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Yu Deng   Nikolay Tzvetkov   Nicola Visciglia

Abstract

We summarize the main ideas in a series of papers ([20], [21], [22], [5]) devoted to the construction of invariant measures and to the long-time behavior of solutions of the periodic Benjamin-Ono equation.

1. Introduction

A lot of papers have been devoted to the Cauchy problem associated with the Benjamin-Ono equation posed on the torus $\mathbb{T}$:

$$\begin{cases} \partial_t u + H\partial_x^2 u + u \cdot \partial_x u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}, \\ u(0, x) = u_0 \in H^s \end{cases} \tag{1.1}$$

where $H^s = H^s(\mathbb{T}; \mathbb{R})$. In particular we quote [13], where it is proved the global well-posedness in the Sobolev spaces H^s, $s \geq 0$. We also refer to [1, 3, 7, 9, 13, 14, 15, 17, 18] for previous results on the topic, as well as to [4] where the Cauchy problem has been studied in functional spaces larger than L^2.

The aim of this paper is to get informations on the long-time behavior of the solutions associated with (1.1). First we introduce some notations. Denote by $\mu_{k/2}$ the gaussian measure induced by the random Fourier series

$$\varphi_{k/2}(x, \omega) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n(\omega)}{|n|^{k/2}} e^{\mathbf{i}nx}. \tag{1.2}$$

In (1.2), $(g_n(\omega))$ is a sequence of centered complex gaussian variables defined on a probability space $(\Omega, \mathcal{A}, p)$ such that $g_n = \overline{g_{-n}}$ and $(g_n(\omega))_{n>0}$ are independent. More precisely, we have that for a suitable constant c, $g_n(\omega) = c(h_n(\omega) + \mathbf{i}l_n(\omega))$, where $h_n, l_n \in \mathcal{N}(0, 1)$ are independent standard real gaussians. It is well–known that $\mu_{k/2}(H^s) = 1$ for every $s < (k-1)/2$, while $\mu_{k/2}(H^{(k-1)/2}) = 0$.

Our main result concerning the long-time behavior of solutions to (1.1), can be summarized in the following theorem.

Theorem 1.1 (See [5],[21], [22]). *Let $k > 1$ be an integer and $s < (k-1)/2$. Then there exists a Borel set $\mathcal{A}_s \subset H^s(\mathbb{T},\mathbb{R})$ with $\mu_{k/2}(\mathcal{A}_s) = 1$, and such that the corresponding solution of the Benjamin-Ono equation with initial data $u_0 \in \mathcal{A}_s$ is* **recurrent**. *More precisely for every $u_0 \in \mathcal{A}_s$ there is a sequence (t_n) (that depends u_0) such that $t_n \overset{n\to\infty}{\to} \infty$ and the corresponding solution of* (1.1) *satisfies* $\lim_{n\to\infty} \|u(t_n) - u_0\|_{H^s} = 0$.

Remark 1.1. We underline that Theorem 1.1 is true also for $k = 1$ (see [4]). The case $k > 3$ has been treated in [21] and [22] . It is worth noticing, as it will be clear in the sequel, that two different and independent type of difficulties appear in the cases $k = 1$ and $k > 3$. Moreover both difficulties meet for $k = 2, 3$. Those cases have been treated in [5].

A trivial consequence of the theorem above is the following deterministic corollary.

Corollary 1.2. *Fix any $s \geq 0$. Then there exists a dense set F_s of H^s such that for every $u_0 \in F_s$ the solution of the Benjamin-Ono equation, with initial condition u_0, is recurrent.*

Remark 1.2. The proof of Corollary 1.2 follows by Theorem 1.1 by choosing $F_s = \mathcal{A}_s$ and by noticing that open sets in H^s have positive measure w.r.t. $\mu_{k/2}$, provided that $s < (k-1)/2$.

Remark 1.3. The main point in our result is that that we get a recurrence property for data which are **not small** and which are **not of low regularity**.

The proof of Theorem 1.1 (and hence Corollary 1.2) follows by the classical **Poincaré Recurrence Theorem** provided that we can construct a family of measures absolutely continuous w.r.t. to $\mu_{k/2}$, which are invariant along the flow associated with (1.1). Hence the main body of the paper is devoted to the construction of those invariant measures. More precisely the proof of Theorem 1.1 follows from the following one.

Theorem 1.3. *Let us fix $k > 1$ and $R > 0$, then there exists*

$$F_{k/2,R}(u) \in L^q(d\mu_{k/2}), \ \forall q \in [1,\infty)$$

such that the measure $d\rho_{k/2,R} = F_{k/2,R}(u)d\mu_{k/2}$ is invariant along the flow associated with (1.1). *Moreover we have the property:*

$$\bigcup_{R>0} \operatorname{supp}(\rho_{k/2,R}) = \operatorname{supp}(\mu_{k/2}).$$

It is worth noticing that a nontrivial part of Theorem 1.3 is the construction of the densities $F_{k/2,R}(u)$, whose existence follows from some delicate probability arguments. For more details see Theorem 2.1 and its proof. The theory of PDEs, combined again with suitable probability arguments, plays a more crucial role along the proof of the invariance of the measures.
The main part of the paper is devoted to the proof of Theorem 1.3.

Comparison with KdV

It is worth noticing that in the case of the KdV equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + u \cdot \partial_x u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}, \\ u(0,x) = u_0 \in H^s \end{cases} \tag{1.3}$$

much stronger informations are available on the long time behavior of solutions. In fact by combining [2] and [12] one can deduce that the KdV flow is **almost periodic** in time for every initial data $u_0 \in H^s$ with $s \geq 0$.
Namely for every solution u to the KdV equation and for every $\varepsilon > 0$ there exists an almost period l_ε (that depends on the solution u) such that for every interval I of size $\geq l_\varepsilon$ there exists $\tau \in I$ such that $\|u(t+\tau) - u(t)\|_{H^s} < \varepsilon$ for every $t \in \mathbb{R}$.
In particular the KdV flow is **recurrent** for every initial data $u_0 \in H^s$, $s = 0, 1, 2, ...$
In view of this strong result available for the KdV flow, Theorem 1.1 and Corollary 1.2 leave open the following questions:

- **Question 1**: Can we take $F_s = H^s$ in Corollary 1.2?

- **Question 2**: Is the flow associated with (1.1) almost periodic in time, at least for small data?

2. Strategy of the proof of Theorem 1.3

The main point is the construction of measures absolutely continuous w.r.t. to $\mu_{k/2}$, which are invariant along the flow of (1.1). The construction of those invariant measures is based on the existence of infinitely many conservation laws preserved along the Benjamin-Ono flow.

2.1. Conservation Laws for the Benjamin-Ono Equation

There is an infinite sequence of conservation laws satisfied by the solutions of the Benjamin-Ono equation (see [11]). More precisely if u is a smooth solution of BO then :

$$\frac{d}{dt} E_{k/2}(u(t)) = 0, \quad k = 0, 1, 2, 3, \dots$$

where

$$E_{k/2}(u) = \| |\partial_x|^{k/2} u \|_{L^2}^2 + R_{k/2}(u), \tag{2.1}$$

and $R_{k/2}(u)$ is a sum of terms homogenous in u of order ≥ 3 (but containing less derivatives). Here is the list of the first conservation laws :

$$
\begin{aligned}
E_0(u) &= \|u\|_{L^2}^2;\\
E_{1/2}(u) &= \|u\|_{\dot H^{1/2}}^2 + \frac{1}{3}\int u^3 dx;\\
E_1(u) &= \|u\|_{\dot H^1}^2 + \frac{3}{4}\int u^2 H(u_x)dx + \frac{1}{8}\int u^4 dx;\\
E_{3/2}(u) &= \|u\|_{\dot H^{3/2}}^2 - \int [\frac{3}{2}u(u_x)^2 + \frac{1}{2}u(Hu_x)^2]dx\\
&\quad - \int [\frac{1}{3}u^3 H(u_x) + \frac{1}{4}u^2 H(uu_x)]dx - \frac{1}{20}\int u^5 dx;\\
E_2(u) &= \|u\|_{\dot H^2}^2 - \frac{5}{4}\int [(u_x)^2 Hu_x + 2uu_{xx}Hu_x]dx\\
&\quad + \frac{5}{16}\int [5u^2(u_x)^2 + u^2 H(u_x)^2 + 2uH(\partial_x u)H(uu_x)]dx\\
&\quad + \int [\frac{5}{32}u^4 H(u_x) + \frac{5}{24}u^3 H(uu_x)]dx + \frac{1}{48}\int u^6 dx.
\end{aligned}
$$

2.2. Construction of Candidate Invariant Measures $d\rho_{k/2,R}$

For $N \geq 1$, $k \geq 0$ and $R > 0$ we introduce the function

$$
F_{k/2,N,R}(u) = \Big(\prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N u))\Big)\chi_R(E_{(k-1)/2}(\pi_N u) - \alpha_N)e^{-R_{k/2}(\pi_N u)}, \qquad (2.2)
$$

where:

- $E_{j/2}$ for $j = 0, ..., k-1$ are the conservation laws described above and $R_{k/2}$ is the the conservation law $E_{k/2}$ without the quadratic part (see (2.1));
- $\alpha_N = \sum_{n=1}^{N} \frac{c}{n}$ for a suitable constant c;
- π_N is the sharp Dirichlet projector, i.e. $\pi_N(\sum_{n\in\mathbb{Z}} c_n e^{nix}) = \sum_{|n|\leq N} c_n e^{inx}$;
- χ_R is a cut-off function defined as $\chi_R(x) = \chi(x/R)$ with $\chi : \mathbb{R} \to \mathbb{R}$ a smooth, compactly supported function such that $\chi(x) = 1$ for every $|x| < 1$.

We have the following result.

Theorem 2.1 (See [19] for $k = 1$ and [20] for $k > 1$)**.** *For every $k = 1, 2, ...,$ for every $R > 0$ there exists a $\mu_{k/2}$ measurable function $F_{k/2,R}(u)$ such that $F_{k/2,N,R}(u)$ converges to $F_{k/2,R}(u)$ in $L^q(d\mu_{k/2})$ for every $1 \leq q < \infty$. In particular $F_{k/2,R}(u) \in L^q(d\mu_{k/2})$. Moreover, if we set $d\rho_{k/2,R} \equiv F_{k/2,R}(u)d\mu_{k/2}$ then we have*

$$
\bigcup_{R>0} \mathrm{supp}(\rho_{k/2,R}) = \mathrm{supp}(\mu_{k/2}).
$$

Remark 2.1. We recall that the idea to construct invariant measures for Hamiltonian PDEs by exploiting the associated conservation laws goes back to [6] in the context of NLS. In this paper the authors exploit the conservation of the Hamiltonian. At the best of our knowledge the idea to construct invariant measures associated with higher order conservation laws goes back to the work of Zhidkov (see [23]). The main novelty (and source of difficulty) in Theorem 2.1, compared with [23], is that in the definition of $F_{k/2,N,R}$ we introduce the renormalizing coefficient α_N along the truncation with the conservation law $E_{(k-1)/2}$. This is necessary in order to guarantee that the limit measures are non-trivial. As a consequence, despite to [23], in Theorem 2.1 the density $F_{k/2,R}(u)$ belongs to $L^q(d\mu_{k/2})$ for $q \in [1,\infty)$, but it does not belong to $L^\infty(d\mu_{k/2})$.

2.3. Invariance of the Measures $d\rho_{k/2,R}$

In order to prove the invariance of $d\rho_{k/2,R}$ along the flow associated with (1.1) we have two key steps:

- the first one is to prove the convergence of solutions to the finite dimensional approximations of (1.1):

$$\begin{cases} \partial_t u + H\partial_x^2 u + \pi_N\Big((\pi_N u)\cdot \partial_x(\pi_N u)\Big) = 0, \\ u(0,x) = u_0 \end{cases} \tag{2.3}$$

  to the true solution of (1.1), as $N \to \infty$. More precisely, if we denote by $\Phi_t^N(u_0)$ the unique global solution to (2.3) and by $\Phi_t(u_0)$ the unique global solution to (1.1) at time t, then the following estimates are needed:

$$\exists s < \sigma < (k-1)/2 \text{ s.t. } \forall\, S > 0,\ \exists\, \bar t = \bar t(S) > 0 \text{ s.t. } \forall\, \varepsilon > 0,$$
$$\Phi_t^N(A) \subset \Phi_t(A) + B^s(\varepsilon),\ \forall\, N > N_0(\varepsilon), \forall\, t \in (-\bar t, \bar t), \forall A \subset B^\sigma(S), \tag{2.4}$$

  where $B^\sigma(R)$ denotes the ball of radius R in H^σ. The proof of (2.4) follows by classical estimates for the Benjamin-Ono equation in the case $k \geq 6$ (see [20]), and it becomes more and more complicated as long as k becomes smaller. In particular as far as we know it is unclear whether or not property (2.4) it is true for $0 < \sigma < 5/4$;

- a second and more essential source of difficulty to prove the invariance of $d\rho_{k/2,R}$, is related with the fact that the energies $E_{k/2}$, that are conserved for the equation (1.1), are no longer conserved for the truncated problems (2.3), as long as $k \geq 2$. A partial and useful substitute of the lack of invariance of $E_{k/2}$ along the truncated flow (2.3) is the following property:

$$\lim_{N\to\infty} \sup_{\substack{t\in[0,\bar t] \\ A\in\mathcal{B}(H^{(k-1)/2-\epsilon})}} \left| \frac{d}{dt} \int_{\Phi_t^N(A)} F_{k/2,N,R}(u)\, d\mu_{k/2} \right| = 0 \tag{2.5}$$

  where $\mathcal{B}(H^\gamma)$ denote the Borel sets in H^γ.
  In the sequel we shall refer to (2.5) as to the **almost invariance property of $F_{k/2,N,R}(u)d\mu_{k/2}$ along the flow Φ_t^N.**

Remark 2.2. The lack of invariance under truncated versions of the equation, of quantities conserved along the infinite dimensional equation, appears in other important situations. See for instance [23] for KdV and [16] for DNLS. In those papers the problem is solved by evaluating the energy growth of individual solutions. In particular in the context of DNLS it is crucial to exploit heavily the deterministic time oscillations of the equation. The main novelty in our approach is that we do not exploit the deterministic time oscillations of the equation in order to get (2.5), but we take full advantage of the random character of the initial data.

Next we focus on the proof of the following proposition.

Proposition 2.2. *We have the following implication*

$$(2.4) \text{ and } (2.5) \Rightarrow \text{invariance of } d\rho_{k/2,R} \text{ for } k>1.$$

Proof. Since now on we assume $k>1$ and R fixed, we shall denote by σ any real number smaller than $(k-1)/2$, $F_N = F_{k/2,N,R}$, $d\mu = d\mu_{k/2}$, $F(u) = F_{k/2,R}(u)$ (see Theorem 2.1). Based on the deterministic theory well-established for the Benjamin-Ono equation we get that for every compact set $K \subset H^\sigma$ there exists $S>0$ such that

$$\{\Phi_t(K) | t \in [0,t_0]\} \subset B^\sigma(S). \tag{2.6}$$

We state the following

Claim: *For every $S>0$ there exists $\bar{t} = \bar{t}(S) > 0$ such that for every compact set $K \subset H^\sigma$, with $K \subset B^\sigma(S)$ we have*

$$\int_K F(u)d\mu \leq \int_{\Phi_t(K)} F(u)d\mu, \ \forall t \in (-\bar{t}, \bar{t}).$$

Based on the claim we shall prove that for every compact $K \subset H^\sigma$ and for every $t_0 \in \mathbb{R}$ we get

$$\int_K F(u)d\mu \leq \int_{\Phi_{t_0}(K)} F(u)d\mu. \tag{2.7}$$

Notice that by an approximation argument of Borel sets by compact sets, and by using the reversibility of the flow, this implies the invariance of $d\rho_{k/2,R} = F(u)d\mu$.

We give the proof of (2.7) only for t_0 positive, the analysis for negative t_0 is completely analogous.

Next we consider $\bar{t} = \bar{t}(S) \in (0, t_0]$ given in the claim above and we choose $\tilde{t}$ such that

$$\tilde{t} \in (0, \bar{t}] \text{ and } \frac{t_0}{\tilde{t}} \in \mathbb{N}.$$

By the claim we get

$$\int_K F(u)d\mu \leq \int_{\Phi_{\tilde{t}}(K)} F(u)d\mu.$$

Notice that by (2.6) we have $\Phi_{\tilde{t}}(K) \subset B^\sigma(S)$, hence we can iterate the estimate above and we obtain

$$\int_{\Phi_{\tilde{t}}(K)} F(u)d\mu \leq \int_{\Phi_{\tilde{t}}(\Phi_{\tilde{t}}(K))} F(u)d\mu = \int_{\Phi_{2\tilde{t}}(K)} F(u)d\mu.$$

By repeating this argument N_0 times, where $N_0\tilde{t} = t_0$, we get

$$\int_{\Phi_{(j-1)\tilde{t}}(K)} F(u)d\mu \leq \int_{\Phi_{j\tilde{t}}(K)} F(u)d\mu, \ \forall j = 1, ..., N$$

and hence by the above chain of inequalities we deduce

$$\int_K F(u)d\mu \leq \int_{\Phi_{t_0}(K)} F(u)d\mu.$$

Next we focus on the proof of the claim. By (2.5) we get

$$\int_{\Phi_t^N(K)} F_N(u)d\mu = \int_K F_N(u)d\mu + o(1), \ \forall t \in \mathbb{R} \tag{2.8}$$

where $\lim_{N\to\infty} o(1) = 0$. Moreover $F_N \to F$ in $L^1(d\mu)$ and hence by (2.8) we get

$$\lim_{N\to\infty} \int_{\Phi_t^N(K)} F_N(u)d\mu = \lim_{N\to\infty} \int_K F_N(u)d\mu = \int_K F(u)d\mu, \ \forall t \in \mathbb{R}. \tag{2.9}$$

By (2.4) we get $\bar{t} = \bar{t}(S) > 0$ such that for every $\epsilon > 0$ there exists a suitable $N_0(\epsilon)$ with the property

$$\sup_{N>N_0(\epsilon)} \int_{\Phi_t^N(K)} F(u)d\mu \le \int_{\Phi_t(K)+B^s(\epsilon)} F(u)d\mu, \ \forall t \in (-\bar{t}, \bar{t}). \tag{2.10}$$

We estimate the l.h.s. as follows:

$$\sup_{N>N_0(\epsilon)} \int_{\Phi_t^N(K)} F(u)d\mu \ge \lim_{N\to\infty} \int_{\Phi_t^N(K)} F(u)d\mu. \tag{2.11}$$

On the other hands K is closed in H^s and, since Φ_t is a diffeomorphism on H^s, also $\Phi_t(K)$ is closed in H^s. As a consequence we deduce

$$\bigcap_{\epsilon>0} (\Phi_t(K) + B^s(\epsilon)) = \Phi_t(K)$$

and hence by the Lebesgue theorem we deduce that the r.h.s. in (2.10) converges to $\int_{\Phi_t(K)} F(u)d\mu$ as $\epsilon \to 0$. By combining this fact with (2.11) we get

$$\lim_{N\to\infty} \int_{\Phi_t^N(K)} F_N(u)d\mu \le \int_{\Phi_t(K)} F(u)d\mu, \ \forall t \in (-\bar{t}, \bar{t}).$$

The proof can be completed by combining the last inequality with (2.9).

□

3. On the invariance of $d\rho_{k/2,R}$ for $k \ge 4$

Following Proposition 2.2, the proof of the invariance of $d\rho_{k/2,R}$ for $k \ge 4$ holds provided that we can show (2.4) (for $\sigma > 5/4$) and (2.5) for $k \ge 4$. More precisely we shall check (2.5) for every $k > 1$. Concerning (2.4) we shall restrict to the case $\sigma > 5/4$. It is unclear if property (2.4) is verified for smaller values of σ. Indeed we shall introduce in section 4 a substitute of (2.4) that will allow us to get invariance of $d\rho_{k/2,R}$ also for $k = 2, 3$.

3.1. On the Approximation of Φ_t by the Truncated Flow Φ_t^N

The main result of this subsection is the following proposition, which is strictly related to the property (2.4).

Proposition 3.1. *Fix $5/4 < s < \sigma < \infty$. For every $M > 0$ there exists $T = T(M) > 0$ such that*

$$\lim_{N\to\infty} \Big(\sup_{\substack{t\in[0,T] \\ u_0\in B^\sigma(M)}} \|\Phi_t(u_0) - \Phi_t^N(u_0)\|_{H^s} \Big) = 0. \tag{3.1}$$

Proof. In the case $\sigma > 3/2$ the proof follows by combining a classical energy estimate, a well-known estimate by Kato-Ponce (see [8]) and the Sobolev embedding $H^{1/2+\delta} \subset L^\infty$. The case $5/4 < \sigma \leq 3/2$ is treated in [22], where the key tool is the technique introduced in [10].

□

3.2. On the Almost Invariance of $F_{k/2,N,R}(u)d\mu_{k/2}$ along Φ_t^N

The main result of this subsection is the following proposition.

Proposition 3.2. *Let $k \geq 2$. Then there exists $\epsilon_0 > 0$ such that (2.5) holds $\forall \epsilon < \epsilon_0$, $t \in \mathbb{R}$, $R > 0$.*

In the sequel, when it is not better specified, we use the notations $F_N(u) = F_{k/2,N,R}(u)$ and $d\mu = d\mu_{k/2}$.

Remark 3.1. Notice that (2.5) is trivially satisfied also for $k = 1$. In fact the energy $E_{1/2}$ is the Hamiltonian of the Benjamin-Ono equation, and it is easy to check that it is preserved also along the truncated flow Φ_t^N.

Proof. The basic idea is to reduce the problem to $t = 0$. More precisely we have the following chain of implications:

$$\lim_{N\to\infty} \left\| \frac{d}{dt}\Big(E_{j/2}\big(\pi_N \Phi_t^N(u)\big)\Big)_{|t=0}\right\|_{L^2(d\mu_{k/2})} = 0, \ \forall j = 1, ..., k$$

$$\Rightarrow \lim_{N\to\infty} \sup_{A\in\mathcal{B}(H^{(k-1)/2-\epsilon})} \left| \frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{k/2,N,R}(u)d\mu_{k/2}\Big)_{|t=0}\right| = 0$$

$$\Rightarrow \lim_{N\to\infty} \sup_{\substack{t\in[0,t_0]\\ A\in\mathcal{B}(H^{(k-1)/2-\epsilon})}} \left| \frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{k/2,N,R}(u)d\mu_{k/2}\Big)\right| = 0.$$

Proof of the second implication. We have

$$\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_N(u)d\mu\Big)_{|t=\bar{t}}$$

$$= \lim_{h\to 0} h^{-1}\Big(\int_{\Phi_{\bar{t}+h}^N(A)} F_N(u)d\mu - \int_{\Phi_{\bar{t}}^N(A)} F_N(u)d\mu\Big)$$

$$= \lim_{h\to 0} h^{-1}\Big(\int_{\Phi_h^N\circ\Phi_{\bar{t}}^N(A)} F_N(u)d\mu - \int_{\Phi_{\bar{t}}^N(A)} F_N(u)d\mu\Big),$$

and hence

$$\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_N(u)d\mu\Big)_{t=\bar{t}} = \frac{d}{dt}\Big(\int_{\Phi_t^N(\tilde{A})} F_N(u)d\mu\Big)_{|t=0}$$

where $\tilde{A} = \Phi_{\bar{t}}^N(A)$.

Proof of the first implication.
First of all we notice the following identity (see [21]):

$$\int_{\Phi_t^N(A)} F_N(u)d\mu = \gamma_N \int_A \prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N \Phi_t^N(u)))\times \tag{3.2}$$

$$\chi_R(E_{(k-1)/2}(\pi_N\Phi_t^N(u)) - \alpha_N)e^{-E_{k/2}(\pi_N(\Phi_t^N(u))}du_1...du_N \times d\mu_N^\perp,$$

where the factor γ_N is given by the following decomposition of $d\mu$ as a product measures:

$$d\mu = \gamma_N e^{-\|\pi_N u\|^2_{H^{k/2}}} du_1 ... du_N \times d\mu_N^\perp$$

where $d\mu_N^\perp$ is supported on the space $E_N^\perp$ which is orthogonal to the real space E_N spanned by $(\cos(nx), \sin(nx))_{1 \leq n \leq N}$. The proof on (3.2) is based on a combination of: 1) Liouville theorem on the invariance of Lebesgue measure for finite dimensional hamiltonian systems; 2) invariance of the measure $d\mu^\perp$ along the linear flow associated with the linear Benjamin-Ono flow; 3) Fubini theorem. By (3.2) we see that to consider the time derivative at time $t = 0$ of the l.h.s. is equivalent to consider the time derivative at time $t = 0$ of the r.h.s., i.e.

$$\int_A G_N(u) \prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N(u)))\chi_R(E_{(k-1)/2}(\pi_N(u)) - \alpha_N) e^{-R_{k/2}(\pi_N u)} d\mu +$$

$$+ \int_A H_N(u) \prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N(u)))\chi_R'(E_{(k-1)/2}(\pi_N(u)) - \alpha_N) e^{-R_{k/2}(\pi_N u)} d\mu +$$

$$\sum_{j_0=0}^{k-2} \int_A L_N^{j_0}(u) \chi_R'(E_{j_0/2}(\pi_N u)) \prod_{\substack{j=0 \\ j \neq j_0}}^{k-2} \chi_R(E_{j/2}(\pi_N u)) \chi_R(E_{(k-1)/2}(\pi_N u) - \alpha_N) \times$$
$$e^{-R_{k/2}(\pi_N u)} d\mu$$

where $G_N(u), H_N(u), L_N^{j_0}(u)$ for $j_0 = 0, ..., k-2$ are respectively defined by

$$G_N(u) = \frac{d}{dt}\Big(E_{k/2}(\pi_N(\Phi_t^N u))\Big)_{|t=0},$$

$$H_N(u) = \frac{d}{dt}\Big(E_{(k-1)/2}(\pi_N(\Phi_t^N u))\Big)_{|t=0},$$

$$L_N^{j_0}(u) = \frac{d}{dt}\Big(E_{j_0/2}(\pi_N(\Phi_t^N u))\Big)_{|t=0}, j_0 = 0, ..., k-2.$$

By using the Cauchy-Schwartz inequality w.r.t. $d\mu$ we conclude the proof of the first implication.

Summarizing the proof of Proposition 3.2 follows provided that one proves

$$\lim_{N \to 0} \left\| \frac{d}{dt}\Big(E_{j/2}\big(\pi_N \Phi_t^N(u)\big)\Big)_{|t=0} \right\|_{L^2(d\mu_{k/2})} = 0, \ \forall j = 1, ..., k.$$

The proof of this fact is quite involved and we skip the details (see [21] and [22]). We just recall the key steps. The first one is to get a representation of the functions $\frac{d}{dt}\Big(E_{j/2}\big(\pi_N \Phi_t^N(u)\big)\Big)_{|t=0}$ as linear combination of multilinear products of the gaussian variables $g_n(\omega)$ that appear in (1.2), i.e.

$$\frac{d}{dt}\Big(E_{j/2}\big(\pi_N \Phi_t^N(u)\big)\Big)_{|t=0} = \sum_{\mathcal{C}_N} c_{j_1,...,j_n} g_{j_1} ... g_{j_n} \tag{3.3}$$

where $c_{j_1,...,j_n}$ are suitable numbers and the dependence on N in (3.3) is hidden in the constraint $\mathcal{C}_N$. The second step is to notice that thanks to integration by parts one can cancel the worst terms in the above representation. In this step some delicate informations on the structure of the conservation laws play a crucial role (see

[11] and [21]). The remaining terms can be estimated, for $k \geq 6$ and k even, via the Minkowski inequality, hence reducing the problem to the analysis of numerical series of the type $\sum_{\mathcal{C}_N} |c_{j_1,\ldots,j_n}|$. In the case $k = 2, 4$ and $k \geq 3$ odd the Minkowski inequality is useless to estimate (3.3), and one needs to exploit the L^2_ω orthogonality of multilinear expressions $g_{j_1} \ldots g_{j_n}$. Hence we reduce to the study of numerical expressions of the type $\sum_{\mathcal{C}'_N} |c_{j_1,\ldots,j_n}|^2$, where $\mathcal{C}'_N$ is a large subset of $\mathcal{C}_N$ where suitable orthogonality relations occur. The analysis on the resonant set $\mathcal{C}_N \backslash \mathcal{C}'_N$ is then done in a straightforward way.

□

4. Invariance of $d\rho_{k/2,R}$ for $k = 2, 3$

The proof of the invariance of $d\rho_{k/2,R}$ for $k = 2, 3$ is more complicated compared to the case $k > 3$, since it is unclear whether or not it is satisfied property (2.4) for $0 < \sigma < 5/4$. For this reason we have to modify the family of approximating problems by introducing a smoothed version of the Dirichlet projectors π_N.
For every fixed $\epsilon \in (0, 1)$ we denote by ψ_ϵ a smooth function $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that:

$$\psi_\epsilon(x) = 1 \text{ for } x \in [0, (1-\epsilon)], \psi_\epsilon(x) = 0 \text{ for } x > 1, \qquad (4.1)$$
$$\|\psi_\epsilon\|_{L^\infty} = 1 \text{ and } \psi_\epsilon(x) = \psi_\epsilon(|x|).$$

We denote by S^ϵ_N the Fourier multiplier:

$$S^\epsilon_N(\sum_{j\in\mathbb{Z}} a_j e^{ijx}) = \sum_{j\in\mathbb{Z}} a_j \psi_\epsilon(\frac{j}{N}) e^{ijx}. \qquad (4.2)$$

We also denote by $\Phi^{\epsilon,N}_t$ the flow on H^s, $s \geq 0$ associated with

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + S^\epsilon_N(S^\epsilon_N u \cdot S^\epsilon_N u_x) = 0, \\ u(0,x) = u_0. \end{cases} \qquad (4.3)$$

In order to prove the invariance of $d\rho_{k/2,R}$ for $k = 2, 3$ we need the following ingredients:

- a modification of the construction of the measures $d\rho_{1,R}$ and $d\rho_{3/2,R}$, where we replace the projectors π_N by the smoothed projectors S^ϵ_N in the sequence of approximating densities (see (2.2));
- to prove that $\Phi^{\epsilon,N}_t$ is a **good approximation** to Φ_t as $N \to \infty$;
- a version of (2.5) where we replace the projectors π_N by S^ϵ_N and the flow Φ^N_t by $\Phi^{\epsilon,N}_t$.

4.1. On the Construction of the Measures $d\rho_{1,R}$ and $d\rho_{3/2,R}$ via S^ϵ_N

We first introduce the modified energies:

$$E^\epsilon_N(u) = \|u\|^2_{\dot{H}^1} - \|S^\epsilon_N u\|^2_{\dot{H}^1} + E_1(S^\epsilon_N u), \qquad (4.4)$$

$$G^\epsilon_N(u) = \|u\|^2_{\dot{H}^{3/2}} - \|S^\epsilon_N u\|^2_{\dot{H}^{3/2}} + E_{3/2}(S^\epsilon_N u), \qquad (4.5)$$

and the approximating modified densities:

$$F^{\epsilon}_{N,R}(u) = \chi_R(\|\pi_N u\|_{L^2}) \times \chi_R(\|\pi_N u\|^2_{\dot H^{1/2}} - \alpha_N + 1/3\int (S^{\epsilon}_N u)^3 dx) \times \exp(\|S^{\epsilon}_N u\|^2_{\dot H^1} - E_1(S^{\epsilon}_N u)), \tag{4.6}$$

$$H^{\epsilon}_{N,R}(u) = \chi_R(\|\pi_N u\|_{L^2}) \times \chi_R(\|\pi_N u\|^2_{\dot H^{1/2}} + 1/3\int (S^{\epsilon}_N u)^3 dx) \times \chi_R(E^{\epsilon}_N(\pi_N u) - \alpha_N) \times \exp(\|S^{\epsilon}_N u\|^2_{\dot H^{3/2}} - E_{3/2}(S^{\epsilon}_N u)). \tag{4.7}$$

Next proposition shows that as $N \to \infty$ the measures $F^{\epsilon}_{N,R} d\mu_1$ (for $\epsilon > 0$ fixed) converge to $d\rho_{1,R}$ and $H^{\epsilon}_{N,R} d\mu_{3/2}$ converge to $d\rho_{3/2,R}$ (in a strong sense).

Proposition 4.1. *Let $R, \sigma > 0$ and $\epsilon_0 > 0$ be fixed, then:*

$$\lim_{N\to\infty} \sup_{A \in \mathcal{B}(H^{1/2-\sigma})} |\int_A F^{\epsilon_0}_{N,R}(u) d\mu_1 - \int_A d\rho_{1,R}| = 0, \tag{4.8}$$

$$\lim_{N\to\infty} \sup_{A \in \mathcal{B}(H^{1-\sigma})} |\int_A H^{\epsilon_0}_{N,R}(u) d\mu_1 - \int_A d\rho_{3/2,R}| = 0. \tag{4.9}$$

For the proof see [5].

4.2. On the Approximation of Φ_t by the Truncated Flow $\Phi^{\epsilon,N}_t$

The following proposition is proved in [5]. It is a simplified version of the result contained in [4].

Proposition 4.2. *Let $0 < \epsilon < 1$, $\sigma > \sigma' > 0$ and $M > 0$ be fixed, so that σ is small enough. We have for some $T = T(\epsilon, \sigma, \sigma', M) > 0$, $C = C(\epsilon, \sigma, \sigma', M) > 0$ that:*

$$\sup_{u_0 \in B_M(H^{1/2-\sigma'})} \sup_{|t| \le T} \|\Phi^{\epsilon,N}_t(u_0) - \Phi_t(u_0)\|_{H^{1/2-\sigma}} \le C N^{-\theta},$$

where $\theta = \theta(\sigma, \sigma') > 0$.

Remark 4.1. Notice that in Proposition 4.2 we have a time of approximation that depends on $\epsilon > 0$ and not only on the size M of the initial data (compare with Proposition 3.1).

4.3. On the Almost Invariance of $F^{\epsilon}_{N,R}(u)d\mu_1$ and $H^{\epsilon}_{N,R}(u)d\mu_{3/2}$ along $\Phi^{\epsilon,N}_t$

We have the following version of Proposition 3.2.

Proposition 4.3. *Let $\sigma, R > 0$ be fixed. Then for every $\delta > 0$ there exists $N = N(\delta) > 0$ and $\epsilon = \epsilon(\delta) > 0$ such that:*

$$|\int_A F^{\epsilon}_{N,R}(u) d\mu_1 - \int_{\Phi^{\epsilon,N}_t A} F^{\epsilon}_{N,R} d\mu_1| \le \delta t, \quad \forall A \in \mathcal{B}(H^{1/2-\sigma}), \quad \forall t \in \mathbb{R},$$

$$|\int_A H^{\epsilon}_{N,R}(u) d\mu_1 - \int_{\Phi^{\epsilon,N}_t A} H^{\epsilon}_{N,R} d\mu_1| \le \delta t, \quad \forall A \in \mathcal{B}(H^{1-\sigma}), \quad \forall t \in \mathbb{R}.$$

Exactly as in the proof of Proposition 3.2 the argument is to reduce the analysis at time $t = 0$, more precisely it is sufficient to prove:

$$\lim_{\epsilon\to 0}\Big(\limsup_{N\to\infty}\Big\|\frac{d}{dt}\Big(E_N^\epsilon(\pi_N\Phi_t^{\epsilon,N}(u))\Big)_{|t=0}\Big\|_{L^2(d\mu_1)}\Big)=0,$$

where the energies $E_N^\epsilon(u)$ are defined by (4.4) and

$$\lim_{\epsilon\to 0}\Big(\limsup_{N\to\infty}\Big\|\frac{d}{dt}\Big(G_N^\epsilon(\pi_N\Phi_t^{\epsilon,N}(u))\Big)_{|t=0}\Big\|_{L^2(d\mu_{3/2})}\Big)=0,$$

where the energies $G_N^\epsilon(u)$ are defined by (4.5). The proof of the estimates above are quite involved and are part of the paper [5].

4.4. Invariance of $d\rho_{1,R}$ and $d\rho_{3/2,R}$

We shall focus on the proof of the invariance of $d\rho_{1,R}$. The proof of the invariance of $d\rho_{3/2,R}$ follows by a similar argument. We shall prove the following proposition.

Proposition 4.4. *Prop. 4.1 + Prop. 4.2 + Prop. 4.3 $\Rightarrow$ Invariance of $d\rho_{1,R}$.*

Proof. By the reversibility of the flow Φ_t and by an approximation argument of Borel sets by compact sets, it is sufficient to prove that

$$\int_A d\rho_{1,R}\leq\int_{\Phi_{\bar t}(A)}d\rho_{1,R}, \tag{4.10}$$

where $A\subset H^{1/2-\sigma}$ is compact, $\sigma>0$ is small enough and $\bar t>0$ is given. We fix $M>0$ such that $A\subset B_M(H^{1/2-\sigma})$ and we choose $L>0$ such that

$$\Phi_t(B_M(H^{1/2-\sigma}))\subset B_L(H^{1/2-\sigma}) \tag{4.11}$$

for every $t\in[0,\bar t]$ (the existence of L follows by [13]). Next we fix $k>0$. By Proposition 4.3 there exist $N_k\in\mathbb{N}$ and $\epsilon_k>0$ such that:

$$\Big|\int_A F_{N,R}^{\epsilon_k}d\mu_1-\int_{\Phi_t^{\epsilon_k,N}(A)}F_{N,R}^{\epsilon_k}d\mu_1\Big|\leq t/k,\qquad\forall N>N_k,\quad\forall t\in\mathbb{R}. \tag{4.12}$$

On the other hand we have by Proposition 4.2 the existence of $t_1=t_1(L,k)>0$ and $C=C(L,k)>0$ such that

$$\sup_{\substack{u\in B_L(H^{1/2-\sigma})\\ t\in[0,t_1]}}\|\Phi_t^{\epsilon_k,N}(u)-\Phi_t(u)\|_{H^{1/2-\sigma'}}\leq CN^{-\theta},$$

and hence

$$\int_{\Phi_t^{\epsilon_k,N}(A)}d\rho_{1,R}\leq\int_{\Phi_t(A)+B_{CN^{-\theta}}(H^{1/2-\sigma'})}d\rho_{1,R},\ \forall t\in[0,t_1]. \tag{4.13}$$

In turn by combining (4.12) with Proposition 4.1 we get the existence of $\tilde N_k\in\mathbb{N}$ such that

$$\Big|\int_A d\rho_{1,R}-\int_{\Phi_t^{\epsilon_k,N}(A)}d\rho_{1,R}\Big|\leq 3t_1/k,\quad\forall N>\tilde N_k,\quad\forall t\in\mathbb{R}. \tag{4.14}$$

By combining (4.13) with (4.14) we get

$$\int_A d\rho_{1,R}\leq\int_{\Phi_t(A)+B_{CN^{-\theta}}(H^{1/2-\sigma'})}d\rho_{1,R}+3t_1/k,\ \forall t\in[0,t_1], \tag{4.15}$$

and by taking the limit as $N \to \infty$ gives:

$$\int_A d\rho_{1,R} \leq \int_{\Phi_t(A)} d\rho_{1,R} + 3t_1/k, \ \forall t \in [0, t_1].$$

It is sufficient to iterate the bound above $[\bar{t}/t_1] + 1$ times and to take the limit as $k \to \infty$ in order to get (4.10) (notice that we can iterate thanks to (4.11)).

□

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Mathematics Department
Princeton University
Fine Hall
Washington road, Princeton
NJ 08544-4200, USA
yudeng@math.princeton.edu

Institut Universitaire de France
and Département de Mathématiques
Université de Cergy-Pontoise
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex, France
nikolay.tzvetkov@u-cergy.fr

Dipartimento di Matematica
Università Degli Studi di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa, Italy
viscigli@dm.unipi.it