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**Uniform Estimates in Homogenization: Compactness Methods and Applications**

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# Uniform Estimates in Homogenization: Compactness Methods and Applications

Christophe Prange

## Abstract

The purpose of this note is to explain how to use compactness to get uniform estimates in the homogenization of elliptic systems with or without oscillating boundary. Along with new results in this direction, we highlight some important applications to pointwise estimates of Green and Poisson kernels, to the homogenization of boundary layer systems and to the boundary control of composite materials.

## 1. Introduction

In this note we aim at describing how to get uniform  $L^p$ , Hölder and Lipschitz bounds for  $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^N$  solving the elliptic system in divergence form

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon &= f + \nabla \cdot F, & x \in \Omega^\varepsilon, \\ u^\varepsilon &= g, & x \in \Gamma^\varepsilon \subset \partial\Omega^\varepsilon, \end{cases} \quad (1.1)$$

where  $\Omega^\varepsilon \subset \mathbb{R}^d$ . The matrix  $A$  is typically assumed to be periodic. On the contrary, *no structure assumption* is made on the oscillations of the boundary  $\partial\Omega^\varepsilon$ . All the techniques we use are designed for *systems* rather than single equations.

The general strategy to get uniform estimates, the so-called *compactness method*, has been developed by Avellaneda and Lin in the late 80's and applied successfully to a number of situations: elliptic or parabolic equations, Dirichlet or Neumann boundary conditions. . . Here, we emphasize in particular the recent results of the paper [30] related to uniform estimates in domains with oscillating boundaries. Our purpose is also to describe some important results that can be achieved thanks to the uniformity of these estimates in  $\varepsilon$ . Needless to say, the contents of this introductory note are far from being exhaustive on the subject, and we refer throughout the text to the original works. There, the reader can find a rich variety of results as well as detailed proofs. For a general introduction to (periodic) homogenization theory, the classical books by Bensoussan, Lions, Papanicolaou [10] and Cioranescu, Donato [14] for instance are nice references.

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*Keywords:* Homogenization, compactness methods, boundary layers, potential theory, Green kernel, Poisson kernel, control of distributed systems.

**Outline of this note.** Below we give several examples as motivations for the results presented in this note. The end of the introduction summarizes our main notations and assumptions. Section 2 is an introduction to compactness methods in homogenization. We review the classical approach of Avellaneda and Lin so as to give an insight into the main features and issues of the compactness method. We also outline the current developments of the compactness method. The recent results of the paper [30] are explained in section 3. The last part 4 deals with applications of the uniform estimates to pointwise bounds on kernels, to uniform estimates in  $L^p$  and to boundary layers in homogenization.

**Pointwise estimates in potential theory.** Let us look at the fundamental solution  $G = G(y, \tilde{y})$  of  $-\nabla \cdot A(y)\nabla$ . What we would like to describe is the large scale behavior of  $G(y, \tilde{y})$ , i.e. the behavior for  $|y - \tilde{y}| \gg 1$ . We introduce the (very small) number  $\varepsilon = 1/|y - \tilde{y}|$  and the rescaled variables  $x = \varepsilon y$ ,  $\tilde{x} = \varepsilon \tilde{y}$ , so that  $|x - \tilde{x}|$  is of order one. The fundamental solution  $G$  satisfies the important scaling property

$$G(y, \tilde{y}) = \varepsilon^{d-2} G^\varepsilon(x, \tilde{x}), \quad (1.2)$$

where  $G^\varepsilon$  is the fundamental solution associated to the operator with highly oscillating coefficients  $-\nabla \cdot A(x/\varepsilon)\nabla$ . Thanks to (1.2), the large scale behavior of  $G$  is related to the local behavior of the kernel  $G^\varepsilon$ . To put it in a nutshell, our initial problem of finding a bound for  $G(y, \tilde{y})$  when  $|y - \tilde{y}| \gg 1$  becomes an homogenization problem for  $G^\varepsilon(x, \tilde{x})$ ,  $|x - \tilde{x}| \sim 1$ .

**Boundary layer I: uniform estimates in  $L^p$ .** Although raised by Bensoussan, Lions and Papanicolaou [10, page xiii] in the 70's, the homogenization of boundary layer systems

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u_{bl}^\varepsilon = 0, & x \in \Omega \\ u_{bl}^\varepsilon = \varphi(x/\varepsilon), & x \in \partial\Omega. \end{cases} \quad (1.3)$$

has been a longstanding open problem. In this system, there are oscillations not only in the coefficients, but also in the Dirichlet boundary data  $\varphi = \varphi(y)$  which is assumed to be periodic in  $y$ . The oscillations on the boundary create strong gradients in a layer of size  $O(\varepsilon)$  close to the boundary, as is reflected by the standard a priori estimate

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \lesssim \varepsilon^{-1/2}. \quad (1.4)$$

The goal of the homogenization is to find a non oscillating homogenized matrix  $\bar{A}$  and boundary data  $\bar{\varphi}$  so that the solution of the homogenized system approximates well  $u_{bl}^\varepsilon$  in the limit  $\varepsilon \rightarrow 0$ .

This problem has attracted a great deal of attention in recent years: see [24, 23, 36] and the related papers [13, 17, 18] for non divergence form equations. Its difficulty lies in the following two facts: on the one hand the a priori bound (1.4) is singular in  $\varepsilon$ , on the other hand the boundary breaks the periodic microstructure, making the behavior of  $u_{bl}^\varepsilon$  very sensitive to the interaction between the periodic lattice and  $\partial\Omega$ .

A first reasonable attempt is to get uniform bounds on  $u_{bl}^\varepsilon$ . Of course if the system is scalar, i.e.  $N = 1$ , the maximum principle gives a uniform bound for  $u_{bl}^\varepsilon$  in  $L^\infty(\Omega)$ . This uniform a priori bound has been extended to  $L^p$  for  $1 < p < \infty$  by Avellaneda and Lin [6] in sufficiently smooth domains (satisfying a uniform exterior

sphere condition). How do these uniform  $L^p$  bounds extend to systems or less regular domains?

**Boundary layer II: asymptotic behavior.** Blowing-up in the vicinity of a point of  $\partial\Omega$  may be a good way to get an insight into the interplay between the microstructure and the boundary  $\partial\Omega$  in the boundary layer system (1.3). One is typically led to the analysis of the system

$$\begin{cases} -\nabla \cdot A(y)\nabla v_{bl} = 0, & y \cdot n > 0, \\ v_{bl} = \varphi(y), & y \cdot n = 0, \end{cases}$$

posed in the half-space  $\{y \cdot n > 0\}$ , with  $n \in \mathbb{S}^{d-1}$ . Of particular importance is the asymptotic behavior of  $v_{bl}(y)$  far from the boundary, i.e. when  $y \cdot n \rightarrow \infty$ . The latter strongly depends on  $n$  and is ultimately related to the homogenized boundary condition  $\bar{\varphi}$ .

The representation via Poisson's kernels gives

$$\begin{aligned} v_{bl}(y) &= \int_{\tilde{y} \cdot n = 0} P(y, \tilde{y}) \varphi(\tilde{y}) d\tilde{y} \\ &= \int_{\tilde{x} \cdot n = 0} \frac{1}{\varepsilon^{d-1}} P(x/\varepsilon, \tilde{x}/\varepsilon) \varphi(\tilde{x}/\varepsilon) d\tilde{x}, \end{aligned}$$

where we have performed the change of variable  $y = x/\varepsilon$ ,  $\tilde{y} = \tilde{x}/\varepsilon$  with  $\varepsilon := 1/(y \cdot n)$ . Therefore analyzing the limit  $y \cdot n \rightarrow \infty$  of  $v_{bl}$  boils down to studying the behavior of the highly oscillation Poisson kernel  $P^\varepsilon(x, \tilde{x}) := \frac{1}{\varepsilon^{d-1}} P(x/\varepsilon, \tilde{x}/\varepsilon)$  associated to the operator  $-\nabla \cdot A(x/\varepsilon)\nabla$ .

**Composite materials and boundary control.** This problem may have triggered much of the interest into uniform estimates in homogenization. In the control of composite materials the oscillations at the microscopic scale make the control tedious to compute. Two approaches are possible to deal with this complexity: either one first simplifies the problem, then computes the control, or one first computes the control and then tries to simplify it. The first approach is where homogenization may help.

One example pointed out by Lions [33] is the optimal control of the wave equation in an heterogeneous environment. For a boundary data  $g \in L^2(\partial\Omega)$ , we call  $y^\varepsilon = y^\varepsilon(t, x; g)$  the solution of the initial value problem

$$\begin{cases} y_{tt}^\varepsilon - \nabla \cdot A(x/\varepsilon)\nabla y^\varepsilon = 0 & \text{in } (0, T) \times \Omega, \\ y^\varepsilon(0, x) = y_0(x) & \text{in } \Omega, \\ y_t^\varepsilon(0, x) = y_1(x), & \text{in } \Omega, \\ y^\varepsilon(t, x) = g(x), & \text{on } \partial\Omega. \end{cases}$$

The optimal control problem reads as follows: for a fixed time  $T > 0$ , minimize the cost function

$$\mathcal{J}^\varepsilon(g) = \int_{\Omega} [y^\varepsilon(T, x; g) - F(x)]^2 dx + \int_{\partial\Omega} g.$$

over all  $g \in L^2(\partial\Omega)$ , where  $F = F(x) \in L^2(\Omega)$  is a prescribed final state. A minimizer  $g^\varepsilon$  of  $\mathcal{J}^\varepsilon$  exists and is unique; it is the control we are looking for. The goal, as described by Lions is to find the limit of the cost functional  $\mathcal{J}^\varepsilon$  and of  $g^\varepsilon$ . Is the limit problem simply the one where the oscillating coefficients  $A(x/\varepsilon)$  have been

replaced by the homogenized ones? The key issue lies in the convergence of  $y^\varepsilon(\cdot; g)$  to  $y^0(\cdot; g)$  in  $L^\infty((0, T); L^2(\Omega))$ .

Avellaneda and Lin [5, 8] have considered a simpler stationary problem ( $\mathcal{P}^\varepsilon$ ): minimize the cost functional

$$\mathcal{I}^\varepsilon(g) = \int_{\Omega} [u^\varepsilon - F(x)]^2 + \int_{\partial\Omega} g^2,$$

over  $g \in L^2(\partial\Omega)$  where  $u^\varepsilon = u^\varepsilon(x; g)$  solves

$$\begin{cases} -\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = 0, & x \in \Omega, \\ u^\varepsilon = g, & x \in \partial\Omega. \end{cases}$$

Is the formal limit problem ( $\mathcal{P}^0$ ), minimize

$$\mathcal{I}^0(g) = \int_{\Omega} [u^0 - F(x)]^2 + \int_{\partial\Omega} g^2$$

over  $g \in L^2(\partial\Omega)$  where  $u^0$  solves

$$\begin{cases} -\nabla \cdot \bar{A} \nabla u^0 = 0, & x \in \Omega, \\ u^0 = g, & x \in \partial\Omega, \end{cases}$$

relevant to analyse the limit  $\varepsilon \rightarrow 0$ ? Again, the main question is to study the convergence of  $u^\varepsilon$  in  $L^2$ . The result of [5, Proposition 4] is that the optimal triple  $(g^\varepsilon, u^\varepsilon(\cdot; g^\varepsilon), \mathcal{I}^\varepsilon(g^\varepsilon))$  for ( $\mathcal{P}^\varepsilon$ ) does not necessary converge to the optimal triple  $(g^0, u^0(\cdot; g^0), \mathcal{I}^0(g^0))$  for ( $\mathcal{P}^0$ ). In fact,  $\mathcal{I}^0$  has to be modified into

$$\tilde{\mathcal{I}}^0(g) = \int_{\Omega} [u^0 - F(x)]^2 + \int_{\partial\Omega} p g^2,$$

where  $p = p(x)$  accounts for the high frequency oscillations of the Poisson kernel (for more details see [8, page 6]). We conclude this paragraph by emphasizing that this problem underlies and motivates a lot of results expounded in this note.

**Main notations and assumptions.** Let  $\lambda > 0$ ,  $0 < \nu_0 < 1$  and  $M_0 > 0$  be fixed in what follows. Notice that  $\Omega^\varepsilon$  may be bounded or not, and may or may not depend explicitly on  $\varepsilon$ ;  $\Gamma^\varepsilon$  may be empty. We assume that the coefficients matrix  $A = A(y) = (A_{ij}^{\alpha\beta}(y))$ , with  $1 \leq \alpha, \beta \leq d$  and  $1 \leq i, j \leq N$  is real, that

$$A \text{ belongs to the class } C^{0, \nu_0} \text{ and } \|A\|_{L^\infty(\mathbb{R}^d)} + [A]_{C^{0, \nu_0}(\mathbb{R}^d)} \leq M_0, \quad (1.5)$$

that  $A$  is uniformly elliptic i.e.

$$\lambda |\xi|^2 \leq A_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\lambda} |\xi|^2, \quad \text{for all } \xi = (\xi_i^\alpha) \in \mathbb{R}^{dN}, \quad y \in \mathbb{R}^d \quad (1.6)$$

(we follow Einstein's convention, repeated subscripts stand for summation) and periodic i.e.

$$A(y + z) = A(y), \quad \text{for all } y \in \mathbb{R}^d, \quad z \in \mathbb{Z}^d. \quad (1.7)$$

We say that  $A$  belongs to the class  $\mathcal{A}^{0, \nu_0}$  if  $A$  satisfies (1.5), (1.6) and (1.7). The boundary is locally described as a graph of a function  $\psi$  taken in the class  $\mathcal{C}_{M_0}^{1, \nu_0}$  defined by

$$\mathcal{C}_{M_0}^{1, \nu_0} := \{\psi \in C^{1, \nu_0}(\mathbb{R}^{d-1}) : 0 \leq \psi \leq M_0, \|\nabla \psi\|_{L^\infty(\mathbb{R}^{d-1})} + [\nabla \psi]_{C^{0, \nu_0}(\mathbb{R}^{d-1})} \leq M_0\}.$$

Throughout this text,

$$D_\psi^\varepsilon(0, r) = D^\varepsilon(0, r) := \{(x', x_d), |x'| < r, \varepsilon\psi(x'/\varepsilon) < x_d < \varepsilon\psi(x'/\varepsilon) + r\},$$

$$\Delta_\psi^\varepsilon(0, r) = \Delta^\varepsilon(0, r) := \{(x', x_d), |x'| < r, x_d = \varepsilon\psi(x'/\varepsilon)\}.$$

The domain  $\Omega^\varepsilon = D^\varepsilon(0, 1)$  and its boundary part  $\Gamma^\varepsilon = \Delta^\varepsilon(0, 1)$  are frequently used to state local boundary estimates. Keep in mind that  $D^\varepsilon(0, 1)$  and thus  $u^\varepsilon$  solving (1.1) depend on  $\psi$ , although we usually do not write explicitly the dependence in  $\psi$ . The reason for this is that all our results hold uniformly for  $\psi$  in the above class  $\mathcal{C}_{M_0}^{1, \nu_0}$ .

## 2. The compactness method in homogenization

Compactness methods are a fairly general tool in analysis. The basic idea is to use compactness in a proof by contradiction to inherit some regularity from a limit problem. These methods originated in the works of De Giorgi [15] and Almgren [1] concerned with the regularity of minimal surfaces (Plateau problem). The approach has become standard in geometric measure theory [11] and for the regularity theory in the calculus of variations [16, 25]. Caffarelli [12] has applied this kind of ideas to the study of the regularity of free boundaries. All these results are achieved by blowing-up in the vicinity of the point where one wants to prove regularity. By doing so, the aim is for instance to approximate a nonlinear problem by a tangent linear one, and use the regularity theory for the linear problem together with compactness to infer regularity properties for the original nonlinear problem. A common denominator of these works is that they proceed in three steps:

**compactness:** used in a proof by contradiction, see for instance [16, Lemma 4.1], [25, Lemma 1.1 page 93], [11, Main Lemma] or [12, Lemma 6],

**iteration:** of step 1, see for example [16, Lemma 7.1], [25, pages 95 – 96], [11, Lemma 17] or [12, Lemma 7],

**conclusion:** of the proof using the result of step 2.

The use of compactness arguments in homogenization problems goes back to the seminal works of Avellaneda and Lin [4, 7, 8]. In order to get Hölder or Lipschitz estimates uniform in  $\varepsilon$  on  $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^N$  weak solution to

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f + \nabla \cdot F \quad \text{in } B(0, 1), \quad (2.1)$$

one proceeds in three steps. The first step is where the *compactness* argument takes place. The basic idea is to take advantage of the convergence toward the constant coefficient homogenized operator to get some estimate at scale  $\theta$  uniform in  $\varepsilon$  for  $0 < \varepsilon < \varepsilon_0(\theta)$ . The second step is the *iteration* of step one, in order to get an estimate at scale  $\theta^k$  uniform in  $\varepsilon$  for  $0 < \varepsilon < \theta^{k-1}\varepsilon_0(\theta)$ . The last step, conclusion, consists in a *blow-up* argument: at small scale  $O(\varepsilon)$  one can rely on classical estimates.

### 2.1. Hölder estimates

Let us see how the compactness method works on the simple example of the Hölder estimate.

**Proposition 1** (Hölder estimate, [4, Lemma 9]). *For all  $\kappa, \kappa' > 0$ , there exists  $C > 0$  so that for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\varepsilon > 0$ , for all  $f \in L^{d/2+\kappa'}(B(0,1))$ ,  $F \in L^{d+\kappa}(B(0,1))$ , for all  $u^\varepsilon \in L^2(B(0,1))$  weak solution to (2.1) the following estimate holds*

$$[u^\varepsilon]_{C^{0,\mu}(B(0,1/2))} \leq C \left\{ \|u^\varepsilon\|_{L^2(B(0,1))} + \|f\|_{L^{d/2+\kappa'}(B(0,1))} + \|F\|_{L^{d+\kappa}(B(0,1))} \right\},$$

where  $\mu := \min(1 - d/(d + \kappa), 2 - d/(d/2 + \kappa'))$ . Notice that  $C$  depends on  $d, N$ , on  $\|A\|_{C^{0,\nu_0}}$  i.e. on  $M_0$ , on  $\lambda$  and  $\kappa$ , but not on  $\varepsilon$ .

For simplicity, we carry out the proof in the case without source terms, i.e.  $f = F = 0$ . Of course, it is enough to prove that there exists  $C > 0$  so that for all  $\varepsilon > 0$

$$\int_{B(0,1)} |u^\varepsilon|^2 \leq 1 \quad \text{implies} \quad [u^\varepsilon]_{C^{0,\mu}(B(0,1/2))} \leq C.$$

For systems it is appropriate to measure the oscillation in terms of integral norms. Indeed Hölder continuity can be characterized in terms of Campanato spaces [25, Theorem 1.2]: when  $\Omega \subset \mathbb{R}^d$  is a Lipschitz domain and  $u \in C^{0,\mu}(\bar{\Omega})$  for  $0 < \mu < 1$ , then

$$[u]_{C^{0,\mu}(\bar{\Omega})}^2 \sim \sup_{x_0 \in \bar{\Omega}, \rho > 0} \rho^{-2\mu} \int_{B(x_0,\rho) \cap \Omega} \left| u - \int_{B(x_0,\rho) \cap \Omega} u \right|^2 dx,$$

where  $\sim$  means that the semi-norms on the left and right hand sides are equivalent.

**First step: compactness.** This is also known as the *improvement* step. Let  $0 < \mu < 1$ . We aim at proving that for some  $\theta$  sufficiently small, there exists  $\varepsilon_0 = \varepsilon_0(\theta) > 0$  so that for all  $0 < \varepsilon < \varepsilon_0$ ,

$$\int_{B(0,\theta)} \left| u^\varepsilon - \int_{B(0,\theta)} u^\varepsilon \right|^2 \leq \theta^{2\mu}. \quad (2.2)$$

The proof follows from a contradiction argument.

Let  $A^0$  be any constant matrix satisfying (1.6). Any weak solution  $u^0$  to

$$-\nabla \cdot A^0 \nabla u^0 = 0 \quad \text{in } B(0, 1/2), \quad (2.3)$$

is bounded in  $C^1(\overline{B(0, 1/4)})$  by its  $L^2(B(0, 1/2))$  norm. Therefore, for all  $\mu < \mu' < 1$ ,

$$\int_{B(0,1/4)} \left| u^0 - \int_{B(0,\theta)} u^0 \right|^2 \leq C_0 \theta^{2\mu'} \int_{B(0,1/2)} |u^0|^2 \leq 2^d C_0 \theta^{2\mu'} \int_{B(0,1)} |u^0|^2 \leq 2^d C_0 \theta^{2\mu'},$$

where  $C_0$  only depends on  $\lambda$ , not on the particular matrix  $A^0$ . This is the key to the contradiction argument. We choose  $0 < \theta < 1/4$  sufficiently small so that

$$2^d C_0 \theta^{2\mu'} < \theta^{2\mu}.$$

For this  $\theta$ , we prove (2.2).

Assume (by contradiction) that there exists a sequence  $\varepsilon_k \rightarrow 0$ , so that for all  $k$ , there exists  $A_k \in \mathcal{A}^{0,\nu_0}$ , and  $u_k^{\varepsilon_k}$  weak solution to

$$-\nabla \cdot A_k(x/\varepsilon_k) \nabla u_k^{\varepsilon_k} = 0 \quad \text{in } B(0, 1),$$

so that

$$\int_{B(0,1)} |u_k^{\varepsilon_k}|^2 \leq 1 \quad (2.4)$$

and

$$\int_{B(0,\theta)} \left| u_k^{\varepsilon_k} - \int_{B(0,\theta)} u_k^{\varepsilon_k} \right|^2 > \theta^{2\mu}. \quad (2.5)$$

The uniform bound (2.4) implies via Cacciopoli's inequality that up to a subsequence

$$\begin{aligned} \nabla u_k^{\varepsilon_k} &\rightharpoonup \nabla u^0 \quad \text{weakly in } L^2(B(0, 1/2)), \\ u_k^{\varepsilon_k} &\rightarrow u^0 \quad \text{strongly in } L^2(B(0, 1/2)). \end{aligned}$$

These convergences and a standard oscillating test function argument (for more details see [30, Theorem 3]) imply that  $u^0$  solves (2.3) with a constant matrix  $A^0$ . Passing to the limit in (2.5) yields

$$\theta^{2\mu} \leq \int_{B(0,\theta)} \left| u^0 - \int_{B(0,\theta)} u^0 \right|^2 \leq 2^d C_0 \theta^{2\mu'} < \theta^{2\mu},$$

which is a contradiction.

*Remark 1* (necessity of the second step). The proof actually shows that for all  $\theta$  sufficiently small, there exists  $\varepsilon_0(\theta)$  so that the estimate (2.2) holds uniformly for  $0 < \varepsilon < \varepsilon_0(\theta)$ . However, we do not have any control on  $\varepsilon_0(\theta)$  in terms of  $\theta$ . This is the reason why we need the second step, the iteration.

**Second step: iteration.** Our goal is to show that for all integer  $k$ , for all  $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ ,

$$\int_{B(0,\theta^k)} \left| u^\varepsilon - \int_{B(0,\theta^k)} u^\varepsilon \right|^2 \leq \theta^{2k\mu}. \quad (2.6)$$

This estimate is of course true for  $k = 1$  because of step 1 above.

Let  $k \geq 1$  and assume (2.6). For  $x \in B(0, 1)$ ,

$$U^\varepsilon(x) := \frac{1}{\theta^{k\mu}} \left\{ u^\varepsilon(\theta^k x) - \int_{B(0,\theta^k)} u^\varepsilon \right\}.$$

It solves

$$-\nabla \cdot A(\theta^k x / \varepsilon) \nabla U^\varepsilon = 0 \quad \text{in } B(0, 1). \quad (2.7)$$

Moreover, by (2.6),

$$\int_{B(0,1)} |U^\varepsilon|^2 \leq 1.$$

Therefore, we can apply the first step to  $U^\varepsilon$  and get for  $\varepsilon/\theta^k < \varepsilon_0$

$$\frac{1}{\theta^{2k\mu}} \int_{B(0,\theta^k)} \left| u^\varepsilon - \int_{B(0,\theta^k)} u^\varepsilon \right|^2 = \int_{B(0,\theta)} \left| U^\varepsilon - \int_{B(0,\theta)} U^\varepsilon \right|^2 \leq \theta^{2\mu}.$$

*Remark 2* (key to the iteration). It is the uniformity of estimate (2.2) for  $0 < \varepsilon < \varepsilon_0$ , which makes this procedure work. Indeed in equation (2.7) one can call  $\varepsilon' := \varepsilon/\theta^k$ , so that step 1 applies for  $\varepsilon' < \varepsilon_0$ , i.e.  $\varepsilon < \theta^k \varepsilon_0$ .



**Third step: blow-up.** Let  $\varepsilon > 0$  be fixed. Either  $\varepsilon > \varepsilon_0$ , in which case the estimate in Proposition (1) follows from the classical Schauder theory, or  $\varepsilon < \varepsilon_0$ . We now focus on the latter. There exists a unique integer  $k$  so that  $\theta^k \leq \varepsilon/\varepsilon_0 < \theta^{k+1}$ .

Let  $0 < r < 1/2$ . Our goal is to prove that

$$\int_{B(0,r)} \left| u^\varepsilon - \int_{B(0,r)} u^\varepsilon \right|^2 \leq Cr^{2\mu},$$

with a constant  $C$  uniform in  $r$  and  $\varepsilon$ . Assume  $r \geq \varepsilon/\varepsilon_0$ . Then, there exists an integer  $1 \leq l \leq k$  so that  $\theta^l \leq r < \theta^{l+1}$ , and

$$\int_{B(0,r)} \left| u^\varepsilon - \int_{B(0,r)} u^\varepsilon \right|^2 \leq C \int_{B(0,\theta^{l-1})} \left| u^\varepsilon - \int_{B(0,\theta^{l-1})} u^\varepsilon \right|^2 \leq C\theta^{2(l-1)\mu} \leq Cr^{2\mu}, \quad (2.8)$$

with  $C$  dependent only on  $d$  and  $\theta$ . Assume  $r < \varepsilon/\varepsilon_0$ . Then, we carry out the blow-up argument. This amounts to considering the auxiliary function

$$U^\varepsilon(y) := \frac{1}{\varepsilon^\mu} u^\varepsilon(\varepsilon y),$$

defined for  $y \in B(0, 2/\varepsilon_0)$ . Since

$$-\nabla \cdot A(y) \nabla \left\{ U^\varepsilon - \int_{B(0,2/\varepsilon_0)} U^\varepsilon \right\} = 0 \quad \text{in } B(0, 2/\varepsilon_0),$$

we have by the classical Schauder theory

$$[U^\varepsilon]_{C^{0,\mu}(\overline{B(0,1/\varepsilon_0)})} \leq C \left\| U^\varepsilon - \int_{B(0,2/\varepsilon_0)} U^\varepsilon \right\|_{L^2(B(0,2/\varepsilon_0))}.$$

Rescaling the latter estimate, we get

$$\begin{aligned} [u^\varepsilon]_{C^{0,\mu}(\overline{B(0,\varepsilon/\varepsilon_0)})} &= [U^\varepsilon]_{C^{0,\mu}(\overline{B(0,1/\varepsilon_0)})} \leq C \left\| U^\varepsilon - \int_{B(0,2/\varepsilon_0)} U^\varepsilon \right\|_{L^2(B(0,2/\varepsilon_0))} \\ &= C\varepsilon^{-\mu} \left( \int_{B(0,2\varepsilon/\varepsilon_0)} \left| u^\varepsilon - \int_{B(0,2\varepsilon/\varepsilon_0)} u^\varepsilon \right|^2 \right)^{1/2} \leq C \end{aligned}$$

by (2.8) with  $r = \varepsilon/\varepsilon_0$ . Thus, we have proved

$$\sup_{0 < r < 1/2} \frac{1}{r^{2\mu}} \int_{B(0,r)} \left| u^\varepsilon - \int_{B(0,r)} u^\varepsilon \right|^2 \leq C.$$

It remains to consider the balls  $B(x, r)$  centered at  $x \in B(0, 1/2)$  and not at 0. This point can be dealt with by simply translating the point  $x$  to 0. Of course, the coefficients of  $A$  are changed. However this change does not affect the proof, since the translated matrix belongs to  $\mathcal{A}^{0,\nu_0}$  and the above constant is uniform in this class.

*Remark 3* (structure of the coefficients). One of the crucial facts in the improvement lemma is that the limit matrix  $A^0$  is a *constant* matrix, so that we can rely on the classical Schauder estimates for systems. Without any structure assumption on  $A \in \mathcal{A}^{0,\nu_0}$ , we know by the abstract H-convergence theory developed by Murat and Tartar in the 70's (see [35] and [14, Chapter 13]) that a subsequence of  $u_k^{\varepsilon_k}$  converges to  $u^0$  weak solution of

$$-\nabla \cdot \bar{A}(x) \nabla u^0 = 0 \quad \text{in } B(0, 1/2),$$

with  $\bar{A} = \bar{A}(x) \in L^\infty(B(0, 1/2))$ . However, in general  $\bar{A}$  is no better than  $L^\infty$ . This regularity is not sufficient for the Schauder estimates to hold for systems.

*Remark 4* (boundary estimate). Following the same scheme as for the interior estimate, Avellaneda and Lin have proved a boundary Hölder estimate uniform in  $\varepsilon$  for  $u^\varepsilon$  solving

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f + \nabla \cdot F, & x \in D_\psi^1(0, 1), \\ u^\varepsilon = g, & x \in \Delta_\psi^1(0, 1), \end{cases}$$

with non oscillating boundary given by  $\psi \in \mathcal{C}_{M_0}^{1,\nu_0}$ . We refer to [4, Section 2.3] for a statement and details of the proof. Remark that  $C^1$  regularity for  $\psi$  is enough for the boundary Hölder estimate to hold.

## 2.2. Lipschitz estimates

In this part we first state the Lipschitz estimate proved by Avellaneda and Lin. We then explain what makes its proof much different from the proof of the Hölder estimate, although both proofs rely on the three steps compactness scheme described above.

**Proposition 2** (Lipschitz estimate, [4, Lemma 16]). *For all  $\kappa > 0$ ,  $0 < \mu < 1$ , there exists  $C > 0$  so that for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\varepsilon > 0$ , for all  $f \in L^{d+\kappa}(B(0, 1))$ , for all  $F \in C^{0,\mu}(B(0, 1))$ , for all  $u^\varepsilon \in L^\infty(B(0, 1))$  weak solution to (2.1), the following estimate holds*

$$\|\nabla u^\varepsilon\|_{L^\infty(B(0,1/2))} \leq C \left\{ \|u^\varepsilon\|_{L^\infty(B(0,1))} + \|f\|_{L^{d+\kappa}(B(0,1))} + \|F\|_{C^{0,\mu}(B(0,1))} \right\}.$$

Notice that  $C$  depends on  $d$ ,  $N$ ,  $M_0$ ,  $\lambda$ ,  $\kappa$  and  $\mu$ .

A first important comment is that one cannot expect to have higher order derivatives of  $u^\varepsilon$  bounded uniformly in  $\varepsilon$ . The reason for this is the following easy computation in dimension  $d = 1$ : from

$$-d_x(A(x/\varepsilon)d_x u^\varepsilon) = 0 \quad \text{in } (-1, 1),$$

we infer that there is a constant  $C > 0$  so that

$$d_x u^\varepsilon = \frac{C}{A(x/\varepsilon)} \quad \text{for } x \in (-1, 1).$$

In particular, it is not possible to show  $C^{1,\mu}$  estimates for  $u^\varepsilon$ . Therefore, an approach based on an integral characterization of  $C^{1,\mu}$  won't work.

The major issue one encounters for the Lipschitz estimate is the lack of (strong) compactness of  $\nabla u^\varepsilon$  in  $L^\infty$ . Such a compactness property is at the heart of the proof by contradiction of step 1. Thus one has to cook up another approach. For simplicity, we take throughout this section  $f = F = 0$ ; the appealing points lie elsewhere. Notice that  $u^\varepsilon - u^\varepsilon(0)$  solves (2.1). By simply rescaling the classical Lipschitz estimate (for non highly oscillating coefficients), we get that

$$\|\nabla u^\varepsilon\|_{L^\infty(B(0,\varepsilon))} = \|\nabla(u^\varepsilon - u^\varepsilon(0))\|_{L^\infty(B(0,\varepsilon))} \leq \frac{C}{\varepsilon} \|u^\varepsilon - u^\varepsilon(0)\|_{L^\infty(B(0,2\varepsilon))}, \quad (2.9)$$

where  $C > 0$  does not depend on  $\varepsilon$ . The idea of the proof is to show, in two steps, that

$$\|u^\varepsilon - u^\varepsilon(0)\|_{L^\infty(B(0,2\varepsilon))} = O(\varepsilon), \quad (2.10)$$

so that the right hand side in (2.9) is bounded by a constant uniformly in  $\varepsilon$ . As we will see, showing a (2.9)-like estimate strongly uses the homogenization properties of the operator  $-\nabla \cdot A(x/\varepsilon)\nabla$  and even more, the periodic structure of the oscillations of  $A$ .

Hereafter, we concentrate on the second step of the compactness method. We emphasize the need for correctors, which prompts the use of more structure (periodicity) than in the proof of the Hölder estimate. Let us show that there exists  $C > 0$  so that for all  $\varepsilon > 0$

$$\|u^\varepsilon\|_{L^\infty(B(0,1))} \leq 1 \quad \text{implies} \quad \|\nabla u^\varepsilon\|_{L^\infty(B(0,1/2))} \leq C.$$

Moreover, without loss of generality, one can assume that  $u^\varepsilon(0) = 0$ .

**Step 1: improvement.** A proof by contradiction makes it possible to show that there is  $\varepsilon_0(\theta) > 0$ , so that for  $0 < \varepsilon < \varepsilon_0(\theta)$ ,

$$\left\| u^\varepsilon(x) - x \cdot \int_{B(0,\theta)} \nabla u^\varepsilon \right\|_{L^\infty(B(0,\theta))} \leq \theta^{1+\mu}. \quad (2.11)$$

Notice that the bound  $\|u^\varepsilon\|_{L^\infty(B(0,1))} \leq 1$  yields enough compactness (via Ascoli-Arzelà type arguments) to pass to the limit in the proof by contradiction.

**Step 2: iteration.** The idea is again to iterate step 1. Let  $k$  be an integer and assume that for  $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ ,

$$\left\| u^\varepsilon(x) - x \cdot \int_{B(0,\theta^k)} \nabla u^\varepsilon \right\|_{L^\infty(B(0,\theta^k))} \leq \theta^{(1+\mu)k}.$$

Then for  $x \in B(0,1)$ , let

$$U^\varepsilon := \frac{1}{\theta^{(1+\mu)k}} \left\{ u^\varepsilon(\theta^k x) - \theta^k x \cdot \int_{B(0,\theta^k)} \nabla u^\varepsilon \right\}.$$

On the one hand  $\|U^\varepsilon\|_{L^\infty(B(0,1))} \leq 1$ , and on the other hand  $U^\varepsilon$  solves

$$-\nabla \cdot A(\theta^k x/\varepsilon)\nabla U^\varepsilon = \frac{1}{\theta^{\mu k}} \partial_{x\alpha} \left( A^{\alpha\beta}(\theta^k x/\varepsilon) \int_{B(0,\theta^k)} \partial_{x\beta} u^\varepsilon \right) \quad \text{in } B(0,1). \quad (2.12)$$

Here, we bump into a new issue. Indeed, the equation (2.12) has a right hand side, which prevents us from applying directly the estimate of the first step (without right hand side). Notice that this issue is not due to the fact that we have assumed  $f = F = 0$ . The reason for the right hand side lies in the oscillations of  $A$ . The way to cope with this difficulty is to modify the expansion for  $u^\varepsilon$ . We have to introduce *correctors*.

Let  $\chi = \chi^\gamma(y) \in M_N(\mathbb{R})$ ,  $y \in \mathbb{T}^d$ , solving the cell problems

$$-\nabla_y \cdot A(y)\nabla_y \chi^\gamma = \partial_{y_\alpha} A^{\alpha\gamma}, \quad y \in \mathbb{T}^d \quad \text{and} \quad \int_{\mathbb{T}^d} \chi^\gamma(y) dy = 0. \quad (2.13)$$

Here we take advantage of the periodic structure for the existence of the correctors  $\chi$ . Since  $\|\chi\|_{L^\infty(\mathbb{T}^d)} < \infty$ , we have  $\varepsilon\chi(x/\varepsilon) \rightarrow 0$  in  $L^\infty(B(0,1))$ , and thus it is easy to modify the first step to get instead of (2.11) the estimate

$$\left\| u^\varepsilon(x) - \left\{ x - \varepsilon\chi(x/\varepsilon) \right\} \cdot \int_{B(0,\theta)} \nabla u^\varepsilon \right\|_{L^\infty(B(0,\theta))} \leq \theta^{1+\mu}. \quad (2.14)$$

We notice that

$$U^\varepsilon := \frac{1}{\theta^{1+\mu}} \left\{ u^\varepsilon(\theta x) - \left\{ \theta x - \varepsilon \chi(\theta x/\varepsilon) \right\} \cdot \int_{B(0,\theta)} \nabla u^\varepsilon \right\}$$

solves

$$-\nabla \cdot A(\theta x/\varepsilon) \nabla U^\varepsilon = 0 \quad \text{in } B(0,1).$$

Since by (2.14)  $\|U^\varepsilon\|_{L^\infty(B(0,1))} \leq 1$ , we get for  $0 < \varepsilon/\theta < \varepsilon_0$ ,

$$\left\| U^\varepsilon(x) - \left\{ x - \varepsilon/\theta \chi(\theta x/\varepsilon) \right\} \cdot \int_{B(0,\theta)} \nabla U^\varepsilon \right\|_{L^\infty(B(0,\theta))} \leq \theta^{1+\mu},$$

i.e.

$$\|u^\varepsilon(x) - x \cdot a_2^\varepsilon + \varepsilon b_2^\varepsilon(x/\varepsilon)\|_{L^\infty(B(0,\theta))} \leq \theta^{2(1+\mu)},$$

with  $a_2^\varepsilon \in \mathbb{R}^{dN}$ ,  $b_2^\varepsilon = b_2^{\varepsilon,\gamma}(y) \in M_N(\mathbb{R})$

$$b_2^{\varepsilon,\gamma}(y) = \chi^\gamma(y) \cdot \int_{B(0,\theta)} \nabla u^\varepsilon + \theta^\mu \chi^\gamma(\theta y) \cdot \int_{B(0,\theta)} \nabla U^\varepsilon,$$

and

$$\begin{aligned} |a_2^\varepsilon| &\leq (C/\theta)(1 + \theta^\mu), \\ \|b_2^\varepsilon\|_{L^\infty(\mathbb{T}^d)} &\leq (C/\theta)(1 + \theta^\mu). \end{aligned}$$

Reiterating this procedure, one can prove that there is a constant  $C > 0$ , for all integer  $k$ , for all  $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ , there are  $a_k^\varepsilon \in \mathbb{R}^{dN}$  and  $b_k^\varepsilon = b_k^{\varepsilon,\gamma}(y) \in M_N(\mathbb{R})$  so that

$$\begin{aligned} |a_k^\varepsilon| &\leq (C/\theta)(1 + \theta^\mu + \dots + \theta^{(k-1)\mu}), \\ \|b_k^\varepsilon\|_{L^\infty(\mathbb{T}^d)} &\leq (C/\theta)(1 + \theta^\mu + \dots + \theta^{(k-1)\mu}), \end{aligned} \tag{2.15}$$

and

$$\|u^\varepsilon(x) - x \cdot a_k^\varepsilon + \varepsilon b_k^\varepsilon(x/\varepsilon)\|_{L^\infty(B(0,\theta^k))} \leq \theta^{k(1+\mu)}. \tag{2.16}$$

**Step 3: blow-up.** The case when  $\varepsilon > \varepsilon_0$  follows of course from the classical estimates. Assume now for the remainder of this section that  $\varepsilon < \varepsilon_0$ . There exists a unique integer  $k$  so that  $\theta^{k+1} \leq \varepsilon/\varepsilon_0 < \theta^k$ . With the estimate (2.16) of step 2 at hand, we can easily estimate the size of  $|u^\varepsilon(x)|$  for  $|x| \lesssim \varepsilon$ . Indeed, it follows from (2.16) that

$$\|u^\varepsilon(x) - x \cdot a_k^\varepsilon + \varepsilon b_k^\varepsilon(x/\varepsilon)\|_{L^\infty(B(0,\varepsilon/\varepsilon_0))} \leq C(\varepsilon/\varepsilon_0)^{1+\mu}, \tag{2.17}$$

with a constant  $C$  depending only on  $\theta$ , neither on  $k$ , nor on  $\varepsilon$ . Therefore, thanks to the uniform bounds (2.15)

$$\|u^\varepsilon\|_{L^\infty(B(0,\varepsilon/\varepsilon_0))} \leq C\varepsilon/\varepsilon_0, \tag{2.18}$$

with  $C > 0$  uniform in  $\varepsilon$ . Now the classical Lipschitz estimate finally yields

$$\|\nabla u^\varepsilon\|_{L^\infty(B(0,\varepsilon/(2\varepsilon_0)))} \leq \frac{C}{\varepsilon} \|u^\varepsilon\|_{L^\infty(B(0,\varepsilon/\varepsilon_0))} \leq C.$$

The bound

$$\|\nabla u^\varepsilon\|_{L^\infty(B(x,\varepsilon/(2\varepsilon_0)))} \leq C$$

for  $x \in B(0,1/2)$  and  $C > 0$  uniform in  $x$  and  $\varepsilon$  simply follows from translating the origin to  $x$ , since the translated coefficients still belong to  $\mathcal{A}^{0,\nu_0}$ .

*Remark 5* (boundary estimate). As for the Hölder estimate, there is a boundary version of the Lipschitz estimate. We refer to [4, Section 3.2] for a statement and details of the proof. Remark that  $C^{1,\nu_0}$  regularity for  $\psi$  is needed for the Lipschitz estimate to hold (see [27, Lemma 1.12] for counter-examples on  $C^1$  domains in  $\mathbb{R}^2$ ).

### 2.3. Related recent works

The method of Avellaneda and Lin has inspired numerous works and it is impossible to be exhaustive. We briefly discuss here some very recent developments in four directions: other types of microstructures (almost periodic, random), different norms, different boundary conditions and other types of equations.

In the periodic setting, works by Shen and his collaborators address uniform  $W^{1,p}$  estimates for elliptic equations [37] and for the system of elasticity [20] in non smooth domains (Lipschitz or  $C^1$ ). The paper by Kenig, Lin and Shen [28] is devoted to uniform  $W^{1,p}$  and Lipschitz estimates for elliptic systems with Neumann boundary conditions.

A number of results valid for elliptic equations has been transposed to parabolic equations. Geng and Shen [19] prove uniform interior Hölder, Lipschitz and  $W^{1,p}$ , as well as boundary Hölder and  $W^{1,p}$  estimates.

In the past few months, and notably at the time where this note has been written, there have been exciting breakthroughs toward relaxing structure assumptions on the coefficient matrix  $A$ .

Uniform (interior and boundary) Hölder estimates for (1.1) have been extended to almost periodic structures. A comprehensive reference on this topic is the paper by Shen [38]. The main novelty of this latter work is to provide quantitative estimates for the almost periodic homogenization.

Armstrong and Smart [3] have been able to prove uniform Lipschitz estimates in some random environments. Their method does not rely on compactness, but on (non optimal) error estimates between  $u^\varepsilon$  and the homogenized solution  $u^0$ . The basic tool of the proof, which replaces the improvement lemma (step 1), is a result (see [3, Lemma 5.1]) telling that if a function  $v$  is sufficiently close to a function  $w$  satisfying a so-called “improvement of flatness property”, then  $v$  itself satisfies the “improvement of flatness property”. Iterating this lemma they prove an estimate down to a mesoscopic scale. The work of Armstrong and Smart emphasizes the need to separate the large scales, where homogenization holds, from the small scales, where classical regularity theory applies.

The paper [3] has inspired two works, which have just come out. Based on the error estimates lately obtained by Shen [38] in almost periodic homogenization, Armstrong and Shen [2] have been able to prove interior and boundary  $W^{1,p}$  and Lipschitz estimates in almost periodic environments. Since this approach relies on error estimates, one has to quantify the almost periodicity. Inspired by [3], Gloria, Neukamm and Otto [26] have proved Lipschitz estimates in quite general random environments. Homogenization is used in a more qualitative way.

### 3. Lipschitz estimates in bumpy half-spaces

This section is devoted to a description of a recent work by Kenig and the author [30], which is devoted to uniform estimates for elliptic systems near oscillating boundaries. The main result of this paper is the following theorem concerned with estimates of  $u^{\alpha,\beta}$  weak solution to

$$\begin{cases} -\nabla \cdot A(x/\alpha)\nabla u^{\alpha,\beta} = f + \nabla \cdot F, & x \in D^\beta(0,1), \\ u^{\alpha,\beta} = 0, & x \in \Delta^\beta(0,1), \end{cases} \quad (3.1)$$

where  $\alpha, \beta > 0$ .

**Theorem 3** (Lipschitz estimate, [30, Theorem 27]). *Let  $0 < \mu < 1$  and  $\kappa > 0$ . There exist  $C > 0, \varepsilon_0 > 0$ , so that for all  $\psi \in \mathcal{C}_{M_0}^{1,\nu_0}$ , for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\alpha, \beta > 0$ , for all  $f \in L^{d+\kappa}(D^\beta(0,1))$ , for all  $F \in C^{0,\mu}(D^\beta(0,1))$ , for all  $u^{\alpha,\beta}$  weak solution to (3.1) the bounds*

$$\|u^{\alpha,\beta}\|_{L^\infty(D^\beta(0,1))} \leq 1, \quad \|f\|_{L^{d+\kappa}(D^\beta(0,1))} \leq \varepsilon_0, \quad \|F\|_{C^{0,\mu}(D^\beta(0,1))} \leq \varepsilon_0$$

imply

$$\|\nabla u^{\alpha,\beta}\|_{L^\infty(D^\beta(0,1/2))} \leq C.$$

Notice that  $C$  and  $\varepsilon_0$  depend on  $d, N, M_0, \lambda, \nu_0, \kappa$  and  $\mu$ . Again, the salient point is the uniformity in  $\alpha$  and  $\beta$  of the constant  $C$ .

*Remark 6* (rescaled estimate). Let  $r > 0$ . For  $u^{\alpha,\beta}$  solution of (3.1) in  $D^\beta(0,r)$  (instead of  $D^\beta(0,1)$ ) and vanishing on  $\Delta^\beta(0,r)$ , the uniformity in  $\alpha$  and  $\beta$  gives the rescaled estimate:

$$\|\nabla u^{\alpha,\beta}\|_{L^\infty(D^\beta(0,r/2))} \leq C \left\{ r^{-1} \|u^{\alpha,\beta}\|_{L^\infty(D^\beta(0,r))} + r^{1-d/(d+\kappa)} \|f\|_{L^{d+\kappa}(D^\beta(0,r))} + r^\mu \|F\|_{C^{0,\mu}(D^\beta(0,r))} \right\}. \quad (3.2)$$

A Hölder estimate uniform in  $\alpha$  and  $\beta$  for (3.1) has also been proved in [30, Proposition 26]. The latter is much easier to establish than the Lipschitz estimate. We need the uniform Hölder estimate in the proof of the Lipschitz estimate.

The main feature of our result is the *lack of structure* in the oscillations of the boundary. In particular,  $\psi$  is neither assumed to be periodic, quasiperiodic nor stationary ergodic. This generalization is in another vein than the recent works [3, 2, 26]. Our point is to remove any structure assumption on the boundary, and not to relax the hypotheses on the coefficients. We are aware of only one similar result in the literature. Gérard-Varet [21, Theorem 11] has proved a Hölder estimate for Stokes's system near an oscillating boundary.

Let us comment on some of the main difficulties of this problem. Since Lipschitz estimates require  $C^{1,\nu_0}$  regularity of the boundary and

$$[\nabla(\beta\psi(x/\beta))]_{C^{0,\nu_0}} \simeq \beta^{-\nu_0},$$

the uniformity in  $\beta$  cannot be achieved by resorting to the boundary Lipschitz estimate of Avellaneda and Lin [4, Lemma 20]. Moreover, flattening the boundary by

$$(x', x_d) \in \mathbb{R}_+^d \longmapsto (x', x_d + \beta\psi(x'/\beta)) \in \{x_d > \beta\psi(x'/\beta)\}$$

can by no mean give the result in general. Indeed, assume for a moment that the equation is scalar and that the coefficient matrix  $A$  is the identity, so that  $-\nabla \cdot A \nabla =$

$-\Delta$  and denote by  $w^\beta$  a solution of (3.1) in that case. Then,  $\tilde{w}^\beta$  defined by

$$\tilde{w}^\beta(x', x_d) := w^\beta(x', x_d + \beta\psi(x'/\beta)) \quad \text{for all } (x', x_d) \in \mathbb{R}_+^d$$

is a weak solution to

$$\begin{cases} -\nabla \cdot \tilde{A}(x/\beta) \nabla \tilde{w}^\beta = 0, & x \in (-1, 1)^{d-1} \times (0, 1), \\ \tilde{w}^\beta = 0, & x \in (-1, 1)^{d-1} \times \{0\}. \end{cases}$$

The new matrix

$$\tilde{A}(x/\beta) := \left( \begin{array}{c|c} \mathbf{I}_{d-1} & -\nabla_{x'} \psi(x'/\beta) \\ \hline 0 & 1 \end{array} \right)^T \left( \begin{array}{c|c} \mathbf{I}_{d-1} & -\nabla_{x'} \psi(x'/\beta) \\ \hline 0 & 1 \end{array} \right)$$

is oscillating at scale  $\beta$ , but a priori with no structure. If  $\psi$  is periodic, then we can apply the theorem of Avellaneda and Lin. However, in the absence of structure in the oscillations of the boundary, flattening the boundary does not help.

From an heuristic point of view, the Lipschitz estimate should follow from the work of Avellaneda and Lin [4] in the extreme case when  $\alpha \ll \beta$ . Indeed in that case, the coefficients oscillate faster than the boundary. Rescaling the boundary at scale 1 yields coefficients oscillating at scale  $\alpha/\beta \ll 1$ . Therefore, in order to focus on the true issues introduced by the oscillating boundary, we carry out the proof of the Lipschitz estimate in the case of a constant coefficients elliptic system

$$\begin{cases} -\nabla \cdot A^0 \nabla u^\varepsilon = 0, & x \in D^\varepsilon(0, 1), \\ u^\varepsilon = 0, & x \in \Delta^\varepsilon(0, 1). \end{cases} \quad (3.3)$$

**Proposition 4** ([30, Proposition 13]). *There exists a constant  $C > 0$  so that for all  $\psi \in C_{M_0}^{1, \nu_0}$ , for all  $\varepsilon > 0$ , for all  $u^\varepsilon$  weak solution to (3.3), if*

$$\|u^\varepsilon\|_{L^\infty(D^\varepsilon(0,1))} \leq 1,$$

then

$$\|\nabla u^\varepsilon\|_{L^\infty(D^\varepsilon(0,1/2))} \leq C.$$

Notice that  $C$  depends on  $d$ ,  $N$ ,  $M_0$ ,  $\lambda$  and  $\nu_0$ .

The proof of Proposition 4 follows the three steps compactness scheme. As above for the interior Lipschitz bound, the main difficulty is to find out and then prove by contradiction (step 1) an estimate which nicely goes through the iteration procedure (step 2). We have noticed in the proof of the interior Lipschitz estimate that the latter may call for the introduction of correctors. This demand also appears here when proving the second step. The correctors deal with the oscillations of the boundary and do not require any structure assumption on the graph. They are called *boundary correctors*.

The original work [30] handles the full situation with oscillating coefficients and oscillating boundary. Glueing the two extreme cases  $\alpha \ll \beta$  and  $\alpha \gg \beta$  together requires in particular refined estimates on the boundary correctors.

**Step 1.** The first thing to do is to identify the limit problem. This is where the boundedness of  $\psi$  is crucial. Indeed, since  $\|\varepsilon\psi(\cdot/\varepsilon)\|_{L^\infty(\mathbb{R}^{d-1})} \leq M_0\varepsilon$ , at the limit  $\varepsilon \rightarrow 0$  the boundary is flat. Remark that the salient information that  $\psi$  is bounded has been lost in the flattening procedure above. Let  $u^0$  be a weak solution to

$$\begin{cases} -\nabla \cdot A^0 \nabla u^0 = 0, & x \in (-1/4, 1/4)^{d-1} \times (0, 1/4), \\ u^0 = 0, & x \in (-1/4, 1/4)^{d-1} \times \{0\}. \end{cases} \quad (3.4)$$

By classical regularity, we have for  $0 < \theta < 1/8$

$$\left\| u^0(x) - \left( \int_{(-\theta, \theta)^{d-1} \times (0, \theta)} \partial_{x_d} u^0 \right) x_d \right\|_{L^\infty((-\theta, \theta)^{d-1} \times (0, \theta))} \leq C_0 \theta^2,$$

where  $C_0 > 0$  does not depend on  $\theta$ . Let  $0 < \mu < 1$  and take  $0 < \theta < 1/8$  sufficiently small so that  $C_0 \theta^2 < \theta^{1+\mu}$ .

Assume by contradiction that there exists a sequence  $\varepsilon_k \rightarrow 0$ ,  $\psi_k \in \mathcal{C}_{M_0}^{1, \nu_0}$  and  $u_k^{\varepsilon_k}$  weak solution to

$$\begin{cases} -\nabla \cdot A^0 \nabla u_k^{\varepsilon_k} = 0, & x \in D_{\psi_k}^{\varepsilon_k}(0, 1), \\ u_k^{\varepsilon_k} = 0, & x \in \Delta_{\psi_k}^{\varepsilon_k}(0, 1), \end{cases}$$

so that  $\|u_k^{\varepsilon_k}\|_{L^\infty(D_{\psi_k}^{\varepsilon_k}(0, 1))} \leq 1$  and

$$\left\| u_k^{\varepsilon_k}(x) - \left( \int_{D^{\varepsilon_k}(0, 1)} \partial_{x_d} u_k^{\varepsilon_k} \right) [x_d - \varepsilon_k \psi_k(x'/\varepsilon_k)] \right\|_{L^\infty(D^{\varepsilon_k}(0, \theta))} > \theta^{1+\mu}. \quad (3.5)$$

The uniform boundary Hölder estimate  $[u_k^{\varepsilon_k}]_{C^{0, \mu}(\overline{D^{\varepsilon_k}(0, 1/2)})} \leq C$  and Cacciopoli's inequality give the compactness needed on  $u_k^{\varepsilon_k}$  to see that a subsequence converges to  $u^0$  solution of (3.4). We skip a few technicalities; those are handled in [30]. One can then pass to the limit in (3.5), which yields a contradiction.

**Step 2 (attempt).** For  $x \in D^{\varepsilon/\theta}(0, 1)$  and  $0 < \varepsilon < \varepsilon_0$ , we define

$$U^\varepsilon(x) := \frac{1}{\theta^{1+\mu}} \left\{ u^\varepsilon(\theta x) - \left( \int_{D^\varepsilon(0, 1)} \partial_{x_d} u^\varepsilon \right) [\theta x_d - \varepsilon \psi(\theta x'/\varepsilon)] \right\}.$$

The first step implies that  $\|U^\varepsilon\|_{L^\infty(D^{\varepsilon/\theta}(0, 1))} \leq 1$ . Moreover,  $U^\varepsilon$  vanishes on  $\Delta^{\varepsilon/\theta}(0, 1)$ . Nevertheless,

$$-\nabla \cdot A^0 \nabla U^\varepsilon = -\frac{1}{\theta^\mu} \left( \int_{D^\varepsilon(0, 1)} \partial_{x_d} u^\varepsilon \right) \nabla_{x'} \cdot A^0 \nabla_{x'} (\psi(\theta x'/\varepsilon)) \quad \text{in } D^{\varepsilon/\theta}(0, 1), \quad (3.6)$$

which prevents us from applying the estimate of step 1 to  $U^\varepsilon$ . In order to address this issue, we have to introduce correctors in the expansion for  $u^\varepsilon$ , the so-called boundary correctors.

**Boundary correctors.** The list of requirements for these correctors is that:

- (1) they vanish on the oscillating boundary,
- (2) they cancel out the right hand side in (3.6) and
- (3) they are nicely estimated in terms of the distance to the oscillating boundary.

For the latter, we are led to introduce a cut-off function  $\Theta$  whose purpose is to limit the effect of  $\psi$  to a small layer close to the boundary. For fixed  $M_0 > 0$ , let  $\vartheta' \in C_c^\infty(\mathbb{R}^{d-1})$  (resp.  $\vartheta_d \in C_c^\infty(\mathbb{R})$ ) a cut-off function compactly supported in  $(-3/2, 3/2)^{d-1}$  (resp. in  $(-3M_0/2, 3M_0/2)$ ), identically equal to 1 on  $(-1, 1)^{d-1}$  (resp. on  $(-M_0, M_0)$ ). We define the cut-off function  $\Theta \in C_c^\infty(\mathbb{R}^d)$  by for all  $x' \in \mathbb{R}^{d-1}$ ,  $y_d \in \mathbb{R}$ ,

$$\Theta(x', y_d) := \vartheta'(x') \vartheta_d(y_d).$$



Notice that  $\Theta$  is compactly supported in  $(-3/2, 3/2)^{d-1} \times (-3M_0/2, 3M_0/2)$ , identically equal to 1 on  $(-1, 1)^{d-1} \times (-M_0, M_0)$  and that for all  $x', \hat{x}' \in \mathbb{R}^{d-1}$ ,  $y_d \in \mathbb{R}$ ,

$$|x'|, |\hat{x}'| \leq 1 \quad \text{implies} \quad \Theta(x', y_d) = \Theta(\hat{x}', y_d). \quad (3.7)$$

The next lemma asserts the existence of the boundary correctors.

**Lemma 5** (boundary corrector, [30, Lemma 10]). *For all  $1/2 < \tau < 1$ , there exists  $C > 0$  so that for all  $\psi \in \mathcal{C}_{M_0}^{1, \nu_0}$ , for all  $0 < \varepsilon < 1$ , the unique weak solution  $v^\varepsilon \in W^{1,2}(D^\varepsilon(0, 2))$  of*

$$\begin{cases} -\nabla \cdot A^0 \nabla v^\varepsilon = \nabla \cdot A^0 \nabla (\psi(x'/\varepsilon) \Theta(x', x_d/\varepsilon)), & x \in D^\varepsilon(0, 2), \\ v^\varepsilon = 0, & x \in \partial D^\varepsilon(0, 2), \end{cases} \quad (3.8)$$

satisfies the following estimate: for all  $x \in D^\varepsilon(0, 3/2)$ ,

$$|v^\varepsilon(x)| \leq \frac{C\delta(x)^\tau}{\varepsilon^\tau}, \quad (3.9)$$

where  $\delta(x) := x_d - \varepsilon\psi(x'/\varepsilon)$ .

We refer to [30, pages 19 – 23] for a proof of this Lemma. The key to estimate (3.9) is the representation of  $v^\varepsilon$  via Green's kernel, an integration by parts and the following estimates of the Green kernel ( $d \geq 3$ ): for all  $x, \tilde{x} \in D^\varepsilon(0, 3/2)$ ,

$$|\nabla_2 \tilde{G}^\varepsilon(x, \tilde{x})| \leq \frac{C\delta(x)^\tau}{|x - \tilde{x}|^{d-1+\tau}}, \quad \text{for } |x - \tilde{x}| \leq \varepsilon, \quad (3.10)$$

$$|\nabla_2 \tilde{G}^\varepsilon(x, \tilde{x})| \leq \frac{C\delta(x)^\tau \delta(\tilde{x})^\tau}{\varepsilon |x - \tilde{x}|^{d-2+2\tau}}, \quad \text{for } |x - \tilde{x}| > \varepsilon. \quad (3.11)$$

Both estimates rely on the uniform Hölder estimate and on the classical Lipschitz estimate applied at microscale.

**Step 2 (continuation).** Using the bound (3.9), one can redo step 1 and show that for  $0 < \theta < 1/8$ , there exists  $\varepsilon_0$ , so that for all  $0 < \varepsilon < \varepsilon_0$ ,

$$\left\| u^\varepsilon(x) - \left( \int_{D^\varepsilon(0,1)} \partial_{x_d} u^\varepsilon \right) [x_d - \varepsilon\psi(x'/\varepsilon) \Theta(x', x_d/\varepsilon) - \varepsilon v^\varepsilon(x)] \right\|_{L^\infty(D^\varepsilon(0,\theta))} \leq \theta^{1+\mu}. \quad (3.12)$$

Now, for  $x \in D^{\varepsilon/\theta}(0, 1)$ , we define

$$U^\varepsilon(x) := \frac{1}{\theta^{1+\mu}} \left\{ u^\varepsilon(\theta x) - \left( \int_{D^\varepsilon(0,1)} \partial_{x_d} u^\varepsilon \right) [\theta x_d - \varepsilon\psi(\theta x'/\varepsilon) \Theta(\theta x', \theta x_d/\varepsilon) - \varepsilon v^\varepsilon(\theta x)] \right\}.$$

Because of (3.12), we have the bound

$$\|U^\varepsilon\|_{L^\infty(D^{\varepsilon/\theta}(0,1))} \leq 1,$$

and  $U^\varepsilon$  solves

$$\begin{cases} -\nabla \cdot A^0 \nabla U^\varepsilon = 0, & x \in D^{\varepsilon/\theta}(0, 1), \\ U^\varepsilon = 0, & x \in \Delta^{\varepsilon/\theta}(0, 1), \end{cases}$$

so that the estimate of step 1 can be applied. We can then show recursively that for every integer  $k$ , for all  $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ , there exists  $a_k^\varepsilon \in \mathbb{R}$  and  $V_k^\varepsilon = V_k^\varepsilon(x)$  such as

$$\begin{aligned} |a_k^\varepsilon| &\leq (C/\theta)[1 + \theta^\mu + \dots + \theta^{(k-1)\mu}], \\ |V_k^\varepsilon(x)| &\leq (C/\theta)[1 + \theta^\mu + \dots + \theta^{(k-1)\mu}] \frac{\delta(x)^\tau}{\varepsilon^\tau}, \end{aligned}$$

where  $C$  does not depend on  $k$  nor on  $\varepsilon$ , and

$$\left\| u^\varepsilon(x) - a_k^\varepsilon \{x_d - \varepsilon\psi(x'/\varepsilon)\Theta(x', x_d/\varepsilon)\} - \varepsilon V_k^\varepsilon(x) \right\|_{L^\infty(D^\varepsilon(0, \theta^k))} \leq \theta^{k(1+\mu)}. \quad (3.13)$$

This estimate basically means that since  $u^\varepsilon$  vanishes on the boundary, it remains small close to the boundary.

**Step 3.** Let  $\varepsilon > 0$  be fixed. If  $\varepsilon > \varepsilon_0$ , then the classical Lipschitz estimate applies. If  $\varepsilon < \varepsilon_0$ , then there exists a unique  $k$  so that  $\theta^k \leq \varepsilon/\varepsilon_0 < \theta^{k+1}$ . We have to estimate  $|\nabla u^\varepsilon|$  in the vicinity of every point  $x_0 \in D^\varepsilon(0, 1/2)$ . Let us start with the particular point  $x_0 = (0, x_{0,d}) \in D^\varepsilon(0, 1/2)$ . There are two situations depending on whether  $x_0$  is far or close to the boundary. In the former case when  $\theta^{l+1}/2 < \delta(x_0) = x_{0,d} - \varepsilon\psi(0) \leq \theta^l/2$  with  $0 \leq l \leq k$ , we are far enough from the boundary to apply a rescaled version of the interior Lipschitz estimate (see Proposition 2) and get

$$\|\nabla u^\varepsilon\|_{L^\infty(B(x_0, \delta(x_0)/4))} \leq \frac{C}{\delta(x_0)} \|u^\varepsilon\|_{L^\infty(B(x_0, \delta(x_0)/2))}.$$

Estimate (3.13) tells that

$$\|u^\varepsilon\|_{L^\infty(B(x_0, \delta(x_0)/2))} \leq C\delta(x_0),$$

where  $C$  only depends on  $\theta$  and  $\varepsilon_0$ . When  $x_0$  is close to the boundary i.e.  $0 \leq \delta(x_0) = x_{0,d} - \varepsilon\psi(0) \leq \theta^{k+1}/2$ , estimate (3.13) implies that

$$\|u^\varepsilon\|_{L^\infty(D^\varepsilon(0, \varepsilon/\varepsilon_0))} \leq C\varepsilon,$$

which together with the classical Lipschitz estimate yields

$$\|\nabla u^\varepsilon\|_{L^\infty(D^\varepsilon(0, \varepsilon/(2\varepsilon_0)))} \leq \frac{C}{\varepsilon} \|u^\varepsilon\|_{L^\infty(D^\varepsilon(0, \varepsilon/\varepsilon_0))} \leq C.$$

Arbitrary points  $x_0 \in D^\varepsilon(0, 1/2)$  not of the form  $(0, x_{0,d})$  are dealt with by translating the origin at  $(x'_0, 0)$ . Notice that the translated graph  $\psi$  still belongs to  $\mathcal{C}_{M_0}^{1, \nu_0}$  and the estimates above are uniform for  $\psi$  in this class.

## 4. Applications

### 4.1. Pointwise bounds on kernels and related estimates

**Pointwise bounds.** One of the main applications of uniform Hölder and Lipschitz estimates is to derive pointwise bounds on the large scale behavior of Green's and Poisson's kernels. Let  $G^{\alpha, \beta} = G^{\alpha, \beta}(x, \tilde{x})$  be Green's kernel associated to the operator  $-\nabla \cdot A(x/\alpha)\nabla$  and to the oscillating domain  $\{x_d > \beta\psi(x'/\beta)\}$ . The Poisson kernel  $P^{\alpha, \beta} = P^{\alpha, \beta}(x, \tilde{x})$  is defined for all  $x \in \{x_d > \beta\psi(x'/\beta)\}$ , for all  $\tilde{x} \in \{x_d = \beta\psi(x'/\beta)\}$  by

$$P^{\alpha, \beta}(x, \tilde{x}) = - \left[ A^*(\tilde{x}/\alpha) \nabla_{\tilde{x}} G^{*, \alpha, \beta}(\tilde{x}, x) \cdot n_{\psi, \beta}(\tilde{x}) \right]^T \quad (4.1)$$

where the starred quantities refer to the adjoint operator  $-\nabla \cdot A^*(x/\alpha)\nabla$ .

**Proposition 6** ([30, Proposition 20 and 21]). *For all  $d \geq 2$ , for all  $0 \leq \mu_1, \mu_2 \leq 1$ , there exists  $C > 0$ , so that for all  $\psi \in \mathcal{C}_{M_0}^{1, \nu_0}$ , for all  $A \in \mathcal{A}^{0, \nu_0}$ , for all  $\alpha, \beta > 0$ , for all  $x, \tilde{x} \in \{x_d > \beta\psi(x'/\beta)\}$ , we have*

$$|G^{\alpha, \beta}(x, \tilde{x})| \leq C \log(1 + |x - \tilde{x}|), \quad \text{if } d = 2, \quad (4.2)$$

$$|G^{\alpha, \beta}(x, \tilde{x})| \leq \frac{C}{|x - \tilde{x}|^{d-2}}, \quad \text{if } d \geq 3, \quad (4.3)$$

and for all  $d \geq 2$ ,

$$|G^{\alpha, \beta}(x, \tilde{x})| \leq C \frac{\delta^{\mu_1}(x) \delta^{\mu_2}(\tilde{x})}{|x - \tilde{x}|^{d-1}}, \quad (4.4)$$

$$|\nabla_1 G^{\alpha, \beta}(x, \tilde{x})| \leq \frac{C}{|x - \tilde{x}|^{d-1}} \min \left\{ 1, \frac{\delta(\tilde{x})}{|x - \tilde{x}|} \right\}, \quad |\nabla_1 \nabla_2 G^{\alpha, \beta}(x, \tilde{x})| \leq \frac{C}{|x - \tilde{x}|^d}, \quad (4.5)$$

$$|P^{\alpha, \beta}(x, \tilde{x})| \leq \frac{C \delta(x)}{|x - \tilde{x}|^d}, \quad |\nabla_1 P^{\alpha, \beta}(x, \tilde{x})| \leq \frac{C}{|x - \tilde{x}|^d} \left\{ 1 + \frac{\delta(x)}{|x - \tilde{x}|} \right\}. \quad (4.6)$$

Notice that  $C$  depends on  $d, N, M_0, \lambda$  and  $\nu_0$ , but is uniform in  $\alpha, \beta > 0$ .

The proof of these bounds is standard and relies on the uniform Hölder and Lipschitz estimates of [30]. On the one hand (4.2), (4.3) and (4.4) for  $0 < \mu_1, \mu_2 < 1$  follow from Hölder's estimate, on the other hand (4.4) (for  $\mu_1$  or  $\mu_2 = 1$ ), (4.5) and (4.6) follow from the Lipschitz estimate. Similar estimates also hold for the Green and Poisson kernels in bounded  $C^{1, \nu_0}$  domains, by replacing  $\delta$  by the distance to the boundary.

In order to show where the uniformity in  $\alpha$  and  $\beta$  comes into play, we sketch the proof of (4.4) for  $\mu_1 = 1, \mu_2 = 0$  and  $d \geq 3$ . The latter estimate is only better than (4.3) when  $\delta(x)$  is small with respect to  $|x - \tilde{x}|$ . Therefore, if  $\delta(x) \geq |x - \tilde{x}|/6$ , (4.4) is a simple consequence of (4.3). Assume now that  $\delta(x) < |x - \tilde{x}|/6$ . In that case, call  $\bar{x}$  the point on the oscillating boundary with coordinates  $\bar{x} = (x', \beta\psi(x'/\beta))$ . Since  $G^{\alpha, \beta}(\cdot + \bar{x}, \tilde{x})$  is a weak solution to

$$-\nabla \cdot A((\cdot + \bar{x})/\alpha) \nabla G^{\alpha, \beta}(\cdot + \bar{x}, \tilde{x}) = 0 \quad \text{in } D_{\psi(\cdot + \bar{x})}^\beta(0, |x - \tilde{x}|/3)$$

vanishing on  $\Delta_{\psi(\cdot + \bar{x})}^\beta(0, |x - \tilde{x}|/3)$ , we have

$$\begin{aligned} |G^{\alpha, \beta}(x, \tilde{x})| &= |G^{\alpha, \beta}(x, \tilde{x}) - G^{\alpha, \beta}(\bar{x}, \tilde{x})| \\ &\leq \sup_{z \in [x, \bar{x}]} |\nabla G^\varepsilon(z, \tilde{x})| |x - \bar{x}| = \sup_{z \in D^\beta(0, |x - \tilde{x}|/6)} |\nabla G^\varepsilon(z + \bar{x}, \tilde{x})| \delta(x) \\ &\leq \frac{6C \delta(x)}{|x - \tilde{x}|} \|G^\varepsilon(\cdot + \bar{x})\|_{L^\infty(D^\beta(0, |x - \tilde{x}|/3))} \leq \frac{C \delta(x)}{|x - \tilde{x}|^{d-1}}. \end{aligned}$$

Here we have used the rescaled Lipschitz estimate (3.2) (true for all  $r > 0$  thanks to the uniformity in  $\alpha$  and  $\beta$ ) with  $r = |x - \tilde{x}|/3$ .

**$L^p$  estimates.** Uniform  $L^p$  estimates for

$$\begin{cases} -\nabla \cdot A(x/\alpha) \nabla u^{\alpha, \beta} = f, & x_d > \beta\psi(x'/\beta), \\ u^{\alpha, \beta} = g, & x_d = \beta\psi(x'/\beta), \end{cases}$$

are immediate corollaries of the pointwise bounds on the Green and Poisson kernels.

**Corollary 7** ( $L^p$  estimate for the boundary value problem). *Let  $f = 0$ . For all  $1 < p < \infty$ , there exists  $C > 0$ , so that for all  $\psi \in C_{M_0}^{1,\nu_0}$ , for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\alpha, \beta > 0$ , for all  $g \in L^p(\{x_d = \beta\psi(x'/\beta)\})$ ,  $u^{\alpha,\beta}$  the solution of (4.1) given by the Poisson integral*

$$u^{\alpha,\beta}(x) = \int_{\tilde{x}_d = \beta\psi(\tilde{x}'/\beta)} P^{\alpha,\beta}(x, \tilde{x})g(\tilde{x})d\tilde{x}$$

*satisfies*

$$\|u^{\alpha,\beta}\|_{L^p(\{x_d > \beta\psi(x'/\beta)\})} \leq C\|g\|_{L^p(\{x_d > \beta\psi(x'/\beta)\})}.$$

**Corollary 8** (gradient  $L^q$  estimate). *Let  $g = 0$ . For all  $\kappa > 0$ , for all  $1 \leq p \leq d + \kappa$ , there exists  $C > 0$ , so that for all  $\psi \in C_{M_0}^{1,\nu_0}$ , for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\alpha, \beta > 0$ , for all  $f \in L^q(\{x_d > \beta\psi(x'/\beta)\})$ , the solution  $u^{\alpha,\beta}$  to (4.1) given by the Green integral*

$$u^{\alpha,\beta}(x) = \int_{\tilde{x}_d > \beta\psi(\tilde{x}'/\beta)} G^{\alpha,\beta}(x, \tilde{x})f(\tilde{x})d\tilde{x}$$

*satisfies*

$$\|\nabla u^{\alpha,\beta}\|_{L^q(\{x_d > \beta\psi(x'/\beta)\})} \leq C\|f\|_{L^p(\{x_d > \beta\psi(x'/\beta)\})},$$

*with  $1/q = 1/p - 1/(d + \kappa)$ .*

These estimates have been proved by Avellaneda and Lin in bounded non oscillating domains: see [4, Theorem 3] for Corollary 7 and [4, Theorem 4] for Corollary 8. The boundary  $L^p$  estimate follows from the bound (4.6), whereas the gradient bound relies on estimate (4.5). Generalizing such estimates to rougher boundaries, Lipschitz for instance instead of  $C^{1,\nu_0}$  is a different story and requires other tools [32, 31].

We also get an Agmon-Miranda maximum principle for systems.

**Corollary 9** (maximum principle for systems, [30, Lemma 23]). *There exists  $C > 0$ , such that for all  $\psi \in C_{M_0}^{1,\nu_0}$ , for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\alpha, \beta > 0$ , for all  $g \in L^\infty(\Delta^\beta(0, 1))$ , for all weak solution  $v^{\alpha,\beta} = v^{\alpha,\beta}(x) \in L^2(D^\beta(0, 1))$  to*

$$\begin{cases} -\nabla \cdot A(x/\alpha)\nabla v^{\alpha,\beta} &= 0, & x \in D^\beta(0, 1), \\ v^{\alpha,\beta} &= g, & x \in \Delta^\beta(0, 1), \end{cases}$$

*we have*

$$\|v^{\alpha,\beta}\|_{L^\infty(D^\beta(0,1/2))} \leq C \left\{ \|v^{\alpha,\beta}\|_{L^2(D^\beta(0,1))} + \|g\|_{L^\infty(\Delta^\beta(0,1))} \right\}.$$

*Notice that  $C$  depends on  $d, N, M_0, \lambda$  and  $\nu_0$ .*

**Expansions of Green and Poisson kernels.** This paragraph deals with systems with non oscillating boundary. In other words, we take for the moment  $\alpha = \varepsilon$  and  $\beta = 1$ . Relying on two scale asymptotic expansions and on uniform estimates, Avellaneda and Lin have been able to derive expansions for the fundamental solution  $G^\varepsilon = G^\varepsilon(x, \tilde{x})$  of  $-\nabla \cdot A(x/\varepsilon)\nabla$  valid for  $|x - \tilde{x}| \gg 1$ : see [9], Lemmas 1, 2, 3 and their corollaries. This work has inspired the derivation of an expansion for Poisson kernel in the half-space [36, Theorem 5.3]. Kenig, Lin and Shen [29] have obtained several (near) optimal expansions for Green's and Poisson's kernels, as well as some of their derivatives in bounded domains. Their work includes in particular a substantial improvement of a result by Avellaneda and Lin [8, Theorem 1] on the

approximation of Poisson's kernel: the Poisson kernel  $P^\varepsilon$  associated to  $-\nabla \cdot A(x/\varepsilon)\nabla$  and  $\Omega$  (sufficiently smooth, at least  $C^{2,\eta}$ ,  $\eta > 0$ ) can be written as

$$P^\varepsilon(x, \tilde{x}) = P^0(x, \tilde{x})\omega^\varepsilon(\tilde{x}) + R^\varepsilon(x, \tilde{x}),$$

where the remainder term is estimated as follows

$$|R^\varepsilon(x, \tilde{x})| \leq \frac{C\varepsilon \log(\varepsilon^{-1}|x - \tilde{x}| + 1)}{|x - \tilde{x}|^d}.$$

The factor  $\omega^\varepsilon(\tilde{x})$  is explicit and accounts for the high frequency oscillation of the Poisson kernel  $P^\varepsilon$ , whereas  $P^0$  is the Poisson kernel associated to the homogenized operator  $-\nabla \cdot \bar{A}\nabla$  and  $\Omega$ . We refer to [29, Theorem 3.8] for more details and the expression for  $\omega^\varepsilon$ .

## 4.2. Homogenization of boundary layers

Much of the recent progress in the analysis of systems of the type of (1.3) with oscillating Dirichlet boundary data has been achieved by studying

$$\begin{cases} -\nabla \cdot A(y)\nabla v_{bl} = 0, & y \cdot n > 0, \\ v_{bl} = \varphi(y), & y \cdot n = 0, \end{cases} \quad (4.7)$$

posed in the half-space  $\{y \cdot n > 0\}$ . As underlined in the introduction, this system comes from blowing-up (1.3) in the vicinity of a point of the boundary  $\partial\Omega$ . So as to identify a possible homogenized boundary condition  $\bar{\varphi}$  for (1.3), one has to investigate the behavior of  $v_{bl}$  far from the boundary, i.e. when  $y \cdot n \rightarrow \infty$ . Depending on whether  $n \in \mathbb{S}^{d-1}$

**(RAT)**: has rational coordinates, i.e.  $n \in \mathbb{R}\mathbb{Q}^d$ ,

**(DIV)**: satisfies a small divisors assumption, i.e. there exists  $c, \tau > 0$  such as for all  $\xi \in \mathbb{Z}^d \setminus \{0\}$ , for all  $i = 1, \dots, d-1$ ,

$$|n_i \cdot \xi| \geq C |\xi|^{-d-\tau} \quad (4.8)$$

where  $(n_1, \dots, n_{d-1}, n)$  forms an orthogonal basis of  $\mathbb{R}^d$ ,

**(GEN)**: or is an arbitrary vector  $\notin \mathbb{R}\mathbb{Q}^d$ , which does not satisfy **(DIV)**,

the asymptotics of  $v_{bl}$  may be very different.

The system (4.7), existence and asymptotics, can be analysed by relying only on energetical arguments (Saint-Venant estimates) in the cases **(RAT)** (see [34]) and **(DIV)** (see [23]). In both cases one can show that the (unique, if some decay of the gradient is prescribed) solution  $v_{bl}$  of (4.7) converges very fast to a constant  $V^\infty \in \mathbb{R}^N$ , when  $y \cdot n \rightarrow \infty$ . For a precise statement, see the original references. To gain an insight into the behavior of  $v_{bl}$  in the general case **(GEN)**, as well as a precise description of the dependence of  $V^\infty$  on  $n$ , the starting point is the representation formula of  $v_{bl}$  in terms of Poisson's kernel  $P = P(y, \tilde{y})$  for the operator  $-\nabla \cdot A(y)\nabla$  and the domain  $\{y \cdot n > 0\}$ ,

$$v_{bl}(y) = \int_{\tilde{y} \cdot n = 0} P(y, \tilde{y})\varphi(\tilde{y})d\tilde{y}.$$

Estimates on the Green and Poisson kernels are crucial in the work of Gérard-Varet and Masmoudi [24, section 2], where the function  $V^\infty = V^\infty(n)$  is studied for

$n$  satisfying **(DIV)**. They prove the homogenization of (1.3) in the case of smooth uniformly convex domains  $\Omega$ . Along with the convergence, their main result [24, Theorem 1] gives a rate of convergence in  $L^2(\Omega)$  between  $u_{bl}^\varepsilon$  and its homogenized limit. In dimension  $d = 2$ , the assumption of uniform convexity makes it possible to approximate the domain  $\Omega$  by polygonal domains with slopes satisfying **(DIV)**. In particular, the uniform convexity rules out the possibility that  $\partial\Omega$  has flat parts with normals  $n$  satisfying **(GEN)**.

The interaction between the microstructure and the boundary has been analysed in full generality in the paper [36]. The asymptotics of  $v_{bl}$  have been investigated in the general case when  $n$  neither satisfies **(RAT)** nor **(DIV)**. The following result is proved: for all  $n$  satisfying **(GEN)**, there exists  $V^\infty \in \mathbb{R}^N$ , so that

$$v_{bl}(y) \xrightarrow{y \cdot n \rightarrow \infty} V^\infty,$$

locally uniformly in the tangential variable. Moreover, explicit examples show that the convergence can be almost arbitrarily slow. The key to the proof of this result is to establish an expansion of the Poisson kernel  $P(y, \tilde{y})$  for  $|y - \tilde{y}| \gg 1$ .

**Theorem 10** ([36, Theorem 5.3]). *There exists an explicit kernel  $P^{exp} = P^{exp}(y, \tilde{y})$  with ergodicity properties along the boundary, and a number  $\eta > 0$  so that for all  $y, \tilde{y} \in \mathbb{R}^d$ ,  $y \cdot n > 0$ ,  $\tilde{y} \cdot n = 0$ ,*

$$|P(y, \tilde{y}) - P^{exp}(y, \tilde{y})| \leq \frac{C}{|y - \tilde{y}|^{d-1+\eta}}. \quad (4.9)$$

Using this theorem, it is then possible to pass to the limit in

$$v_{bl}(y) = \int_{\tilde{y} \cdot n = 0} P^{exp}(y, \tilde{y}) \varphi(\tilde{y}) d\tilde{y} + \int_{\tilde{y} \cdot n = 0} \{P(y, \tilde{y}) - P^{exp}(y, \tilde{y})\} \varphi(\tilde{y}) d\tilde{y}$$

when  $y \cdot n \rightarrow \infty$ . The first term converges to a constant  $V^\infty \in \mathbb{R}^N$  thanks to ergodicity on the boundary, while the second term tends to 0 thanks to (4.9).

The proof of Theorem 10 uses in a crucial way the uniform boundary Lipschitz estimate [4, Lemma 20]. Let  $\tilde{x}$  be fixed on the boundary, and  $x \in \mathbb{R}^d$  so that  $x \cdot n > 0$  and  $|x - \tilde{x}| = 1$ . The main idea, in the spirit of the paper [9] on large scale expansions of fundamental solutions, is to 2-scale expand the Green kernel  $G^{*,\varepsilon}(\cdot, x)$  associated to the operator  $-\nabla \cdot A^*(x/\varepsilon) \nabla$  and to the domain  $\{x \cdot n > 0\}$ :

$$G^{*,\varepsilon}(\cdot, x) = G^{\varepsilon, exp}(\cdot) + R^\varepsilon(\cdot).$$

The remainder  $R^\varepsilon$  then satisfies the system

$$\begin{cases} -\nabla \cdot A(z/\varepsilon) \nabla R^\varepsilon &= F^\varepsilon, & z \in D(\tilde{x}, 1/2), \\ R^\varepsilon &= H^\varepsilon, & z \in \Delta(\tilde{x}, 1/2). \end{cases}$$

By the uniform Lipschitz estimate, we get

$$\|\nabla R^\varepsilon\|_{L^\infty(D(\tilde{x}, 1/4))} \leq C \left\{ \|R^\varepsilon\|_{L^\infty(D(\tilde{x}, 1/2))} + \|F^\varepsilon\|_{L^{d+\kappa}(D(\tilde{x}, 1/2))} + \|H^\varepsilon\|_{C^{1,\mu}(\Delta(\tilde{x}, 1/2))} \right\},$$

where  $D(\tilde{x}, r) := B(\tilde{x}, r) \cap \{z \cdot n > 0\}$ . Each term on the right hand side can be showed to be of order  $O(\varepsilon^\eta)$ , for some  $\eta > 0$ . Therefore, by the definition of the Poisson kernel (4.1),

$$\left| P^\varepsilon(x, \tilde{x}) - P^0(x, \tilde{x}, \tilde{x}/\varepsilon) - \varepsilon P^1(x, \tilde{x}, \tilde{x}/\varepsilon) - \varepsilon^2 P^2(x, \tilde{x}, \tilde{x}/\varepsilon) \right| \lesssim \varepsilon^\eta,$$

which yields Theorem 10 by rescaling in the variables  $y = x/\varepsilon$  and  $\tilde{y} = \tilde{x}/\varepsilon$ .

**Toward oscillating half-spaces.** We have in mind to extend the previous analysis to the system

$$\begin{cases} -\nabla \cdot A(y)\nabla v = 0, & y_d > \psi(y'), \\ v = v_0(y), & y_d = \psi(y'), \end{cases} \quad (4.10)$$

posed in the oscillating half-space  $\{y_d > \psi(y')\}$ , with  $\psi \in C_{M_0}^{1,\nu_0}$ . Given that we have a uniform Lipschitz estimate for domains with oscillating boundaries (see Theorem 3), is it possible to expand in a similar way the Poisson kernel associated to  $-\nabla \cdot A(y)\nabla$  and to the domain  $\{y_d > \psi(y')\}$ ?

We take  $\alpha = \beta =: \varepsilon$ . Thanks to local error estimates in  $L^\infty$ , we have been able to prove:

**Theorem 11** ([30, Theorem 22]). *There exists  $C > 0$ , so that for all  $\psi \in C_{M_0}^{1,\nu_0}$ , for all  $A \in \mathcal{A}^{0,\nu_0}$ , for all  $\varepsilon > 0$ , for all  $x, \tilde{x} \in \{x_d > \varepsilon\psi(x'/\varepsilon)\}$ ,*

$$|G^\varepsilon(x, \tilde{x}) - G^0(x, \tilde{x})| \leq \frac{C\varepsilon}{|x - \tilde{x}|^{d-1}}.$$

where  $G^0$  is the Green kernel associated to the homogenized operator  $-\nabla \cdot \bar{A}\nabla$  and to the flat space  $\mathbb{R}_+^d$ . Notice that  $C$  depends on  $d, N, M_0, \lambda$  and  $\nu_0$ .

Of course, this statement translates into a result in the fast variables: for all  $y, \tilde{y} \in \{y_d > \psi(y')\}$ ,

$$|G(y, \tilde{y}) - G^0(y, \tilde{y})| \leq \frac{C}{|y - \tilde{y}|^{d-1}}.$$

Let us describe in a loose way the limitations which prevent us from going further in the expansion in general. For fixed  $x$ , the first term  $G^0(x, \cdot)$  in the asymptotic expansion of  $G^\varepsilon$  does not vanish on the oscillating boundary, and worse it is highly oscillating. These oscillations lead to bad Lipschitz estimates, singular in  $\varepsilon$ .

To overcome this problem, an idea in the spirit of the works on oscillating boundaries in fluid mechanics (see for instance among a lot of other papers [21, 22]) is to replace the oscillating boundary by a flat one with an ad hoc boundary condition. This boundary condition may not necessarily be of Dirichlet type, as was first taken for  $G^0$  above. Another boundary condition on the flat boundary (for example of Navier type) may yield another zero-order term, which better approximates the first-order derivatives of  $G^\varepsilon$  than does  $G^0$ . One of the issues is that determining the right boundary condition involves being able to study the behavior of  $v_{bl}$  far from the boundary.

Studying the asymptotics of  $v_{bl}$  requires in turn some structure (periodicity, quasiperiodicity or stationary ergodicity for example) on the oscillations of the boundary. Consider the simple problem

$$\begin{cases} -\Delta v = 0, & y_2 > 0, \\ v = v_0, & y_2 = 0, \end{cases} \quad (4.11)$$

posed in  $\mathbb{R}_+^2$ , with boundary data  $v_0 \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$  non decaying at space infinity. The formula

$$v(y_1, y_2) = \int_{\mathbb{R}} \frac{y_2}{\pi((t - y_1)^2 + y_2^2)} v_0(t) dt$$

gives a bounded solution to (4.11). This solution is the unique solution of (4.11) in the class of bounded functions. Gérard-Varet and Masmoudi [22, Proposition

[11] point out an example of a  $v_0 \in L^\infty(\mathbb{R})$  (one can smooth it to be in  $C^\infty(\mathbb{R})$  as well), so that  $v(0, y_2)$  does not converge when  $y_2 \rightarrow \infty$ . This boundary data  $v_0$  for which convergence does not hold lacks ergodicity properties, which would ensure the convergence of the averages  $v(0, y_2)$  when the “time”  $y_2$  tends to  $\infty$ .

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