\dots)E_0^w(\dots) + \mathcal{O}(h^2).$$

A simple matrix calculus combined with symbolic calculus shows that we must solve

$$E_{0,i,i} u_{i,j} - u_{i,j} E_{0,j,j} = E_{1,i,j}$$

for all $i < j = 1, \dots, q_0$, where $(E_{k,i,j})_{i,j}$ denote the coefficients of the matrix $E_k$. We take

$$u_{i,j}(y, z, \eta, \xi) = \frac{1}{2(i-j)} E_{1,i,j}(y, z, \eta, \xi)$$

for $i \neq j$ and $u_{i,i} = 0$ for $i = 1, \dots, q_0$. Then $u$ solves our problem. We refer to [11] for a rigorous proof. □

## 5. Spatial and finite-time localization

The starting point of this section is the following formula

$$\mathcal{P}_{N_0}(\lambda)^{-1} = ih^{-1} \int_0^T e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} dt + \mathcal{P}_{N_0}(\lambda)^{-1} e^{-ih^{-1}T\mathcal{P}_{N_0}(\lambda)}, \tag{5.1}$$

where we would like to neglect the second term of the right hand side. For this purpose, we need a sort of Egorov Lemma for our problem, but we can not apply standard results as our operators are $h$-pseudo in the variable $z$ and $h^2$-pseudo in the variable $y$. Nevertheless, working with operator valued symbol we can establish the results we need. For $t > 0$, let $\phi_t : T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1}$ be defined by

$$\phi_t(y, z, \eta, \xi) = \exp(tH_{p_{\eta,y}})(z, \xi).$$

**Lemma 5.1.** *Let $\omega_1, \omega_2 \in S^0(T^*\mathbb{R}^{n-1})$ such that* $\operatorname{supp}\omega_2 \cap \phi_t(\operatorname{supp}(\omega_1)) = \emptyset$ *and $\omega_1$ is compactly supported, then*

$$\|\omega_2^w(y, z, h^2 D_y, hD_z) e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} \omega_1^w(y, z, h^2 D_y, hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty),$$

*for all $\alpha > 1/2$.*

**Proof.** As $e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} = \mathrm{diag}(e^{-ih^{-1}t(P_q(h)-\lambda)}, q = 1,\dots,q_0)$, it suffices to prove that

$$\omega_2^w(y,z,h^2D_y,hD_z)e^{-ih^{-1}tP_q(h)}\omega_1^w(y,z,h^2D_y,hD_z) = \mathcal{O}(h^\infty).$$

For this purpose we build a parametrix for $e^{-ih^{-1}tP_q(h)}\omega_1^w(y,z,h^2D_y,hD_z)$. As in [4], it suffices to find $F(t) = f^w(t,y,z,h^2D_y,hD_z)$ such that

$$ih\partial_t F(t) = [P_q(h), F(t)] \text{ and } f_{|t=0} = \omega_1. \tag{5.2}$$

We look for $f$ under the form $f = \sum_{j\geq 0} h^j f_j$. From Proposition 4.1, we have $P_q(h) = p_q^w(y,z,h^2D_y,hD_z)$ with $p_q = \sum_{j\geq 0} h^j p_{q,j}(y,z,\eta,\xi)$. Using the $h$-pseudo-differential symbolic calculus with respect to $z$ and $h^2$-pseudo-differential symbolic calculus with respect to $y$, equation (5.2) combined with the asymptotic expansions of $p_q$ and $f$ yields

$$i\partial_t f_n = \sum_{j+m+k+2l=n+1} (p_{q,k}\sharp_j\Box_l f_m - f_m\sharp_j\Box_l p_{q,k}),\ \forall n\in\mathbb{N}, \tag{5.3}$$

where

$$p\sharp_j\Box_l f = \sum_{\alpha+\beta=j}\sum_{\gamma+\delta=l}\frac{(-1)^{\alpha+\gamma}}{(2i)^{j+l}\alpha!\beta!\gamma!\delta!}(\partial_y^\gamma\partial_\eta^\delta\partial_z^\alpha\partial_\xi^\beta p)(\partial_\eta^\gamma\partial_y^\delta\partial_\xi^\alpha\partial_z^\beta f).$$

We solve these transport equations by induction and we show that $\mathrm{supp}\, f_n \subset \phi_{-t}(\mathrm{supp}\,\omega_1)$. For instance, in the case $n = 0$, we have to solve

$$i\partial_t f_0 = p_{q,0}\sharp_1\Box_0 f_0 - f_0\sharp_1\Box_0 p_{q,0} = iH_{p_{\eta,y}} f_0.$$

Combined with the initial condition $f_{0|t=0} = \omega_1$, this yields $f_0 = \omega_1\circ\phi_t$. □

From the preceding lemma, we can deduce a convenient approximated formula for the scattering amplitude. First we introduce the analogous operators to $R_\pm$ on the sphere. Let us denote $R_-^s : L^2(\mathbb{R}_y\times S_\omega^{n-3})^{q_0} \to L^2(\mathbb{R}^2_{x,y}\times S_\omega^{n-3})$ the operator defined by $R_-^s(\varphi_1,\dots,\varphi_{q_0}) = \sum_{q=1}^{q_0}\varphi_q\Phi_q$ and let $R_+^s = (R_-^s)^*$.
As in [8], [15] we can prove micro-local resolvent estimates for our problem (see [11] Lemma 4.1 for a precise statement). Working as in [15], it follows from these estimates, Lemma 5.1 and formulas (4.1) and (5.1) that for $T_1, T_0 > 0$ large enough the scattering amplitude is given by

$$T(\omega,\omega',\lambda b,b) = c_0(h)\,\mathrm{U}^*\int_{T_1}^{T_0}\int_{\mathbb{R}^{n-2}} R_+^s G(z,t,\omega,\omega',\lambda,h)R_-^s\,dzdt\,\mathrm{U} + \mathcal{O}(h^{N_0-2}),$$

with $G(z,t,\omega,\omega',\lambda,h) = (g_{pq}(z,t,\omega,\omega',\lambda,h))_{p,q=1,\dots,q_0}$ and

$$\begin{aligned} g_{pq}(z,t,\omega,\omega',\lambda,h) = \sum_{k=1}^{q_0} & e_{p-}(z,\lambda,\omega',h)a_{pk}^w(\dots) \\ & e^{-ih^{-1}t(p_k^w(\dots)-\lambda_k)}b_{kq}^w(\dots)e_{q+}(z,\lambda,\omega,h). \end{aligned} \tag{5.4}$$

Here, we have used the following notations

$$e_{q,\pm}(z,\lambda,\omega,h) = (\lambda_q)^{\frac{n-4}{4}}e^{\pm ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle},\ c_0(h) = (1/2)h^{-(n-1)}(2\pi)^{-(n-3)} \tag{5.5}$$

and $(a_{p,q})_{p,q}$, $(b_{p,q})_{p,q}$ denote some symbols in the class $S^0$. Moreover, as $z \mapsto e_{p\pm}(z,\lambda,\omega,h)$ is micro-localized near $\xi^2 = \lambda_q$, Lemma 5.1 shows that for $p \neq q$, $\|g_{pq}(\omega,\omega',\lambda,h)\|_{L^2,L^2} = \mathcal{O}(h^\infty)$ uniformly with respect to $t \in [0,T]$ and $z \in \mathbb{R}^{n-2}$, and it follows that

$$T(\omega,\omega',\lambda b,b) = c_0(h)\sum_{q=1}^{q_0} \mathrm{U}^* f_q(\omega,\omega',\lambda,h) \otimes \Pi_q \mathrm{U} + \mathcal{O}(h^{N_0-2}), \tag{5.6}$$

with $f_q(\omega,\omega',\lambda,h) = \int_0^{T_0}\int_{\mathbb{R}^{n-2}} g_{qq}(z,t,\omega,\omega',\lambda,h)dzdt$.

## 6. Approximation of the evolution

Let us denote $P_k(h) = p_k^w(y,z,h^2D_y,hD_z)$, which is self-adjoint on $L^2(\mathbb{R}^{n-1}_{y,z})$ with domain $L^2(\mathbb{R}_y, H^2(\mathbb{R}^{n-2}_z))$. Let $\varphi \in C_0^\infty(\mathbb{R}_y)$ and $\psi(t) \in L^2(\mathbb{R}^{n-1}_{y,z})$ be defined by

$$\psi(t) = e^{-ih^{-1}tP_k(h)}(b_{kq}^w(y,z,h^2D_y,hD_z)e_{q+}(z,\lambda,\omega,h)\varphi(y)).$$

We look for $\psi(t)$ under the form $\tau^w_{q,k,l}(t,y,z,h^2D_y,h)\varphi$ with $\tau_{q,k,l}(t,.) \in S^{1/2}(\mathbb{R}^n_{y,\eta},h^2)$. Forgetting the index $q,k,l$, it comes

$$\begin{cases} ih\partial_t\tau^w(t,y,z,h^2D_y,h) - P_k(h)\tau^w(t,y,z,h^2D_y,h) = 0 \\ \tau(0,y,z,\eta,h) = b_{kq}^w(y,z,\eta,hD_z)e_{q+}(z,\lambda,\omega,h). \end{cases} \tag{6.1}$$

From the symbolic calculus in $S^0(\mathbb{R}^2_{y,\eta},\mathcal{L}(L^2(\mathbb{R}_z)),h^2) \times S^{1/2}(\mathbb{R}^2_{y,\eta},\mathcal{L}(\mathbb{R},L^2(\mathbb{R}_z)),h^2)$ it follows that

$$p_k^w(y,z,h^2D_y,hD_z)\tau^w(t,y,z,h^2D_y,h) = \eta_k^w(t,y,z,h^2D_y,h)$$

with $\eta_k(t,.) \in S^1(\mathbb{R}^n_{y,z,\eta})$ and

$$\eta_k(t,y,z,\eta,h) = \sum_{\alpha,\beta\in\mathbb{N}} \frac{h^{2(\alpha+\beta)}(-1)^\alpha}{(2i)^{\alpha+\beta}\alpha!\beta!}\partial_y^\alpha\partial_\eta^\beta p_k(y,z,\eta,hD_z,h)\partial_\eta^\alpha\partial_y^\beta\tau(t,y,z,\eta,h).$$

Formally, we obtain $\eta_k(t,y,z,\eta,h) = L(y,z,\eta,hD_y,hD_z,hD_\eta,h)\tau(t,y,z,\eta,h)$ with

$$L(y,z,\eta,y^*,z^*,\eta^*) = \sum_{\alpha,\beta,m\in\mathbb{N}} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!}\partial_y^\alpha\partial_\eta^\beta p_{k,m}(y,z,\eta,z^*)(\eta^*)^\alpha(y^*)^\beta.$$

On the other hand, we have also an expansion (in powers of $h$) for the initial data $b_{kq}^w(y,z,\eta,hD_z)e_{q+}(z,\lambda,\omega,h)$. Thus (6.1) gives

$$\begin{cases} (ih\partial_t - L^w(y,z,\eta,hD_y,hD_z,hD_\eta,h))\tau(t,y,z,\eta,h) = 0 \\ \tau(0,y,z,\eta,h) = \sum_{\gamma\in\mathbb{N}} h^\gamma c_{q,k,l,\gamma}(y,z,\eta)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} \end{cases} \tag{6.2}$$

for some explicit symbols $c_{q,k,l,\gamma}$. Hence, our problem is reduced to apply Maslov theory (see [9]) to the preceding system. As in [15], we can suppose that supp $c_{q,k,l,\gamma} \subset \Sigma_{q,l,-}$ with

$$\Sigma_{q,l,-} = \{(y,\eta,Z_{q,\infty}(s,\eta,y,\tilde{z},\omega)),\ (y,\eta,\tilde{z}) \in \mathcal{Z}_{q,l},\ -S_1 < s < -S_0\}$$

and

$$\mathcal{Z}_{q,l} = \{(y,\eta,z) \in \mathbb{R}^2 \times \Upsilon_\omega \,|\, |z - \tilde{z}_{q,l}(\eta,y)| < \epsilon\}$$

for some large $S_1, S_0 > 0$ and small $\epsilon > 0$. For $N \in \mathbb{N}$ to be chosen large enough, let $L_N \in S^0(T^*\mathbb{R}^n, \langle y^*\rangle^N \langle \eta^*\rangle^N)$ be defined by

$$L_N(y,z,\eta,y^*,z^*,\eta^*) = \sum_{|\alpha+\beta+m|\leq N} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!} \partial_y^\alpha \partial_\eta^\beta p_{k,m}(y,z,\eta,z^*)(\eta^*)^\alpha (y^*)^\beta$$

and $\mathcal{L}_N(h) = L_N^w(y,z,\eta,hD_y,hD_z,hD_\eta,h)$. Its principal symbol $l_0$ is given by

$$l_0(y,z,\eta,y^*,z^*,\eta^*) = |z^*|^2 + V(\eta,y,z)$$

so that the corresponding Hamiltonian system is

$$\begin{cases} \dot{Z} = 2Z^*, \quad \dot{Z}^* = -\nabla_z V(\Theta,Y,Z) \\ \dot{Y} = 0, \quad \dot{Y}^* = -\nabla_y V(\Theta,Y,Z) \\ \dot{\Theta} = 0, \quad \dot{\Theta}^* = -\nabla_x V(\Theta,Y,Z) \end{cases} \tag{6.3}$$

Remark that $(Y,Z,\Theta,Z^*)$ is independent of the initial value of $Y^*$ and $\Theta^*$. We denote by $(Z,Z^*)(t,\tilde{\eta},\tilde{y},\tilde{z},\tilde{z}^*)$ the solution of the two first equations of (6.3) such that $(Z,Z^*,Y,\Theta)_{|t=0} = (\tilde{z},\tilde{z}^*,\tilde{y},\tilde{\eta})$. Remark that $(Z,Z^*)(t,\tilde{\eta},\tilde{y},\tilde{z},\tilde{z}^*) = \exp(tH_{p_{\tilde{\eta},\tilde{y}}})(\tilde{z},\tilde{z}^*)$.

**Lemma 6.1.** *For $(\tilde{y},\tilde{z},\tilde{\eta}) \in \Sigma_{q,l,-}$ and $T_1 < t < T_0$, the point $(Y,Z,\Theta)(t,\tilde{y},\tilde{z},\tilde{\eta},\sqrt{\lambda_q}\omega)$ is non-focal in the Maslov sense;*

$$D_q(t,\tilde{y},\tilde{z},\tilde{\eta}) := \det \frac{\partial(Y,Z,\Theta)}{\partial(\tilde{y},\tilde{z},\tilde{\eta})}(\tilde{y},\tilde{z},\tilde{\eta},\sqrt{\lambda_q}\omega) \neq 0.$$

**Proof.** This is a direct consequence of Assumption 3. □

**Proposition 6.2.** *Let $c_l \in C^\infty(\mathbb{R}^n_{y,z,\eta})$ be supported in $\Sigma_{q,l,-}$ for some $l \in \{1,\dots,l_q\}$. Suppose additionally that $\operatorname{supp}_z c_l$ is compact and $(y,\eta) \mapsto c_l(y,z,\eta)$ is constant at infinity. Then there exists some functions $\tau_{q,k,l,j} \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_{y,z,\eta})$, $j \in \mathbb{N}$ such that $\operatorname{supp}_z \tau_{q,k,l,j}$ is compact, $(y,\eta) \mapsto \tau_{q,k,l,j}(y,z,\eta)$ is constant at infinity and*

$$\begin{aligned}(e^{-ih^{-1}t\mathcal{L}_N(h)} c_l e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle})(t,y,z,\eta) = \\ e^{ih^{-1}S_{q,l}(t,y,\tilde{z},\eta) - i\mu_{q,l}\pi/2} |D_q(t,y,\tilde{z},\eta)|^{-1/2} \sum_{j=1}^N h^j \tau_{q,k,l,j}(t,y,\tilde{z},\eta) \\ + h^{N+1} r_N(t,y,z,\eta,h)\end{aligned} \tag{6.4}$$

*with $r_N \in S^{1/2}(\mathbb{R}^n,h^2)$, $z = Z(t,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega)$, $S_{q,l}$ is the action along the trajectory joining $\tilde{z}$ and $z$*

$$\begin{aligned} S_{q,l}(t,y,\tilde{z},\eta) = \int_0^t (|Z^*(s,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega)|^2 - V(\eta,y,Z(s,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega))ds \\ + \sqrt{\lambda_q}\langle \tilde{z},\omega\rangle \end{aligned} \tag{6.5}$$

*and $\mu_{q,l}$ is the path index of this trajectory. Moreover, $\tau_{q,k,l,0}(t,y,\tilde{z},\eta) = c_l(y,\tilde{z},\eta)$ and $\mu_{q,l}$ is independent on $(y,\eta)$.*

**Proof.** We apply Maslov strategy (see [9]) to the preceding system. Using again Lemma 6.1, it is clear that $(y,z) = (y, Z(t,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega),\eta)$ is non-focal in the Maslov sense for $T_1 < t < T_0$ and $(y,\tilde{z},\eta) \in \Sigma_{q,l,-}$ so that we can build an approximate solution $\tau_{ap}$ under the form

$$\tau_{ap}(t,y,z,\eta,h) = e^{ih^{-1}S_{q,l}(t,y,\tilde{z},\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde{z},\eta)|^{-1/2}\sum_{j=0}^{N} h^j \tau_{q,k,l,j}(t,y,\tilde{z},\eta)$$

where $S_{q,l}$ defined by (6.5), $\mu_{q,l}$ is the path index of the trajectory joining $\tilde{z}$ and $z$, $\tau_{q,k,l,j}$ are smooth with respect to $(t,y,z,\eta)$ and independent of large $(y,\eta)$. Moreover, denoting $l_1(y,z,\eta,y^*,z^*,\eta^*)$ the sub-principal symbol of $\mathcal{L}_N(h)$, it follows from Theorem 10.5 in [9] that

$$\tau_{q,k,l,0}(t,y,z,\eta) = \exp\left(\int_0^t M(s)ds\right)\chi(y,\eta)c_l(y,\tilde{z},\eta),$$

with

$$M(s) = \frac{1}{2}\left(\frac{\partial^2 l_0}{\partial z\partial z^*} + \frac{\partial^2 l_0}{\partial y \partial y^*} + \frac{\partial^2 l_0}{\partial\eta\partial\eta^*} - 2l_1\right)((Y,Z,\Theta,Y^*,Z^*,\Theta^*)(s,y,\tilde{z},\eta,\sqrt{\lambda_q}\omega))$$

Using the fact that the sub-principal symbol $p_{q,1}$ of $P_q(h)$ is null, an easy calculation shows that $l_1$ vanishes along the Hamiltonian flow so that $M(s)$ is identically zero, and then

$$\tau_{q,k,l,0}(t,y,z,\eta) = c_l(y,\tilde{z},\eta).$$

Thanks to Assumption 3 and using Definition 7.4 in [9] of the path index, it is clear that $\mu_{q,l}$ is locally constant with respect to $(y,\eta)$.
Moreover, $\tau_{q,k,l,j}$ being compactly supported with respect to $(y,\eta)$, it follows from Theorem 3.15 in [9] that $(ih\partial_t - \mathcal{L}_N(h))\tau_{ap} = h^{N+1}r_{N+1}(t,y,z,\eta)$ with $r_{N+1}$ in $S^{1/2}(\mathbb{R}_t \times \mathbb{R}^n_{y,z,\eta}, h^2)$ and the proof is complete. □

Let us fix $N \in \mathbb{N}$ to be chosen large enough at the end of the paper. From the preceding proposition it follows that one can find some functions $\tau_{q,k,l,j} \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_{y,z,\eta})$, $l = 1,\dots,l_q$, $j = 1,\dots,N$ such that

$$\tau_{q,k,l}(t,y,z,\eta,h) = e^{ih^{-1}S_{q,l}(t,y,\tilde{z},\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde{z},\eta)|^{-1/2}\sum_{j=1}^{N} h^j \tau_{q,k,l,j}(t,y,\tilde{z},\eta)$$

with $z = Z(t,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega)$, satisfies

$$(ih\partial_t - P_k(h))\tau_{q,k,l}(t,y,z,h^2D_y,h) = h^{N+1}r^w_N(t,y,z,h^2D_y,h)$$

with $r_N(t,.) \in S^{1/2}(\mathbb{R}^n_{y,z,\eta}, h^2)$ and $\tau_{q,k,l,0}(t,y,\tilde{z},\eta) = c_{q,k,l,0}(y,\tilde{z},\eta)$. Remark that for large $(y,\eta)$, $S_{q,l}$ and $\tau_{q,k,l,j}$ depend only on $z$. Hence, the symbols $\tau_{q,k,l}$ belongs to $S^{1/2}(\mathbb{R}^n_{y,z,\eta}, h^2)$. Using this approximate solution, we obtain

$$f_q(\omega,\omega',\lambda,h) = \sum_{j=0}^{N} h^j \sum_{k=1}^{q_0}\sum_{l=1}^{l_q}\int_{T_1}^{T_0} e^{ih^{-1}t\lambda_k}\int_{\mathbb{R}^{n-2}_z}\mathcal{G}^w_{q,k,l,j}(t,z,\lambda,y,h^2D_y,h)dtdz + \mathcal{O}(h^N) \tag{6.6}$$

with

$$\begin{aligned}\mathcal{G}_{q,k,l,j}(t,z,y,\eta,h) =& e_{q-}(z,\lambda,\omega',h)a^w_{qk}(y,z,\eta,hD_z)\sigma^w_{q,l,+}(y,\eta,z)\\ & e^{ih^{-1}S_{q,l}(t,y,\tilde{z},\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde{z},\eta)|^{-1/2}\tau_{q,k,l,j}(t,y,\tilde{z},\eta).\end{aligned}$$

and $z = Z(t,\eta,y,\tilde{z},\sqrt{\lambda_q}\omega)$. Using the fact that $V$ is compactly supported with respect to $z$, we can intertwine the integrals in (6.6) so that

$$f_q(\omega,\omega',\lambda,h) = \widetilde{F}^w_q(\omega,\omega',y,h^2D_y,\lambda,h) + \mathcal{O}(h^N)$$

with

$$\widetilde{F}_q(\omega,\omega',y,\eta,\lambda,h) = \sum_{j=0}^{N} h^j \sum_{k=1}^{q_0}\sum_{l=1}^{l_q}\int_{T_1}^{T_0} e^{ih^{-1}t\lambda_k}\int_{\mathbb{R}^{n-2}_z}\mathcal{G}_{q,k,l,j}(t,z,y,\eta,h)dzdt. \tag{6.7}$$

The end of the proof consists to apply the stationary phase method to obtain the main term of the right-hand side of (6.7) and to conjugate this quantity by the unitary transformations we performed. This can be found in [11].

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