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Julien Sabin

Abstract

We review some recent results obtained with Mathieu Lewin [21] concerning the nonlinear Hartree equation for density matrices of infinite trace, describing the time evolution of quantum systems with infinitely many particles. Our main result is the asymptotic stability of a large class of translation-invariant density matrices which are stationary solutions to the Hartree equation. We also mention some related result obtained in collaboration with Rupert Frank [13] about Strichartz estimates for orthonormal systems.

1. Introduction

The Hartree equation is a type of nonlinear Schrödinger equation of the form

$$\begin{cases} i\partial_t\gamma &= [-\Delta_x + w * \rho_\gamma, \gamma], \quad t \in \mathbb{R},\, x \in \mathbb{R}^d, \\ \gamma_{|t=0} &= \gamma_0, \end{cases} \tag{1.1}$$

where $\gamma = \gamma(t)$ is a bounded, self-adjoint, operator on $L^2_x(\mathbb{R}^d)$, $w : \mathbb{R}^d \to \mathbb{R}$ is a smooth, fast decaying interaction potential, and $\rho_\gamma : \mathbb{R}^d \to \mathbb{R}$ is the density associated to γ, formally defined as $\rho_\gamma(x) = \gamma(x,x)$ for all $x \in \mathbb{R}^d$, where $\gamma(\cdot,\cdot)$ denotes the integral kernel of γ. This equation models the time evolution of a non-relativistic quantum system with density matrix γ_0 at the initial time, in which the particles are interacting through the potential w. In the case $\operatorname{Tr}|\gamma_0| < \infty$, it can be derived from many-body quantum mechanics [2, 1, 10, 14, 3] in a mean-field or semi-classical limit. The global well-posedness theory in the energy space (that is, for initial data such that $\gamma_0 \geqslant 0$ and $\operatorname{Tr}(1-\Delta)\gamma_0 < +\infty$) has been studied in the works [5, 6, 9, 27].

In [19, 21] we addressed the question of the well-posedness of (1.1) and of the large time behaviour of its solutions, in the case where $\operatorname{Tr}|\gamma_0| = +\infty$, that is for quantum systems with an infinite number of particles. This context was strongly motivated from the study of the dynamics of (1.1) around a class of stationary solutions which are *translation-invariant*. Indeed, consider any density matrix of the form

$$\gamma_{\text{ref}} = g(-i\nabla),$$

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with $g \in (L^1 \cap L^\infty)(\mathbb{R}^d, \mathbb{R})$, i.e. $\gamma_{\rm ref}$ is the Fourier multiplier by the function g. Then, the density of this operator satisfies

$$\rho_{\gamma_{\rm ref}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi)\, d\xi, \quad \forall x \in \mathbb{R}^d,$$

and hence is uniform in space. In particular, the commutator $[-\Delta + w * \rho_{\gamma_{\rm ref}}, \gamma_{\rm ref}]$ vanishes and $\gamma(t) \equiv \gamma_{\rm ref}$ is a stationary solution to (1.1). Notice that the operator $\gamma_{\rm ref}$ is never compact if $g \neq 0$ and thus is a first example of an infinite-trace solution to (1.1). It represents a quantum system invariant by translation, and there are several physical examples of such states:

- *Fermi sea at zero temperature and chemical potential* $\mu > 0$:

$$g(\xi) = \mathbb{1}(0 \leqslant |\xi|^2 \leqslant \mu) \quad \text{and} \quad \gamma_{\rm ref} = \mathbb{1}(-\Delta \leqslant \mu); \tag{1.2}$$

- *Fermi gas at positive temperature* $T > 0$ *and chemical potential* $\mu \in \mathbb{R}$:

$$g(\xi) = \frac{1}{e^{(|\xi|^2-\mu)/T} + 1} \quad \text{and} \quad \gamma_{\rm ref} = \frac{1}{e^{(-\Delta-\mu)/T} + 1}; \tag{1.3}$$

- *Bose gas at positive temperature* $T > 0$ *and chemical potential* $\mu < 0$:

$$g(\xi) = \frac{1}{e^{(|\xi|^2-\mu)/T} - 1} \quad \text{and} \quad \gamma_{\rm ref} = \frac{1}{e^{(-\Delta-\mu)/T} - 1}; \tag{1.4}$$

- *Boltzmann gas at positive temperature* $T > 0$ *and chemical potential* $\mu \in \mathbb{R}$:

$$g(\xi) = e^{-(|\xi|^2-\mu)/T} \quad \text{and} \quad \gamma_{\rm ref} = e^{(\Delta+\mu)/T}. \tag{1.5}$$

We are interested in the dynamics of (1.1) around these stationary states, that is for initial data of the form $\gamma_0 = \gamma_{\rm ref} + Q_0$, where Q_0 is small (for instance finite-rank). In particular, since $\gamma_{\rm ref}$ is not trace-class, the operator γ_0 is not trace-class either and we are naturally lead to study (1.1) for non-trace-class initial data. The dynamics of infinite quantum systems in interaction has already been studied by Hainzl, Lewin, and Sparber [16] in a relativistic setting, and by Cancès and Stoltz [7] for crystals. Both these works show the global well-posedness of their respective equations in the energy space, and in particular leave open the question of the large time behaviour of the solutions.

In a first work [19], we showed the global existence of solutions to (1.1) around a large class of stationary states $\gamma_{\rm ref}$, in the adequate energy space. The operator $\gamma_{\rm ref}$ has not only an infinite trace, it also has an infinite (total) energy, which is defined as

$$\mathcal{E}_{\rm tot}(\gamma) = {\rm Tr}(-\Delta)\gamma + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\gamma(x) w(x-y) \rho_\gamma(y)\, dx\, dy.$$

Hence, a perturbation of the form $\gamma_0 = \gamma_{\rm ref} + Q_0$ (with Q_0 finite-rank for instance) also has an infinite (total) energy. The total energy $\mathcal{E}_{\rm tot}$, which is formally conserved along the flow of (1.1), is thus useless regarding the global well-posedness of (1.1). Instead, the correct object to consider is the *relative energy* with respect to $\gamma_{\rm ref}$. In [19], we show the global well-posedness of (1.1) for perturbations of $\gamma_{\rm ref}$ with finite relative energy with respect to $\gamma_{\rm ref}$. At positive temperature, the relative energy uses the notion of *relative entropy*, whose definition for general $\gamma_{\rm ref}$ has been given in a companion work [20]. Finally, let us mention that a crucial argument in the proof of global well-posedness is the use of *Lieb–Thirring inequalities at positive density*, developed by Frank, Lewin, Lieb and Seiringer in [11].

In [21], we studied the long time behaviour of solutions (1.1) with γ_0 close enough to $\gamma_{\rm ref}$. In particular, we show that $\gamma(t) \to \gamma_{\rm ref}$ a $t \to \infty$ in some sense, meaning that the stationary states $\gamma_{\rm ref}$ are asymptotically stable under the flow of the nonlinear Hartree equation. The goal of this review article is to explain the difficulty of such a result and the strategy of its proof.

The article is organized as follows. In Section 2 we explain how to describe the long time behaviour of small solutions to (1.1) in the simpler case $\gamma_{\rm ref} = 0$, $\mathrm{Tr}\,|\gamma_0| < \infty$. In Section 3, we discuss dispersion outside the trace-class (that is, for infinite quantum systems), using a new family of Strichartz estimates. Finally, in Section 4, we state our main result (Theorem 4) and give elements of its proof.

2. Dispersion of small solutions in the trace-class without background

In this section, we explain how to describe the long time behaviour of small solutions to (1.1) with $\mathrm{Tr}\,|\gamma_0| < \infty$. In particular, the background here is trivial: $\gamma_{\rm ref} = 0$. We use well-known methods in dispersive PDEs involving Strichartz estimates, and we explain the difficulty of their generalization to the infinite-trace setting.

2.1. Rank-one case

We begin with the case where γ_0 is a rank-one operator: we may write[1] $\gamma_0 = |u_0\rangle\langle u_0|$, with $u_0 \in L^2_x(\mathbb{R}^d)$. In this very specific setting, Equation (1.1) is equivalent to

$$\begin{cases} i\partial_t u &= (-\Delta_x + w * |u|^2)u, \quad t \in \mathbb{R},\, x \in \mathbb{R}^d, \\ u_0 &= u_0 \in L^2_x(\mathbb{R}^d). \end{cases} \tag{2.1}$$

This means that the solution $\gamma(t)$ to (1.1) stays of rank one for all times, with $\gamma(t) = |u(t)\rangle\langle u(t)|$, where u satisfies (2.1). The study of the large-time behaviour of solutions to (2.1) is a very well-known topic in dispersive PDEs, and we have the following result.

Proposition 1 (Dispersion of small rank-one solutions without background)**.** *Let $d \geqslant 2$ and $w \in L^{d/2}_x(\mathbb{R}^d) \cap L^\infty_x(\mathbb{R}^d)$. Then, there exists $\varepsilon_0 > 0$ such that for all $u_0 \in L^2_x(\mathbb{R}^d)$ with $\|u_0\|_{L^2_x} \leqslant \varepsilon_0$, there exists a unique $u \in C^0_t(\mathbb{R}, L^2_x(\mathbb{R}^d))$ solution to (2.1) which verifies $u \in L^4_t(\mathbb{R}, L^{2d/(d-1)}_x(\mathbb{R}^d))$. Furthermore, there exist $u_\pm \in L^2_x(\mathbb{R}^d)$ such that*

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{it\Delta}u_\pm \right\|_{L^2_x} = 0. \tag{2.2}$$

In particular, we see that for initial data u_0 sufficiently close to 0, the solution behaves like a free solution for large times and thus converges weakly to 0 as $t \to \infty$. We thus have

$$\gamma(t) = |u(t)\rangle\langle u(t)| \underset{t\to\infty}{\rightharpoonup} 0 = \gamma_{\rm ref},$$

which is the result we anounced, in the special case $\gamma_{\rm ref} = 0$, $\mathrm{rank}(\gamma_0) = 1$.

[1] Here and everywhere, we use Dirac's notation $|u\rangle\langle v|$ for the operator $f \mapsto \langle v, f\rangle u$. Our scalar product is always anti-linear with respect to the left argument.

Proof of Proposition 1. The proof uses techniques which are now standard in nonlinear dispersive PDEs, based on the use of Strichartz estimates. First of all, the fact that $w \in L^\infty_x(\mathbb{R}^d)$ implies very easily that for any $u_0 \in L^2_x(\mathbb{R}^d)$, there exists a unique $u \in C^0_t(\mathbb{R}, L^2_x(\mathbb{R}^d))$ solution to (2.1). Here, the solution is globally defined in time thanks to the conservation of the L^2_x-norm: $\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}$ for all t. To prove dispersion for small solutions, we write the Duhamel formulation of (2.1),

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(w * |u(s)|^2)u(s)\,ds,$$

and use Strichartz estimates [25, 17] to infer that

$$\left\|e^{it\Delta}u_0\right\|_{L^4_t L^{2d/(d-1)}_x} \lesssim \|u_0\|_{L^2_x}\,,$$

$$\left\|\int_0^t e^{i(t-s)\Delta}(w * |u(s)|^2)u(s)\,ds\right\|_{L^4_t L^{2d/(d-1)}_x} \lesssim \left\|(w * |u|^2)u\right\|_{L^{4/3}_t L^{2d/(d+1)}_x}\,, \tag{2.3}$$

Young's inequality implies that

$$\left\|(w * |u|^2)u\right\|_{L^{4/3}_t L^{2d/(d+1)}_x} \lesssim \|w\|_{L^{d/2}_x}\,\|u\|^3_{L^4_t L^{2d/(d-1)}_x}\,,$$

showing that for all $T > 0$,

$$\|u\|_{L^4_t([-T,T],L^{2d/(d-1)}_x(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2_x} + \|u\|^3_{L^4_t([-T,T],L^{2d/(d-1)}_x(\mathbb{R}^d))}\,.$$

A standard continuation argument then shows that, for $\varepsilon_0 > 0$ small enough, there exists $C(\varepsilon_0) > 0$ such that for all $\|u_0\|_{L^2_x} \leqslant \varepsilon_0$ and for all $T > 0$, we have

$$\|u\|_{L^4_t([-T,T],L^{2d/(d-1)}_x(\mathbb{R}^d))} \leqslant C(\varepsilon_0).$$

As a consequence, we infer that $u \in L^4_t(\mathbb{R}, L^{2d/(d-1)}_x(\mathbb{R}^d))$. This implies that the solution scatters as $t \to \pm\infty$, again by standard arguments. □

2.2. General case

We now see that the previous method extends in a straightforward manner to the case $\gamma_{\mathrm{ref}} = 0$, $\mathrm{Tr}\,|\gamma_0| < \infty$. In this case, we may write the spectral decomposition of γ_0:

$$\gamma_0 = \sum_j \lambda_j |u_{j,0}\rangle\langle u_{j,0}|,$$

where $(u_{j,0})_j$ is an orthonormal system in $L^2_x(\mathbb{R}^d)$ and $(\lambda_j) \subset \mathbb{R}$ is such that $\mathrm{Tr}\,|\gamma_0| = \sum_j |\lambda_j| < \infty$. Then, Equation (1.1) is equivalent to the following system of coupled equations:

$$\begin{cases} i\partial_t u_j &= \left(-\Delta + w * \left(\sum_k \lambda_k |u_k|^2\right)\right) u_j, \\ (u_j)_{|t=0} &= u_{j,0}, \end{cases} \quad j \in \mathbb{N}. \tag{2.4}$$

This means that for all times t, the solution $\gamma(t)$ to (1.1) can be written as

$$\gamma(t) = \sum_j \lambda_j |u_j(t)\rangle\langle u_j(t)|,$$

where $(u_j)_j = (u_j(t))_j$ is solution to (2.4) (notice that this system remains orthonormal for all times t). The operator $\gamma(t)$ has the same eigenvalues as the initial data γ_0, only its eigenvectors change with time. Even though the system (2.4) looks more complicated that (2.1), we can prove a result analogue to Proposition 1 in this case.

Proposition 2 (Dispersion of small trace-class solutions without background)**.** *Let $d \geqslant 2$ and $w \in (L^{d/2}_x \cap L^\infty_x)(\mathbb{R}^d)$. Then, there exists $\varepsilon_0 > 0$ such that for all γ_0 with $\operatorname{Tr}|\gamma_0| \leqslant \varepsilon_0$, there exists a unique system $(u_j)_j \subset C^0_t(\mathbb{R}, L^2_x(\mathbb{R}^d))$ solution to (2.4) which verifies $\sum_k \lambda_k |u_k|^2 \in L^2_t(\mathbb{R}, L^{d/(d-1)}_x(\mathbb{R}^d))$. Furthermore, there exist $(u_{j,\pm})_j \subset L^2_x(\mathbb{R}^d)$ such that*

$$\forall j \in \mathbb{N}, \quad \lim_{t \to \pm\infty} \left\| u_j(t) - e^{it\Delta} u_{j,\pm} \right\|_{L^2_x} = 0. \tag{2.5}$$

Again, the fact that each $u_j(t)$ behaves like a free solution for $t \to \pm\infty$ implies in particular that

$$\gamma(t) = \sum_j \lambda_j |u_j(t)\rangle\langle u_j(t)| \xrightarrow[t \to \pm\infty]{} 0 = \gamma_{\mathrm{ref}},$$

which is the expected result in the case $\gamma_{\mathrm{ref}} = 0$, $\operatorname{Tr}|\gamma_0| < \infty$.

Proof of Proposition 2. Again, the global well-posedness in $L^2_x(\mathbb{R}^d)$ follows from the fact that $w \in L^\infty_x(\mathbb{R}^d)$. It is thus enough to show that

$$\sum_k \lambda_k |u_k|^2 \in L^2_t(\mathbb{R}, L^{d/(d-1)}_x(\mathbb{R}^d))$$

for small enough initial data. We can show this using the triangle inequality,

$$\begin{aligned} \left\| \sum_k \lambda_k |u_k|^2 \right\|_{L^2_t L^{d/(d-1)}_x} &\lesssim \sum_k |\lambda_k| \, \|u_k\|^2_{L^4_t L^{2d/(d-1)}_x} \\ &\lesssim \sum_k |\lambda_k| + \left(\sum_k |\lambda_k| \, \|u_k\|^2_{L^4_t L^{2d/(d-1)}_x} \right)^3, \end{aligned}$$

where in the last line we used the Duhamel formulation for each u_k and Strichartz estimates, as in the proof of Proposition 1. By a continuation argument, we get the result if $\sum_k |\lambda_k| \leqslant \varepsilon_0$ with $\varepsilon_0 > 0$ small enough. □

The previous proof shows that the rank-one techniques extend to the trace-class case essentially through the triangle inequality. However, if $\sum_k |\lambda_k| = +\infty$, the previous proof does not work at all. Moreover, we implicitly used that $\sum_k |\lambda_k| < +\infty$ to define the density $\rho_\gamma = \sum_k \lambda_k |u_k|^2$: it is a well-defined L^1_x-function by the triangle inequality and the fact that the (u_k) are normalized:

$$\|\rho_\gamma\|_{L^1_x} \leqslant \sum_k |\lambda_k| \, \|u_k\|^2_{L^2_x} = \sum_k |\lambda_k|.$$

When $\sum_k |\lambda_k| = +\infty$, the definition of the density $\sum_k \lambda_k |u_k|^2$ is not clear. The goal of the next section is to introduce the tools used to overcome these difficulties.

3. Strichartz estimates outside the trace-class

In this section, we discuss Strichartz estimates for initial data outside the trace-class, which were proved by Frank, Lewin, Lieb, and Seiringer [12], and later extended by Frank and the author [13].

3.1. Strichartz estimates for orthonormal systems

As we saw in the previous section, the meaning of (1.1) for γ outside the trace-class is not clear, already because the density ρ_γ is not a well-defined object. Thus, before trying to solve (1.1) outside the trace-class, it is natural to study the free equation in this setting. We look at solutions to

$$\begin{cases} i\partial_t\gamma &= [-\Delta,\gamma], \\ \gamma_{|t=0} &= \gamma_0, \end{cases} \tag{3.1}$$

with typically $\operatorname{Tr}|\gamma_0| = +\infty$. The solution to (3.1) can be written as

$$\gamma(t) = e^{it\Delta}\gamma_0 e^{-it\Delta}.$$

If γ_0 is not trace-class but still compact, we may write its spectral decomposition

$$\gamma_0 = \sum_j \lambda_j |u_{j,0}\rangle\langle u_{j,0}|,$$

for some orthonormal system $(u_{j,0})_j$ in $L^2_x(\mathbb{R}^d)$. This implies that

$$\gamma(t) = \sum_j \lambda_j |e^{it\Delta}u_{j,0}\rangle\langle e^{it\Delta}u_{j,0}|\,,$$

and we are interested in giving a meaning to its density

$$\rho_{\gamma(t)} := \sum_j \lambda_j \left|e^{it\Delta}u_{j,0}\right|^2.$$

The key result is the following [12, Thm. 1], [13, Thm. 8].

Theorem 1 (Strichartz inequalities for orthonormal systems)**.** *Let $d \geqslant 1$, $p, q \geqslant 1$ such that*

$$\frac{2}{p} + \frac{d}{q} = d, \qquad 1 \leqslant q < 1 + \frac{2}{d-1}.$$

Then, for any (possibly infinite) orthonormal system $(u_{j,0})_j \subset L^2_x(\mathbb{R}^d)$ and for any set of coefficients $(\lambda_j)_j \subset \mathbb{C}$, we have

$$\left\| \sum_j \lambda_j \left|e^{it\Delta}u_{j,0}\right|^2 \right\|_{L^p_t(\mathbb{R},L^q_x(\mathbb{R}^d))} \lesssim \left(\sum_j |\lambda_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}. \tag{3.2}$$

Remark 3. Theorem 1 was proved for the first time in [12], for the range $1 \leqslant q \leqslant 1 + 2/d$. It was extended to the range $1 \leqslant q < 1 + 2/(d-1)$ in [13].

Remark 4. Theorem 1 shows that the density $\rho_{\gamma(t)}$ is a well-defined object in $L^p_t L^q_x$, if $\sum_j |\lambda_j|^{2q/(q+1)} < \infty$. When $q > 1$, we have $2q/(q+1) > 1$ and it is possible that $\sum_j |\lambda_j| = +\infty$ while $\sum_j |\lambda_j|^{2q/(q+1)} < \infty$. Notice that for a given t, the density $\rho_{\gamma(t)}$ may not be well-defined; it belongs to L^q_x only for almost every t. The key input to avoid using the triangle inequality is the orthonormality of the $(u_{j,0})_j$.

Remark 5. Theorem 1 also gives information about the dispersive properties of the free evolution of infinite quantum systems: for $\operatorname{Tr}|\gamma_0| = +\infty$, the density $\rho_{\gamma(t)}$ is "small" for large t in the sense that it belongs to $L^p_t(\mathbb{R})$ (with values in $L^q_x(\mathbb{R}^d)$).

Remark 6. For a given p, q in the range given by Theorem 1, the exponent $2q/(q+1)$ on the right side of (3.2) is optimal: the inequality is wrong if we replace $2q/(q+1)$ by some $\alpha > 2q/(q+1)$. This was proved in [12]. They also showed that if (3.2) holds, then one must have $q < 1 + 2/(d-1)$. In this sense, Theorem 1 is optimal.

Remark 7. When $d \geqslant 3$ for instance, Strichartz estimates for a single function are valid on a larger set of exponents, namely for $1 \leqslant q \leqslant 1 + 2/(d-2)$, as was proved by Keel and Tao [17]. When $1 + 2/(d-1) \leqslant q \leqslant 1 + 2/(d-2)$, an inequality of the type (3.2) is still true, up to replacing the exponent $2q/(q+1)$ on the right side by some $\alpha \in [1, 2q/(q+1))$. The case $\alpha = 1$ corresponds to applying the triangle inequality to the left side of (3.2) and using the Strichartz estimates for a single function. The question of the optimal α in this range remains open, however one can show that at the endpoint $q = 1 + 2/(d-2)$, $\alpha = 1$ is optimal: the triangle inequality is the best we can do at the endpoint.

Remark 8. One can formulate Theorem 1 only in terms of the operator $\gamma(t)$. To do so, we define the Schatten space of order $\alpha \geqslant 1$ as

$$\mathfrak{S}^\alpha := \{A \in \mathcal{K}(L^2(\mathbb{R}^d)),\ \mathrm{Tr}\,|A|^\alpha < \infty\},$$

where $\mathcal{K}(L^2(\mathbb{R}^d))$ denotes the space of all compact operators on $L^2(\mathbb{R}^d)$. The space $\mathfrak{S}^\alpha$ is endowed with the norm

$$\|A\|_{\mathfrak{S}^\alpha} := (\mathrm{Tr}\,|A|^\alpha)^{1/\alpha}.$$

Using these notations, Theorem 1 is equivalent to the fact that for all $\gamma_0 \in \mathfrak{S}^\alpha$, we have the inequality

$$\left\|\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}\right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \lesssim \|\gamma_0\|_{\mathfrak{S}^\alpha}. \tag{3.3}$$

In the context of the free equation, the picture is now clear: Strichartz estimates allow us to give a meaning to the density of solutions which are not trace-class, and these operators disperse in the sense that their density belongs to some $L^p_t(\mathbb{R})$. In Section 3.4, we will use the Strichartz estimates to extend these results to the nonlinear case of Equation (1.1), still in the context without background. Before doing so, we explain some elements of the proof of Theorem 1 in [13] (Section 3.2), and we also make a link between Theorem 1 and the Strichartz estimates for the kinetic transport equation (Section 3.3).

3.2. Elements of proof

The proof of Theorem 1 given in [13] relies on the proof used in the original paper of Strichartz [25]. By the TT^* method, the fact that $e^{it\Delta}$ is bounded from L^2_x to $L^p_t L^q_x$ is equivalent to the fact that the operator $e^{i(t-s)\Delta}$ (seen as an operator in both space and time variables) is bounded from $L^{p'}_t L^{q'}_x$ to $L^p_t L^q_x$. To prove it, Strichartz uses a complex interpolation method: he introduces an analytic family of operators (G_z) depending on a complex parameter z belonging to a strip in the complex plane $-1 - d/2 \leqslant \mathrm{Re}\, z \leqslant 0$. This family satisfies three properties:

- The operator $e^{i(t-s)\Delta}$ is an element of this family: $G_{-1} = e^{i(t-s)\Delta}$;
- For all $\eta \in \mathbb{R}$, $\|G_{i\eta}\|_{L^2_{t,x} \to L^2_{t,x}} \lesssim C(\eta)$;

- For all $\eta \in \mathbb{R}$, $\left\| G_{-1-d/2+i\eta} \right\|_{L^1_{t,x} \to L^\infty_{t,x}} \lesssim C(\eta)$.

Here, the function $\eta \mapsto C(\eta)$ has at most an exponential growth at infinity. The family (G_z) is built such that for all z, G_z is a (space-time) Fourier multiplier by a function $(\omega, \xi) \mapsto G_z(\omega, \xi)$. Hence, the last two properties follow from the estimates

$$\forall \eta \in \mathbb{R}, \quad \|G_{i\eta}\|_{L^\infty_{\omega,\xi}} \lesssim C(\eta), \quad \left\| \check{G}_{-1-d/2+i\eta} \right\|_{L^\infty_{t,x}} \lesssim C(\eta),$$

where $\check{G}$ denotes the inverse Fourier transform of G. Using Stein's interpolation theorem [23], Strichartz infers from these three properties that $G_{-1} = e^{i(t-s)\Delta}$ is a bounded operator from $L^{2(d+4)/(d+4)}_{t,x}$ to $L^{2+4/d}_{t,x}$, which is equivalent to the boundedness of $e^{it\Delta}$ from L^2_x to $L^{2+4/d}_{t,x}$. The starting point of the strategy of [13] is to notice that (3.2) is equivalent to the estimate

$$\left\| W(t) e^{i(t-s)\Delta} \overline{W(s)} \right\|_{\mathfrak{S}^{\frac{2q}{q-1}}(L^2_{t,x})} \lesssim \|W\|^2_{L_t^{\frac{2p}{p-2}} L_x^{\frac{2q}{q-2}}}, \tag{3.4}$$

for all $W \in L_t^{\frac{2p}{p-2}} L_x^{\frac{2q}{q-2}}$. This last estimate is shown by complex interpolation, using again three properties:

- $W(t) G_{-1} \overline{W(s)} = W(t) e^{i(t-s)\Delta} \overline{W(s)}$;
- For all $\eta \in \mathbb{R}$, $\left\| W(t) G_{i\eta} \overline{W(s)} \right\|_{L^2_{t,x} \to L^2_{t,x}} \lesssim C(\eta) \|W\|^2_{L^\infty_{t,x}}$;
- For all $\eta \in \mathbb{R}$, $\left\| W(t) G_{-1-d/2+i\eta} \overline{W(s)} \right\|_{\mathfrak{S}^2(L^2_{t,x})} \lesssim C(\eta) \|W\|^2_{L^2_{t,x}}$.

Each of these properties follow from their corresponding one proved by Strichartz. For the first two, this is obvious. To prove the last one, we estimate the Hilbert–Schmidt norm in the following fashion

$$\begin{aligned} &\left\| W(t) G_{-1-d/2+i\eta} \overline{W(s)} \right\|^2_{\mathfrak{S}^2(L^2_{t,x})} \\ &\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |W(t,x)|^2 \left| \check{G}_{-1-d/2+i\eta}(t-s, x-y) \right|^2 |W(s,y)|^2 \, dx \, dy \, ds \, dt \\ &\qquad\qquad \leqslant C(\eta) \|W\|^2_{L^2_{t,x}}. \end{aligned}$$

Using a variant of Stein's interpolation theorem in Schatten spaces, we deduce (3.4) for $q = 2 + 4/d$. The proof for general q's is done by modifying a little bit the proof of Strichartz: instead of using the estimate

$$\forall \eta \in \mathbb{R}, \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \left| \check{G}_{-1-d/2+i\eta}(t,x) \right| \lesssim C(\eta),$$

we use a more general estimate which is implicitly present in the article of Strichartz:

$$\forall \eta \in \mathbb{R}, \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \left| \check{G}_{-\lambda_0+i\eta}(t,x) \right| \lesssim C(\eta) |t|^{\lambda_0 - 1 - d/2},$$

valid for all $\lambda_0 > 1$. We conclude in the same fashion, except that we estimate the Hilbert–Schmidt norm using the Hardy–Littlewood–Sobolev inequality. Notice that this method can be used to show that the original article of Strichartz actually contains the full range of Strichartz estimates (and not only the one for $p = q$ as it is stated in his paper), except for the Keel–Tao endpoint.

In [13], we emphasize the generality of our method: if some functional inequality is proved by interpolating a $L^2 \to L^2$-bound with a $L^1 \to L^\infty$-bound, then it *automatically* implies a more general inequality for systems of orthonormal functions. We applied it to several contexts including the restriction estimates to hypersurfaces of Stein [24] and Strichartz [25] and the uniform Sobolev inequalities of Kenig, Ruiz, and Sogge [18].

3.3. Link with Strichartz estimates for the transport equation

In this section we explain briefly the link between Theorem 1 and the Strichartz estimates for the kinetic transport equation

$$\begin{cases} i\partial_t f + 2v \cdot \nabla_x f = 0, \quad t \in \mathbb{R},\, x \in \mathbb{R}^d,\, v \in \mathbb{R}^d, \\ f_{t=0} = f_0, \end{cases} \tag{3.5}$$

where $f = f(t,x,v) \geqslant 0$ is a phase-space distribution. Equation (3.5) also enjoys Strichartz estimates, which are expressed in terms of the density of the solution f,

$$\rho_{f(t)}(x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv, \quad \forall x \in \mathbb{R}^d,$$

by the estimate

$$\left\| \rho_{\gamma(t)} \right\|_{L^p_t L^q_x} \lesssim \| f_0 \|_{L^{\frac{2q}{q+1}}_{x,v}}, \tag{3.6}$$

for all $d, p, q \geqslant 1$ satisfying $2/p + d/q = d$ and $q < 1 + 2/(d-1)$. These estimates were proved by Castella and Perthame [8] and Keel and Tao [17]. The similarity with (3.2) is striking: it is valid for the same range of exponents, and the inequalities themselves are very similar, up to identifying $\rho_{\gamma(t)}$ with $\rho_{f(t)}$ and the Schatten space $\mathfrak{S}^{2q/(q+1)}$ with $L^{2q/(q+1)}_{x,v}$. Later on, Bennett, Bez, Gutierrez, and Lee [4] translated the method of [12] in the commutative setting of the transport equation to provide a new proof of Strichartz estimates for (3.5) and to prove that the endpoint $q = 1 + 2/(d-1)$ also fails for the transport equation, a fact which was left open in [17]. However, the precise relation between these two types of Strichartz estimates has never been worked out. The following result sheds some light on this relation.

Lemma 9. *The Strichartz estimates given by Theorem 1 imply the Strichartz estimates for the kinetic transport equation.*

Proof. Let us prove the Strichartz estimate for the transport equation for any f_0 belonging to the Schwartz class. To such f_0 we may associate its semi-classical (standard) quantization, which is an operator γ_0 on $L^2_x(\mathbb{R}^d)$ defined by

$$\gamma_0 \varphi(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f_0(x, h\xi) \widehat{\varphi}(\xi) e^{ix\cdot\xi}\, d\xi, \quad \forall x \in \mathbb{R}^d,$$

for all $\varphi \in L^2_x(\mathbb{R}^d)$ and for all $h > 0$. It is a well-known fact in the theory of pseudo-differential operators that

$$\|\gamma_0\|^r_{\mathfrak{S}^r} \sim \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f_0(x,\xi)|^r\, dx\, d\xi \quad (h \to 0).$$

Furthermore, an explicit computation yields

$$\rho_{\gamma(t)}(x) = \frac{h^{-d}}{(2\pi)^{3d/2}} \left(\rho_{f(t/h)} * \mathcal{F}\left[e^{-it|\cdot|^2}\right]\right)(x),$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, where $\gamma(t) = e^{it\Delta}\gamma_0 e^{-it\Delta}$ and $\mathcal{F}$ denotes the Fourier transform. A change of variables leads to

$$(2\pi)^d h^{d-\frac{1}{p}} \left\|\rho_{\gamma(t)}\right\|_{L^p_t L^q_x} = (2\pi)^{-d/2} \left\|\rho_{f(t)} * \mathcal{F}\left[e^{-ith|\cdot|^2}\right]\right\|_{L^p_t L^q_x},$$

and hence by Fatou's lemma together with Theorem 1

$$\left\|\rho_{f(t)}\right\|_{L^p_t L^q_x} \lesssim \liminf_{h\to 0} h^{d-\frac{1}{p}} \left\|\rho_{\gamma(t)}\right\|_{L^p_t L^q_x} \lesssim \liminf_{h\to 0} h^{d-\frac{1}{p}} \left\|\gamma_0\right\|_{\mathfrak{S}^{\frac{2q}{q+1}}} \simeq \left\|f_0\right\|_{L^{\frac{2q}{q+1}}_{x,v}},$$

which is the desired Strichartz estimate. □

We thus see that Strichartz estimates for systems of orthonormal functions may be seen as the missing link between Strichartz estimates for the Schrödinger and the transport equations.

3.4. Application: dispersion outside the trace-class

As a natural consequence of a new family of Strichartz estimates, we now prove a well-posedness result for (1.1) for initial data outside of the trace-class.

Theorem 2 (Global existence of infinite-trace solutions)**.** *Let $d \geqslant 1$, $p, q \geqslant 1$ such that $2/p + d/q = d$ and $q < 1 + 2/(d-1)$. Let $w \in L^{q'}_x(\mathbb{R}^d)$. Then, for any $\gamma_0 \in \mathfrak{S}^{2q/(q+1)}$, there exists a unique $\gamma \in C^0_t(\mathbb{R}, \mathfrak{S}^{2q/(q+1)})$ solution to* (1.1) *such that $\rho_\gamma \in L^p_{t,loc}(\mathbb{R}, L^q_x(\mathbb{R}^d))$.*

Proof. Let $T, R > 0$. We apply a fixed-point theorem to the function

$$F : (\gamma,\rho) \mapsto \Big(t \mapsto e^{it\Delta}\gamma_0 e^{-it\Delta} - i\int_0^t e^{i(t-s)\Delta}[w * \rho(s), \gamma(s)]e^{i(s-t)\Delta}\, ds,$$
$$(t,x) \mapsto \rho\left[e^{it\Delta}\gamma_0 e^{-it\Delta} - i\int_0^t e^{i(t-s)\Delta}[w * \rho(s), \gamma(s)]e^{i(s-t)\Delta}\, ds\right](x)\Big),$$

on the ball

$$B_R = \Big\{(\gamma,\rho) \in C^0_t([0,T], \mathfrak{S}^{\frac{2q}{q+1}}) \times L^p_t([0,T], L^q_x(\mathbb{R}^d)),$$
$$\|\gamma\|_{C^0_t \mathfrak{S}^{\frac{2q}{q+1}}} + \|\rho\|_{L^p_t L^q_x} \leqslant R\Big\}.$$

Here, we used the notation $\rho[A](x) := \rho_A(x)$. We have to separate the unknown γ and ρ because the density of $\gamma \in \mathfrak{S}^{2q/(q+1)}$ is a not a well-defined object. It will be well-defined for solutions to (1.1) however, as in Theorem 1. To estimate F, we use Theorem 1 together with the inhomogeneous Strichartz estimates [12, Cor. 1]. The latter read

$$\left\|\rho\left[\int_0^t e^{i(t-s)\Delta}R(s)e^{i(s-t)\Delta}\, ds\right]\right\|_{L^p_t L^q_x} \lesssim \left\|\int_{\mathbb{R}} e^{is\Delta}|R(s)|e^{-is\Delta}\, ds\right\|_{\mathfrak{S}^{\frac{2q}{q+1}}}, \tag{3.7}$$

for p, q in the same range as in the statement of Theorem 1. This leads to

$$\|F(\gamma,\rho)\|_{C^0_t \mathfrak{S}^{\frac{2q}{q+1}} \times L^p_t L^q_x} \lesssim \|\gamma_0\|_{\mathfrak{S}^{\frac{2q}{q+1}}} + T^{1/p'}R^2, \quad \forall(\gamma,\rho) \in B_R.$$

Hence, if $\|\gamma_0\|_{\mathfrak{S}^{\frac{2q}{q+1}}} \lesssim R$ and $T = T(R) > 0$ is small enough, the function F stabilizes B_R. Under similar conditions, one can easily show that this is a contraction on B_R. The Banach–Picard fixed point theorem then ensures the existence of a solution to (1.1) in the announced space. Uniqueness can be proved with similar estimates. To get global solutions, we first notice that local well-posedness shows that we can extend the solution as long as its Schatten norm $\mathfrak{S}^{2q/(q+1)}$ remains bounded. Furthermore, using the results of [26], we infer that there exists a unitary operator $U(t)$ such that for all times of existence t, we have $\gamma(t) = U(t)\gamma_0 U(t)^*$. As a consequence, any Schatten norm is preserved along the flow and solutions are globally defined in time. $\square$

This result does not give any information on the long time behaviour of small solutions. This is the content of the following theorem.

Theorem 3 (Dispersion of small, infinite-trace solutions without background)**.** *Let $d \geqslant 2$ and $w \in (L^{d/2}_x \cap L^d_x)(\mathbb{R}^d)$. Then, there exists $\varepsilon_0 > 0$ such that for any $\gamma_0 \in \mathfrak{S}^{2d/(2d-1)}$ with $\|\gamma_0\|_{\mathfrak{S}^{2d/(2d-1)}} \leqslant \varepsilon_0$, there exists a unique $\gamma \in C^0_t(\mathbb{R}, \mathfrak{S}^{2d/(2d-1)})$ solution to* (1.1) *satisfying $\rho_\gamma \in L^2_t(\mathbb{R}, L^{d/(d-1)}_x(\mathbb{R}^d))$. In particular, we have $\gamma(t) \rightharpoonup 0$ as $t \to \pm\infty$.*

This shows the asymptotic stability of $\gamma_{\mathrm{ref}} = 0$ under perturbations in $\mathfrak{S}^{2d/(2d-1)}$ which may not be trace-class, and thus can contain an infinite number of particles.

Proof of Theorem 3. By Theorem 2 applied to $q = d/(d-1)$, we only have to show that $\rho_\gamma \in L^2_t(\mathbb{R}, L^{d/(d-1)}_x(\mathbb{R}^d))$ when $\|\gamma_0\|_{\mathfrak{S}^{2d/(2d-1)}}$ is small enough. To do so, we write the Duhamel formulation of (1.1) and take its density:

$$\rho_{\gamma(t)} = \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} + \rho\left[-i\int_0^t e^{i(t-s)\Delta}[w * \rho_{\gamma(s)}, \gamma(s)]e^{i(s-t)\Delta}\,ds\right].$$

The first term can be estimated by Theorem 1, while one may be tempted to use the inhomogeneous Strichartz estimate (3.7) on the second term. If we do so, we obtain

$$\left\|\rho\left[-i\int_0^t e^{i(t-s)\Delta}[w * \rho_{\gamma(s)}, \gamma(s)]e^{i(s-t)\Delta}\,ds\right]\right\|_{L^2_t L^{\frac{d}{d-1}}_x} \lesssim \left\|\int_{\mathbb{R}} e^{is\Delta}|[w * \rho_{\gamma(s)}, \gamma(s)]|e^{-is\Delta}\,ds\right\|_{\mathfrak{S}^{\frac{2d}{2d-1}}}. \quad (3.8)$$

The operator $|[w * \rho_{\gamma(s)}, \gamma(s)]|$ does not seem to belong to $L^1_t\mathfrak{S}^{2d/(2d-1)}$ since we only have $w * \rho_\gamma \in L^2_t L^\infty_x$ and $\gamma \in L^\infty_t \mathfrak{S}^{2d/(2d-1)}$. Hence, the finiteness of the right side of (3.8) is not clear at all. It is interesting to compare this estimate with the estimate (2.3) that we used in the proof of the same result in the rank one case. In (2.3), we use that $u \in L^4_t$ to infer that $(w * |u|^2)u \in L^{4/3}_t$. In another words, the time decay of u is equivalent to the time decay of $|u|^2$. In our more general setting, $\rho_\gamma \in L^2_t$ but it is wrong to say that $\gamma \in L^2_t$. This is the reason why the inhomogeneous Strichartz estimates (3.7) are weaker than their "single function" equivalent (2.3), and we cannot show dispersion of small solutions directly from them. To go around the lack of decay of $\gamma(s)$, we use the best information that we know about it, namely

that it is a solution to (1.1). Iterating the Duhamel formulation leads to the formula

$$\gamma(t) = e^{it\Delta}\gamma_0 e^{-it\Delta} - i\int_0^t e^{i(t-t_1)\Delta}[w*\rho_{\gamma(t_1)}, e^{it_1\Delta}\gamma_0 e^{-it_1\Delta}]e^{i(t_1-t)\Delta}\,dt_1 + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 \times \times e^{i(t-t_1)\Delta}\left[w*\rho_{\gamma(t_1)}, e^{i(t_1-t_2)\Delta}[w*\rho_{\gamma(t_2)}, \gamma(t_2)]e^{i(t_2-t_1)\Delta}\right]e^{i(t_1-t)\Delta}.$$

The second term can be shown to decay, as we will see shortly. The third term, however, still has a $\gamma(t_2)$ term which does not decay. We get rid of it by again injecting the value of $\gamma(t_2)$ given by Duhamel's formula, and so on. Iterating this procedure an infinite number of times in order to eliminate all the terms involving a $\gamma(t_n)$ leads to the following expansion of $\rho_{\gamma(t)}$ into a Dyson series:

$$\rho_{\gamma(t)} = \sum_{n,m\geqslant 0}\rho\left[e^{it\Delta}\mathcal{W}_V^{(n)}(t)\gamma_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\right], \tag{3.9}$$

where the wave operators $\mathcal{W}_V^{(n)}(t)$ are defined as

$$\mathcal{W}^{(n)}(t) = (-i)^n\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n \times \times e^{-it_1\Delta}V(t_1)e^{it_1\Delta}\cdots e^{-it_n\Delta}V(t_n)e^{it_n\Delta}.$$

Here, we used the notation $V = w*\rho_\gamma$ for the potential generated by ρ_γ. The equation (3.9) is a closed equation on ρ_γ where the solution $\gamma(t)$ only appears through its initial data γ_0. The key dispersive estimates that we need are

Proposition 10. *Let $d\geqslant 2$ and $p,q\geqslant 1$ such that $2/p + d/q = d$ and $q\leqslant 1+2/d$. Then, for all $V\in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^d))$, for all $n,m\geqslant 0$, and for all $\gamma_0\in\mathfrak{S}^{2q/(q+1)}$, we have*

$$\left\|\rho\left[e^{it\Delta}\mathcal{W}_V^{(n)}(t)\gamma_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\right]\right\|_{L_t^pL_x^q} \leqslant \frac{C^{n+m}\left\|V\right\|_{L_t^{p'}L_x^{q'}}^{n+m}}{(n!)^{\frac{1}{2q'}}(m!)^{\frac{1}{2q'}}}\left\|\gamma_0\right\|_{\mathfrak{S}^{\frac{2q}{q+1}}}, \tag{3.10}$$

for some constant C independent of V, n, m, γ_0.

For $n = 0 = m$, Proposition 10 reduces to Theorem 1 in the restricted range $q\leqslant 1+2/d$ (by convention, $\mathcal{W}^{(0)} = 1$). Hence, Proposition 10 may be seen as a family of generalized Strichartz estimates, which play the role of inhomogeneous Strichartz estimates. The proof of Proposition 10 uses duality arguments and mimics the proof of [12, Thm. 3]. A huge drawback of (3.10) compared to usual inhomogeneous Strichartz estimates is that the space $L_t^{p'}L_x^{q'}$ in which V lives on the right side of (3.10) has to be the dual of the space $L_t^pL_x^q$ where ρ lives of the left side of (3.10). In particular, in our nonlinear context of (3.9) where $V = w*\rho_\gamma$, we see that V has the same time decay as ρ_γ and our method imposes $p = p'$, which is why we have to take $p = 2$. Hence, using Proposition 10 in the case $p = 2$, $q = d/(d-1)$, to estimate (3.9), we get

$$\left\|\rho_\gamma\right\|_{L_t^2L_x^{d/(d-1)}} \lesssim \left\|\gamma_0\right\|_{\mathfrak{S}^{2d/(2d-1)}} + \mathcal{O}\left(\left\|\rho_\gamma\right\|_{L_t^2L_x^{d/(d-1)}}^2\right),$$

which by a continuation argument shows the desired result. □

4. Dispersion of small solutions outside the trace-class with background

We have seen that the trivial background $\gamma_{\text{ref}} = 0$ is asymptotically stable under small perturbations that are not necessarily trace-class. The purpose of this section is to present our main result in [21], which treats the case $\gamma_{\text{ref}} \neq 0$.

Theorem 4 (Dispersion around a non-trivial background in $d = 2$)**.** *Let $d = 2$, w and g be smooth, decaying enough functions, and let $\gamma_{\text{ref}} = g(-i\nabla)$. We assume that w and g are such that γ_{ref} is linearly stable under the flow* (1.1)*. Then, there exists $\varepsilon_0 > 0$ such that for any $\gamma_0 \in \gamma_{\text{ref}} + \mathfrak{S}^{4/3}$ with $\|\gamma_0 - \gamma_{\text{ref}}\|_{\mathfrak{S}^{4/3}} \leqslant \varepsilon_0$, there exists a unique $\gamma \in C^0_t(\mathbb{R}, \gamma_{\text{ref}} + \mathfrak{S}^{4/3})$ solution to* (1.1) *such that $\rho_\gamma \in \rho_{\gamma_{\text{ref}}} + L^2_t(\mathbb{R}, L^2_x(\mathbb{R}^d))$. Furthermore, there exist profiles $Q_\pm \in \mathfrak{S}^4$ such that*

$$\lim_{t\to\pm\infty} \left\|\gamma(t) - \gamma_{\text{ref}} - e^{it\Delta} Q_\pm e^{-it\Delta}\right\|_{\mathfrak{S}^4} = 0. \tag{4.1}$$

In particular, we see that (4.1) implies that $\gamma(t) \rightharpoonup \gamma_{\text{ref}}$ as $t \to \pm\infty$. Furthermore, $\gamma(t)$ behaves asymptotically as $t \to \pm\infty$ as a free solution around γ_{ref}.

Remark 11. The assumption of linear stability will be explained in the next section.

Remark 12. The fact that we only dealt with $d = 2$ will be explained when discussing the proof of Theorem 4.

Remark 13. We allow perturbations of γ_{ref} that belong to $\mathfrak{S}^{4/3}$ and in particular that may not be trace-class. This means that we may perturb an infinite number of particles of the Fermi gas and still return to the same Fermi gas for large times.

Remark 14. Our theorem applies to the Fermi gases (1.3), (1.4) and (1.5). However, we will see in the next section that it does not apply to the Fermi sea at zero temperature (1.2). The question whether this last translation invariant state is unstable remains open.

4.1. Linear stability

When studying the nonlinear stability of some non-trivial stationary solution, it is useful to first understand the linearization of the equation around the stationary state. More explicitly, writing $\gamma(t) = \gamma_{\text{ref}} + Q(t)$, the Hartree equation (1.1) on γ is equivalent to the following equation on Q:

$$\begin{cases} i\partial_t Q &= [-\Delta, Q] + [w * \rho_Q, Q] + [w * \rho_Q, \gamma_{\text{ref}}], \\ Q_{|t=0} &= Q_0 \in \mathfrak{S}^{4/3}. \end{cases} \tag{4.2}$$

This equation differs from (1.1) only by the term $[w * \rho_Q, \gamma_{\text{ref}}]$. In particular, if this term were not present, we already saw in Section 3.4 that solutions go to zero for large times. This was a consequence of the dispersion induced by the term $[-\Delta, Q]$ and the nonlinear term $[w * \rho_Q, Q]$ which is quadratic in Q and hence of lower order (since we deal with small solutions) was treated as a perturbation. Here, the problem is radically different since the term $[w * \rho_Q, \gamma_{\text{ref}}]$ is of the same order in Q as the dispersive term $[-\Delta, Q]$. Hence, we have to check that it does not destroy the dispersion induced by $-\Delta$, and to do so we investigate the large time behaviour

of solutions to the following linear equation

$$\begin{cases} i\partial_t Q &= [-\Delta, Q] + [w * \rho_Q, \gamma_{\mathrm{ref}}], \\ Q_{|t=0} &= Q_0 \in \mathfrak{S}^{4/3}, \end{cases} \tag{4.3}$$

which is nothing but the linearization of (1.1) around γ_{ref}. The Duhamel formulation of (4.3) is

$$Q(t) = e^{it\Delta} Q_0 e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta} [w * \rho_{Q(s)}, \gamma_{\mathrm{ref}}] e^{i(s-t)\Delta}\, ds.$$

Defining the linear operator

$$\mathcal{L}(\rho) := \rho \left[i \int_0^t e^{i(t-s)\Delta} [w * \rho_{Q(s)}, \gamma_{\mathrm{ref}}] e^{i(s-t)\Delta}\, ds \right]$$

leads to the following equation satisfied by ρ_Q,

$$\rho_Q = \rho_{e^{it\Delta} Q_0 e^{-it\Delta}} - \mathcal{L}(\rho_Q)$$

and hence

$$\rho_Q = (1 + \mathcal{L})^{-1} \rho_{e^{it\Delta} Q_0 e^{-it\Delta}}.$$

The Strichartz estimates show that $\rho_{e^{it\Delta} Q_0 e^{-it\Delta}} \in L^2_{t,x}$ if $Q_0 \in \mathfrak{S}^{4/3}$. To prove that $\rho_Q \in L^2_{t,x}$, it is thus sufficient that the operator $1 + \mathcal{L}$ is invertible on $L^2_{t,x}$. The fact that $1 + \mathcal{L}$ is invertible on $L^2_{t,x}$ is what we call linear stability in the statement of Theorem 4.

The operator $\mathcal{L}$ is a space-time Fourier multiplier by the function

$$(\omega, \xi) \in \mathbb{R} \times \mathbb{R}^d \mapsto \widehat{w}(\xi) m_g(\omega, \xi)$$

where we used the notation

$$m_g(\omega, \xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(t|\xi|^2) \check{g}(2t\xi) e^{-it\omega}\, dt.$$

Here, ω denotes the Fourier variable dual to t and ξ the Fourier variable dual to x. The function m_g is well-known in the condensed matter physics litterature, where it is called the Lindhard function [15, Sec. 4.4]. As a Fourier multiplier, the operator $1 + \mathcal{L}$ is invertible on $L^2_{t,x}$ if and only if

$$\inf_{(\omega,\xi) \in \mathbb{R} \times \mathbb{R}^d} |1 + \widehat{w}(\xi) m_g(\omega, \xi)| > 0.$$

This condition is a quantum analogue of the condition that appears in the study of Landau damping (see Eq. (2.3) and the following condition **(L)** in [22]). In [21, Cor. 1], we give two explicit conditions under which $1 + \mathcal{L}$ is invertible.

Proposition 15. *The operator $1 + \mathcal{L}$ is invertible on $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^d)$ if one of the two following conditions is satisfied:*

1. $\|\check{g}\|_{L^1(\mathbb{R}^2)} \|\widehat{w}\|_{L^\infty(\mathbb{R}^2)} < 4\pi$;
2. *g is a radial function, $g(\xi) = f(|\xi|^2)$ for all ξ, with $f' < 0$ a.e. and*

$$\max\left(\varepsilon_g \widehat{w}(0)_+, \|\check{g}\|_{L^1(\mathbb{R}^2)} \|(\widehat{w})_-\|_{L^\infty(\mathbb{R}^2)} \right) < 4\pi,$$

where $0 \leqslant \varepsilon_g \leqslant \|\check{g}\|_{L^1(\mathbb{R}^2)}$ is defined as

$$\varepsilon_g := -\frac{1}{8\pi} \inf_{a \in \mathbb{R}} \int_0^\infty r \check{g}(r) \cos(ar)\, dr.$$

Remark 16. The first condition is a perturbative condition since we have the uniform bound for all $(\omega, k) \in \mathbb{R} \times \mathbb{R}^d$,

$$|\widehat{w}(k) m_g(\omega, k)| \leqslant (4\pi)^{-1} \|\check{g}\|_{L^1(\mathbb{R}^2)} \|\widehat{w}\|_{L^\infty(\mathbb{R}^2)}.$$

The second condition is weaker than the first one, except that we have to assume that g is radially decreasing. In the defocusing case, that is when $\widehat{w} \geqslant 0$ (see [19]), this condition amounts to

$$\varepsilon_g \widehat{w}(0)_+ < 4\pi.$$

Remark 17. The Fermi sea at zero temperature (1.2) does not satisfy any of these two conditions. We can furthermore show that in this case, the operator $1 + \mathcal{L}$ is not invertible on $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2)$. Whether this shows that there are solutions to (4.3) for which $\rho_Q \notin L^2_{t,x}$ remains an open problem.

Under the conditions listed in Proposition 15, we see that solutions to (4.3) disperse like solutions to the free equation, in the sense that their density belongs to $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^d)$. In the next section, we close the argument to prove Theorem 4 by explaining how to go from linear stability to the full nonlinear stability.

4.2. Elements of proof

The strategy to prove Theorem 4 is the same as the one to prove Theorem 3, that is in the case without background. We start by expanding the density of the solution to the Hartree equation as a Dyson series,

$$\rho_{\gamma(t)} = \sum_{n,m \geqslant 0} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_0 \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right],$$

with $V := w * \rho_Q$. In the proof of Theorem 3 without background, we had that γ_0 belonged to the Schatten space $\mathfrak{S}^{4/3}$. Here, we rather have that $\gamma_0 = \gamma_{\mathrm{ref}} + Q_0$ with $Q_0 \in \mathfrak{S}^{4/3}$. Hence, a consequence of the proof of Theorem 3 is the estimate

$$\left\| \sum_{n,m \geqslant 0} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) Q_0 \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] \right\|_{L^2_{t,x}} \lesssim \|Q_0\|_{\mathfrak{S}^{4/3}} + \mathcal{O}\left(\|\rho_Q\|^2_{L^2_{t,x}} \right).$$

We thus only have to estimate the terms involving γ_{ref}, where the initial data Q_0 does not appear. First of all, we notice that

$$\sum_{n+m=0} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_{\mathrm{ref}} \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] = \gamma_{\mathrm{ref}},$$

$$\sum_{n+m=1} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_{\mathrm{ref}} \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] = -\mathcal{L}(\rho_Q),$$

and hence

$$\rho_Q = (1 + \mathcal{L})^{-1} \left(\sum_{n,m \geqslant 0} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) Q_0 \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] \right.$$
$$\left. + \sum_{n+m \geqslant 2} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_{\mathrm{ref}} \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] \right).$$

It thus only remains to treat the terms involving γ_{ref}, with $n + m \geqslant 2$. The key result in this direction is [21, Lemma 3].

Lemma 18. *Let $d \geqslant 1$, $g : \mathbb{R}^d \to \mathbb{R}$ such that $\check{g} \in L^1(\mathbb{R}^d)$, $1 < q \leqslant 1 + 2/d$ and p such that $2/p + d/q = d$. Let $V \in L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{R}^d)$. Then, for all $n, m \in \mathbb{N}$ such that*

$$n + m + 1 \geqslant 2q',$$

we have

$$\left\| \sum_{n,m \geqslant 0} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_{\mathrm{ref}} \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] \right\|_{L_t^p L_x^q} \leqslant C \left\| \check{g} \right\|_{L^1} \frac{C^{n+m} \left\| V \right\|_{L_t^{p'} L_x^{q'}}^{n+m}}{(n!)^{\frac{1}{2q'}} (m!)^{\frac{1}{2q'}}},$$

for some constant C independent of V, n, m, g.

The previous lemma gives an information similar to Proposition 10, except that it only holds for large values of $n+m$. This is where the restriction on the dimension comes from. We already saw in Theorem 3 that we could not treat the dimension $d = 1$ since we need $p = 2$ as explained after Proposition 10, and thus $q = \infty$ which is forbidden by the restriction $q \leqslant 1 + 2/d = 3$. In dimension $d \geqslant 2$, $p = 2$ implies $q = d/(d-1)$, and hence the restriction $n + m + 1 \geqslant 2q'$ becomes $n + m \geqslant 2d - 1$. In $d = 2$, this is $n + m \geqslant 3$ and hence only the terms with $n + m = 2$ remain. In $d = 3$, we have to treat the terms with $n + m = 2, 3, 4$. As the dimension increases, the number of terms remaining gets bigger and this is why the higher dimensions are harder to treat. The estimate on the term with $n + m = 2$ in $d = 2$ was given by [21, Prop. 4].

Proposition 19. *For w and g smooth and decaying enough, we have the estimate*

$$\left\| \sum_{n+m=2} \rho \left[e^{it\Delta} \mathcal{W}_V^{(n)}(t) \gamma_{\mathrm{ref}} \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta} \right] \right\|_{L_{t,x}^2} \leqslant C(g, w) \left\| \rho_Q \right\|_{L_{t,x}^2}^2 .$$

This last result together with our previous remarks lead to the global estimate

$$\left\| \rho_Q \right\|_{L_{t,x}^2} \lesssim \left\| Q_0 \right\|_{\mathfrak{S}^{4/3}} + \mathcal{O}\left(\left\| \rho_Q \right\|_{L_{t,x}^2}^2 \right)$$

which again by a continuation argument closes the proof of Theorem 4.

Conclusions and perspectives

We explained the results contained in [21] which prove the asymptotic stability of translation-invariant quantum density matrices under the nonlinear Hartree flow (Theorem 4). A key element of the proof is a generalization of Strichartz estimates to density matrices (Theorem 1), which was discovered in [12] and extended later in [13]. In the nonlinear context of the Hartree equation, these estimates have to be completed by another set of estimates (Proposition 10 and Lemma 18), which proof mimics ideas contained in [12]. This new set of estimates play a role analogue to inhomogeneous Strichartz estimates.

To go further into the study of the stability/instability of these translation-invariant states, one needs to understand better the properties of the linearized equation. In particular, the existence of small, non-decaying solutions to this equation remains a challenging problem.

References

[1] C. Bardos, L. Erdős, F. Golse, N. Mauser, and H.-T. Yau, *Derivation of the Schrödinger-Poisson equation from the quantum N-body problem*, C. R. Math. Acad. Sci. Paris, 334 (2002), pp. 515–520.

[2] C. Bardos, F. Golse, A. Gottlieb, and N. Mauser, *Mean field dynamics of fermions and the time-dependent Hartree-Fock equation*, J. Math. Pures Appl. (9), 82 (2003), pp. 665–683.

[3] N. Benedikter, M. Porta, and B. Schlein, *Mean-field evolution of fermionic systems*, Comm. Math. Phys., 331 (2014), pp. 1087–1131.

[4] J. Bennett, N. Bez, S. Gutierrez, and S. Lee, *On the Strichartz estimates for the kinetic transport equation*, arXiv preprint arXiv:1307.1600, (2013).

[5] A. Bove, G. Da Prato, and G. Fano, *An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction*, Commun. Math. Phys., 37 (1974), pp. 183–191.

[6] ——, *On the Hartree-Fock time-dependent problem*, Commun. Math. Phys., 49 (1976), pp. 25–33.

[7] E. Cancès and G. Stoltz, *A mathematical formulation of the random phase approximation for crystals*, Ann. Inst. H. Poincaré (C) Anal. Non Linéaire, 29 (2012), pp. 887–925.

[8] F. Castella and B. Perthame, *Estimations de Strichartz pour les équations de transport cinétique*, CR Acad. Sci. Paris Sér. I Math, 322 (1996), pp. 535–540.

[9] J. Chadam, *The time-dependent Hartree-Fock equations with Coulomb two-body interaction*, Commun. Math. Phys., 46 (1976), pp. 99–104.

[10] A. Elgart, L. Erdős, B. Schlein, and H.-T. Yau, *Nonlinear Hartree equation as the mean field limit of weakly coupled fermions*, J. Math. Pures Appl., 83 (2004), pp. 1241–1273.

[11] R. Frank, M. Lewin, E. Lieb, and R. Seiringer, *A positive density analogue of the Lieb-Thirring inequality*, Duke Math. J., 162 (2012), pp. 435–495.

[12] R. L. Frank, M. Lewin, E. H. Lieb, and R. Seiringer, *Strichartz inequality for orthonormal functions*, J. Eur. Math. Soc., (2013). In press.

[13] R. L. Frank and J. Sabin, *Restriction theorems for orthonormal functions, strichartz inequalities, and uniform sobolev estimates*, arXiv preprint arXiv:1404.2817, (2014).

[14] J. Fröhlich and A. Knowles, *A microscopic derivation of the time-dependent Hartree-Fock equation with Coulomb two-body interaction*, J. Stat. Phys., 145 (2011), pp. 23–50.

[15] G. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid*, Cambridge University Press, 2005.

[16] C. Hainzl, M. Lewin, and C. Sparber, *Existence of global-in-time solutions to a generalized Dirac-Fock type evolution equation*, Lett. Math. Phys., 72 (2005), pp. 99–113.

[17] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., 120 (1998), pp. 955–980.

[18] C. E. Kenig, A. Ruiz, and C. D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J., 55 (1987), pp. 329–347.

[19] M. Lewin and J. Sabin, *The Hartree equation for infinitely many particles. I. Well-posedness theory*, Comm. Math. Phys., (2013). To appear.

[20] M. Lewin and J. Sabin, *A family of monotone quantum relative entropies*, Lett. Math. Phys., 104 (2014), pp. 691–705.

[21] M. Lewin and J. Sabin, *The Hartree equation for infinitely many particles. II. Dispersion and scattering in 2D*, Analysis and PDE, 7 (2014), pp. 1339–1363.

[22] C. Mouhot and C. Villani, *On Landau damping*, Acta Math., 207 (2011), pp. 29–201.

[23] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc., 83 (1956), pp. 482–492.

[24] E. M. Stein, *Oscillatory integrals in Fourier analysis*, in Beijing lectures in harmonic analysis (Beijing, 1984), vol. 112 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355.

[25] R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J., 44 (1977), pp. 705–714.

[26] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys., 110 (1987), pp. 415–426.

[27] S. Zagatti, *The Cauchy problem for Hartree-Fock time-dependent equations*, Ann. Inst. H. Poincaré Phys. Théor., 56 (1992), pp. 357–374.

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