Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Biarritz, 3–7 juin 2013

Mathieu Lewin

Derivation of Hartree's theory for mean-field Bose gases

J. É. D. P. (2013), Exposé n° VII, 21 p.

<http://jedp.cedram.org/item?id=JEDP_2013____A7_0>

Journées *Équations aux dérivées partielles*
Biarritz, 3–7 juin 2013
GDR 2434 (CNRS)

Derivation of Hartree's theory for mean-field Bose gases

Mathieu Lewin

Dérivation de la théorie de Hartree pour des gaz de bosons dans le régime de champ moyen

Résumé

Dans cet article, nous présentons des résultats obtenus avec Phan Thành Nam, Nicolas Rougerie, Sylvia Serfaty et Jan Philip Solovej. Nous considérons un système de N bosons qui interagissent avec un potentiel d'intensité $1/N$ (on parle de régime de champ moyen). Dans la limite où $N \to \infty$, nous montrons que le premier ordre du développement des valeurs propres du Hamiltonien à N corps est donné par la théorie non linéaire de Hartree, alors que l'ordre suivant est donné par l'opérateur de Bogoliubov. Nous discutons également en détails du phénomène de condensation de Bose-Einstein dans de tels systèmes.

Abstract

This article is a review of recent results with Phan Thành Nam, Nicolas Rougerie, Sylvia Serfaty and Jan Philip Solovej. We consider a system of N bosons with an interaction of intensity $1/N$ (mean-field regime). In the limit $N \to \infty$, we prove that the first order in the expansion of the eigenvalues of the many-particle Hamiltonian is given by the nonlinear Hartree theory, whereas the next order is predicted by the Bogoliubov Hamiltonian. We also discuss the occurrence of Bose-Einstein condensation in these systems.

Contents

The author has received financial support from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement MNIQS 258023) and from the French Ministry of Research (ANR-10-BLAN-0101).

MSC 2000: 35Q40, 81Q99.
Keywords: Hartree theory, mean-field limit, Bose-Einstein condensation, quantum de Finetti theorem.

The purpose of this article is to review recent results obtained in collaboration with Phan Thành Nam and Nicolas Rougerie in [40] and, in the last section, with Phan Thành Nam, Sylvia Serfaty and Jan Philip Solovej in [42]. In these works we studied the behavior of a quantum system containing a large number N of bosons, in the *mean-field regime* corresponding to having interactions of order $1/N$. The first term in the expansion of the eigenvalues of the many-particle Hamiltonian is given by Hartree's theory, whereas the second term is predicted by Bogoliubov's theory.

1. Derivation of Hartree's energy

1.1. The quantum many-particle Hamiltonian

We consider here the simplest case of N spinless non-relativistic particles leaving in the whole space $\mathbb{R}^d$ with $d \geq 1$. More general situations can be dealt with using our method and we refer to [40, 42] for details. The corresponding many-particle Hamiltonian reads

$$H_N := \sum_{j=1}^{N} \Big(-\Delta_{x_j} + V(x_j) \Big) + \frac{1}{N-1} \sum_{1 \leq k < \ell \leq N} w(x_k - x_\ell). \tag{1.1}$$

It is an operator acting on the subspace

$$\mathfrak{H}^N := L^2_s((\mathbb{R}^d)^N)$$

of symmetric functions in $L^2((\mathbb{R}^d)^N)$, that is, which satisfy

$$\Psi(x_{\sigma(1)}, ..., x_{\sigma(N)}) = \Psi(x_1, ..., x_N)$$

for every permutation σ of the indices $1, ..., N$. This symmetry requirement is the mathematical expression of the fact that our particles are *bosons*.[1] For fermions we would have to require *anti*-symmetry and everything would be completely different.

In (1.1), the first term corresponds to the sum of the kinetic energies of the particles, V is an external potential which is applied to the system and which could as well be $V \equiv 0$ (the system is then translation-invariant), and w is a two-body interaction. Everywhere in this paper we make the assumption that w is even, $w(-x) = w(x)$. We will also assume that $V = f_1 + f_2 + V_+$ and $w = f_3 + f_4$ with $f_i \in L^{p_i}(\mathbb{R}^d)$ and

[1] In the particular case considered here, the first eigenfunction of H_N on $L^2(\mathbb{R}^{dN})$ is always symmetric. If we are only interested in the lowest eigenvalue, we could therefore drop the symmetry condition. However the constraint matters for the higher eigenvalues, as well as for more general Hamiltonians which are not based on the Laplacian.

$\max(1, d/2) < p_i < \infty$, or with $p_i = \infty$ and $f_i \to 0$ at infinity. These conditions imply that $V - V_+$ and w are perturbations of the Laplacian, which are relatively compact in the quadratic form sense. We may want to consider the case where the system is confined and we use $0 \leq V_+ \in L^{d/2}_{\rm loc}(\mathbb{R}^d)$ to describe this situation. Here we will therefore discuss two particular cases:

- **confined case:** $V_+(x) \to \infty$ when $|x| \to \infty$;

- **unconfined case:** $V_+ \equiv 0$.

Under our assumptions on V and w, the quadratic form associated with H_N is bounded from below, and H_N can be realized as a self-adjoint operator on $\mathfrak{H}^N = L^2_s((\mathbb{R}^d)^N)$ by Friedrich's theorem. The domain of the quadratic form is exactly the Sobolev space $H^1_s((\mathbb{R}^d)^N)$ in the unconfined case and it is smaller in the confined case. This allows us to define the bottom of the spectrum of H_N as follows:

$$E(N) := \inf \sigma(H_N) = \inf_{\substack{\Psi \in \mathfrak{H}^N \\ \int |\Psi|^2 = 1}} \langle \Psi, H_N \Psi \rangle. \tag{1.2}$$

Later we will consider the higher eigenvalues $\lambda_j(H_N)$ of H_N, but we stick to $E(N)$ for the moment.

The fact that we are considering the *mean-field regime* is apparent in the factor $1/(N-1)$ in front of the interaction term in (1.1). It has the effect of keeping the single particle energy and the interaction energy of the same order of magnitude, so that one may expect a well-defined limit problem. Note that this factor could be replaced by any constant behaving like $1/N$ in the limit $N \to \infty$, without changing the result; the use of $1/(N-1)$ only simplifies some expressions. While this is certainly not the only scaling one may consider, it is simple and instructive, and has been very often considered in the past as a model case for the rigorous derivation of mean-field theories in many-body physics.

1.2. Hartree states

On the contrary to fermions, it is possible to put all the bosons in the same quantum state. This amounts to taking a wave function Ψ of the special form

$$\Psi(x_1, ..., x_N) = u^{\otimes N}(x_1, ..., x_N) = u(x_1) \cdots u(x_N)$$

with $u \in L^2(\mathbb{R}^d) := \mathfrak{H}$ and $\int_{\mathbb{R}^d} |u|^2 = 1$. Such functions are called *Hartree states* [32]. We recall that $|\Psi(x_1, ..., x_N)|^2$ is interpreted as the density of probability for the positions of the N particles. For a Hartree state, the corresponding probability factorizes into a product of independent densities: $|u(x_1)|^2 \cdots |u(x_N)|^2$. Similarly, $|\widehat{\Psi}(p_1, ..., p_N)|^2$ is the probability for the momenta of the N particles and we have here $|\widehat{\Psi}(p_1, ..., p_N)|^2 = |\widehat{u}(p_1)|^2 \cdots |\widehat{u}(p_N)|^2$.

The Hartree states form a manifold in the sphere of square-integrable symmetric functions and, therefore, even if the many-particle energy is given by a linear operator, we end up with a nonlinear function of u:

$$\begin{aligned}
&\frac{\left\langle u^{\otimes N}, H_N u^{\otimes N}\right\rangle}{N}\\
&\quad=\int_{\mathbb{R}^d}\left(|\nabla u(x)|^2+V(x)|u(x)|^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}w(x-y)|u(x)|^2|u(y)|^2\,dx\,dy\\
&\quad:=\mathcal{E}_{\rm H}(u).
\end{aligned}$$

The nonlinear functional $\mathcal{E}_{\rm H}$ is called the *Hartree energy*. The lowest energy that can be reached with Hartree states is $Ne_{\rm H}$ where

$$e_{\rm H}:=\inf_{\|u\|=1}\mathcal{E}_{\rm H}(u). \tag{1.3}$$

Since $\Psi=u^{\otimes N}$ can be used as a trial state, it is obvious that $E(N)\leq Ne_{\rm H}$ for all $N\geq 2$. Note that any minimizer u_0 of (1.3) (when there is one) is a weak solution to the nonlinear equation

$$\Big(-\Delta+V+w*|u_0|^2\Big)u_0(x)=\mu_{\rm H}\,u_0(x). \tag{1.4}$$

Here $w*|u|^2$ is called the *mean-field potential.*

The main result of [40] is that the many-body quantum energy is, to first order in N, given by the energy of Hartree states.

Theorem 1 (Validity of Hartree's theory [40])**.** *Under the previous assumptions on V and w, we have*

$$\boxed{\lim_{N\to\infty}\frac{E(N)}{N}=e_{\rm H}.} \tag{1.5}$$

The theorem justifies *a posteriori* that the chosen scaling in front of the interaction places us in the mean-field regime. It says that the quantum energy is, to first order in N, given by Hartree states.

There are many similar results in the literature but the previous theorem seems to be the first which does not rely on any specific property of V and w. The confined case was previously treated in [25, 66, 56, 67]. Bosonic atoms corresponding to $d=3$ and

$$V(x)=-\frac{1}{t|x|},\qquad w(x)=\frac{1}{|x|}$$

were considered in [9, 60, 6, 7, 37] but the proof uses some particular properties of the Coulomb potential. In [49, 50], Lieb, Thirring and Yau studied the case of bosons stars in which $V\equiv 0$, $w(x)=-1/|x|$ and $-\Delta$ is replaced by the fractional Laplacian $\sqrt{1-\Delta}$. Our approach also applies to this case as well [40]. Other results include the homogeneous Bose gas [58], trapped Bose gases [31] and the Lieb-Liniger model in a random external potential [44, 59].

Recently, the experimental realization of Bose-Einstein condensates has motivated the study of a more subtle, so called *Gross-Pitaevskii*, limit [46, 45]. The effective theory obtained in this limit is the cubic nonlinear Schrödinger equation, corresponding to taking $w(x-y)=4\pi a\delta(x-y)$ where a is the scattering length of the potential w. We will not consider this limit here, but we hope that our method will in the future be useful to deal with it as well.

That the *linear* many-particle quantum system is well described by a *nonlinear* theory in the mean-field limit $N\to\infty$ has important physical implications. It is the main explanation for the occurrence of *symmetry breaking* which cannot be well described in a linear model. For instance, it has been observed in the laboratory that in a Bose gas which is rotated along an axis, vortices appear and they tend to place themselves on a triangular lattice. This phenomenon is very well explained in the framework of the nonlinear Hartree and Gross-Pitaevskii theories (see, e.g., [2, 1, 3]).

Let us remark that there are many works on the related (but still different) derivation of the time-dependent Hartree theory from the time-dependent Schrödinger equation associated with the Hamiltonian H_N, see for instance [33, 27, 62, 8, 21, 22, 4, 24, 26, 57, 38, 55, 41]. In this case one starts close to a Hartree state at time zero, and then proves that the Schrödinger flow stays close to the corresponding trajectory of the Hartree state. This is *a priori* different from the time-independent case considered here.

Simple proof when $\widehat{w} \geq 0$

There is a simple proof of Theorem 1 in the particular case when $\widehat{w}$ is non-negative and integrable. We quickly outline it for the convenience of the reader. The main trick is the observation that, in this case,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)f(x)f(y)\,dx\,dy = (2\pi)^{d/2}\int_{\mathbb{R}^d}\widehat{w}(k)|\widehat{f}(k)|^2\,dk \geq 0$$

for any measure f such that $\widehat{f} \in L^\infty(\mathbb{R}^d)$. For $f = \sum_{k=1}^N \delta_{x_k} - g$, this leads to the estimate

$$\sum_{1\leq k<\ell\leq N} w(x_k - x_\ell) \geq \sum_{k=1}^N w * g(x_k) - \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)g(x)g(y)\,dx\,dy - \frac{N}{2}w(0).$$

Applied to a given $\Psi \in L^2_s((\mathbb{R}^d)^N)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} &\sum_{1\leq k<\ell\leq N} w(x_k - x_\ell)|\Psi(x_1,...,x_N)|^2\,dx_1\cdots dx_N\\ &\geq N\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)\rho_\Psi(x)g(y)\,dx\,dy\\ &\qquad -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)g(x)g(y)\,dx\,dy - \frac{N}{2}w(0)\\ &\geq \frac{N^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)\rho_\Psi(x)\rho_\Psi(y)\,dx\,dy - \frac{N}{2}w(0),\end{aligned}$$

where

$$\rho_\Psi(x) := \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}|\Psi(x,x_2,...,x_N)|^2\,dx_2\cdots dx_N$$

is the *density of particles* in the system and where in the last line we have optimized with respect to g (that is, we have taken $g = N\rho_\Psi$). If we now use the Hoffman-Ostenhof inequality [35]

$$\sum_{j=1}^N\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}|\nabla_{x_j}\Psi(x_1,...,x_N)|^2\,dx_1\cdots dx_N \geq N\int_{\mathbb{R}^d}\left|\nabla\sqrt{\rho_\Psi}(x)\right|^2 dx,$$

which is a simple consequence of the Cauchy-Schwarz inequality, we get that

$$\langle\Psi, H_N\Psi\rangle \geq N\mathcal{E}_{\rm H}(\sqrt{\rho_\Psi}) - \frac{N}{2(N-1)}w(0) \geq Ne_{\rm H} - \frac{N}{2(N-1)}w(0)$$

and the result follows. We have even shown that $E(N) = Ne_{\rm H} + O(1)$.

The proof of the validity of Hartree's theory is very elementary when $\widehat{w} \geq 0$. When $\widehat{w}$ has no particular sign, the argument is completely different and it is the purpose of this article to explain the strategy of [40].

2. Bose-Einstein condensation

2.1. Density matrices and the quantum de Finetti theorem

We have claimed that the bottom $E(N)$ of the spectrum of the Hamiltonian H_N is always described by Hartree's energy in the limit $N \to \infty$. It is of course important to understand what is happening to states as well. This question is much more complicated than it looks like. As we will explain in detail in Section 3, a wave function Ψ_N which has an energy of the order of $E(N)$ is *never* close to a Hartree state $u^{\otimes N}$ in norm, except when $w \equiv 0$. The relationship between Ψ_N and $u^{\otimes N}$ has to be expressed in a different way.

The standard technique to compare Ψ_N with Hartree states, is to use *density matrices*, which are the non-commutative equivalent of marginals in probability theory. For any $1 \leq k \leq N$, we define the *k-particle density matrix* $\gamma_\Psi^{(k)}$ of an N-body state Ψ by its integral kernel

$$\gamma_\Psi^{(k)}(x_1, ..., x_k; y_1, ..., y_k)$$
$$= \int_{\mathbb{R}^d} dz_{k+1} \cdots \int_{\mathbb{R}^d} dz_N \Psi(x_1, ..., x_k, z_{k+1}, ..., z_N)\overline{\Psi(y_1, ..., y_k, z_{k+1}, ..., z_N)}. \quad (2.1)$$

If we take $x_1 = y_1, ..., x_k = y_k$, then we exactly recover the usual marginal of the probability density $|\Psi|^2$. If we apply the Fourier transform, we see that

$$\widehat{\gamma_\Psi}^{(k)}(p_1, ..., p_k; q_1, ..., q_k)$$
$$= \int_{\mathbb{R}^d} dr_{k+1} \cdots \int_{\mathbb{R}^d} dr_N \widehat{\Psi}(p_1, ..., p_k, r_{k+1}, ..., r_N)\overline{\widehat{\Psi}(q_1, ..., q_k, r_{k+1}, ..., r_N)}$$

and therefore $\widehat{\gamma_\Psi}^{(k)}(p_1, ..., p_k; p_1, ..., p_k)$ is exactly the kth marginal of the momentum density of probability $|\widehat{\Psi}|^2$. Therefore, $\gamma_\Psi^{(k)}$ contains an information of our quantum state Ψ both in direct and Fourier space. The integral kernel $\gamma_\Psi^{(k)}$ defines an operator acting on $\mathfrak{H}^k = L^2_s((\mathbb{R}^d)^k)$, also denoted by $\gamma_\Psi^{(k)}$ for simplicity.

Equivalently, we can define $\gamma_\Psi^{(k)}$ by duality:

$$\langle \Psi, A \otimes \mathbb{1}_{\mathfrak{H}^{N-k}} \Psi \rangle_{\mathfrak{H}^N} = {\rm Tr}_{\mathfrak{H}^k}\Big(A\gamma_\Psi^{(k)}\Big)$$

for any bounded operator A on $\mathfrak{H}^k$. This is the same as saying that $\gamma_\Psi^{(k)}$ is the partial trace of the rank-one projection $|\Psi\rangle\langle\Psi|$ with respect to the last $N-k$ variables:

$$\gamma_\Psi^{(k)} = {\rm Tr}_{k+1\to N}|\Psi\rangle\langle\Psi|.$$

The advantage of using density matrices becomes clear when we look at the energy of Ψ, which can be expressed only in terms of the one– and two–particle density matrices:

$$\frac{\langle \Psi, H_N \Psi\rangle}{N} = {\rm Tr}_{\mathfrak{H}}\,(-\Delta + V)\gamma_\Psi^{(1)} + \frac{1}{2}{\rm Tr}_{\mathfrak{H}^2}\Big(w\gamma_\Psi^{(2)}\Big) = \frac{1}{2}{\rm Tr}_{\mathfrak{H}^2}\Big(H_2\gamma_\Psi^{(2)}\Big). \quad (2.2)$$

We see that, thanks to the mean-field factor $1/(N-1)$ in front of the interaction, the expression of the energy is even completely independent of N. The N dependence is hidden in the constraint that $\gamma_\Psi^{(2)}$ must arise from an N-body state Ψ. If we introduce the set of so-called N-representable two-particle density matrices

$$\mathcal{P}_N^{(2)} := \left\{0 \leq \gamma^{(2)} \leq 1 \ : \ \exists \Psi \in \mathfrak{H}^N,\ \gamma^{(2)} = \gamma_\Psi^{(2)}\right\},$$

then we have

$$\frac{E(N)}{N} = \inf_{\gamma^{(2)} \in \mathcal{P}_N^{(2)}} \frac{\mathrm{Tr}_{\mathfrak{H}^2} H_2 \gamma^{(2)}}{2}.$$

It seems now reasonable to believe that the limit of the ground state energy will be determined by the limit of the set $\mathcal{P}_N^{(2)}$ when $N \to \infty$. In infinite dimension, the main issue is of course the choice of an adequate topology to describe the limit. In the confined case the problem will be compact and we can use a strong topology, whereas in the unconfined case, particles can escape to infinity and we have to use a weak topology. Understanding the set $\mathcal{P}_N^{(2)}$ requires to look at all the (similarly defined) sets $\mathcal{P}_N^{(k)}$ for all $k \geq 1$, however.

For any N-particle wave function Ψ_N, we have defined a family $\gamma_{\Psi_N}^{(1)}, ..., \gamma_{\Psi_N}^{(N)}$ of density matrices. For each k, we have $\gamma_{\Psi_N}^{(k)} \geq 0$ and $\mathrm{Tr}_{\mathfrak{H}^k} \gamma_{\Psi_N}^{(k)} = 1$ and thus $(\gamma_{\Psi_N}^{(k)})_N$ is a bounded sequence of operators in the trace class. So by extracting weak limits, we may assume that

$$\gamma_{\Psi_{N_j}}^{(k)} \rightharpoonup_* \gamma^{(k)} \tag{2.3}$$

weakly-$*$ for every $k \geq 1$. This means

$$\lim_{j \to \infty} \mathrm{Tr}\Big(K \gamma_{\Psi_{N_j}}^{(k)}\Big) = \mathrm{Tr}\Big(K \gamma^{(k)}\Big)$$

for every *compact* operator on $\mathfrak{H}^k$.

Let us assume for the moment that that the limits are all strong. Partial traces are continuous for the strong topology of the trace-class, and therefore we obtain in the limit an infinite hierarchy of operators $(\gamma^{(k)})_{k \geq 1}$, which all satisfy $\mathrm{Tr}_{\mathfrak{H}^k} \gamma^{(k)} = 1$ and which is *consistent* in the sense that

$$\mathrm{Tr}_{k+1} \gamma^{(k+1)} = \gamma^{(k)}$$

for every $k \geq 1$.

When we increase the number N of particles, our intuition is that the system can become more and more complicated and this is certainly true at the level of the N particle wave function Ψ_N. But the surprising fact is that the situation goes in the opposite direction if we look at the set $\mathcal{P}_N^{(k)}$ of k particle density matrices for a *fixed* $k \geq 1$. This set actually *decreases* with N and it becomes trivial in the limit: nothing else but convex combinations of Hartree states remain. This property, which is true for any (strongly convergent) sequence of bosonic wave functions Ψ_N, is called the *quantum de Finetti theorem* and it is the main theoretical explanation of the occurrence of Bose-Einstein condensation in the the mean-field regime.

Theorem 2 (Quantum de Finetti). *Let $\mathfrak{K}$ be any separable Hilbert space and denote by $\mathfrak{K}^k := \bigotimes_s^k \mathfrak{K}$ the corresponding bosonic k-particle space. Consider a hierarchy $\{\gamma^{(k)}\}_{k=1}^\infty$ of non-negative self-adjoint operators, where each $\gamma^{(k)}$ acts on $\mathfrak{K}^k$.*

We assume that the hierarchy is consistent *in the sense that*

$$\mathrm{Tr}_{k+1}\gamma^{(k+1)} = \gamma^{(k)} \tag{2.4}$$

for all k. We also assume that $\mathrm{Tr}_{\mathfrak{K}^k}\gamma^{(k)} = 1$ for all $k \geq 1$.

Then there exists a unique Borel probability measure μ on the sphere $S\mathfrak{K}$ of $\mathfrak{K}$, invariant under the group action of S^1, such that

$$\gamma^{(k)} = \int_{S\mathfrak{K}} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\mu(u) \tag{2.5}$$

for all $k \geq 1$.

The result is the quantum equivalent of the famous Hewitt-Savage theorem for classical systems [16, 17, 20, 34, 19, 53]. The latter deals with a hierarchy of symmetric probability measures $\mu^{(k)}$ on Ω^N such that $\mu^{(k)}(A) = \mu^{(k+n)}(A \times \Omega^n)$ for any $k, n \geq 0$ and any measurable set $A \subset \Omega^k$. The quantum de Finetti Theorem 2 was proved in [63, 36] (see [30, 14] for related content).

Another way to state the theorem is that the only extreme points of the set of infinite consistent hierarchies of density matrices, are Hartree states. The existence of the measure μ then follows from classical results in convex analysis by Choquet [13].

In order to deal with unconfined systems, we have to consider the case where the limit in (2.3) is only weak. Then the consistency of the hierarchy is lost, since partial traces are not weakly continuous. Indeed, when passing to the weak limit we find by Fatou's lemma for trace-class operators

$$\begin{aligned}\gamma^{(k)} = \operatorname*{w-lim}_{j\to\infty} \gamma^{(k)}_{\Psi_{N_j}} &= \operatorname*{w-lim}_{j\to\infty} \mathrm{Tr}_{k+1}\gamma^{(k+1)}_{\Psi_{N_j}} \\ &\geq \mathrm{Tr}_{k+1}\operatorname*{w-lim}_{j\to\infty} \gamma^{(k+1)}_{\Psi_{N_j}} = \mathrm{Tr}_{k+1}\gamma^{(k+1)},\end{aligned}$$

where w-lim denotes the weak-$*$ limit in the trace-class. However, the set of all the infinite hierarchies satisfying this inequality has no interesting structure. We have to keep track of the fact that our limiting sequence has been obtained from a sequence of N-particle states. Then the result is the following:

Theorem 3 (**Weak quantum de Finetti** [40])**.** *Let $\mathfrak{K}$ be any separable Hilbert space and denote by $\mathfrak{K}^k := \bigotimes_s^k \mathfrak{K}$ the corresponding bosonic k-particle space. Let Ψ_N be any sequence of normalized wave functions in $\mathfrak{K}^N$ such that*

$$\gamma^{(k)}_{\Psi_N} \rightharpoonup_* \gamma^{(k)}$$

weakly-$*$ in the trace class as $N \to \infty$, for all $k \geq 1$. Then there exists a unique Borel probability measure μ on the unit ball $B\mathfrak{K}$ *of $\mathfrak{K}$, invariant under the group action of S^1, such that*

$$\gamma^{(k)} = \int_{B\mathfrak{K}} d\mu(u)\, |u^{\otimes k}\rangle\langle u^{\otimes k}| \tag{2.6}$$

for all $k \geq 0$.

This version of the quantum de Finetti theorem is called 'weak' because it deals with weak limits, but it is actually stronger than Theorem 2! That we find a de Finetti measure μ living on the unit ball instead of the unit sphere of our ambient Hilbert space $\mathfrak{K}$ is not surprising as we are looking at weak limits. The reader should keep in mind the case of

$$\Psi_N = u_N^{\otimes N} \quad \text{with} \quad u_N \rightharpoonup u$$

for which the limiting measure μ turns out to be the uniform measure on the one-dimensional circle $\{e^{i\theta}u\}_{\theta\in[0,2\pi)}$, which lives in the unit ball $B\mathfrak{K}$ and not in the unit sphere, when $\|u\| < 1$.

Ammari and Nier have recently proved in [4, 5] results that imply Theorem 3. In analogy with semi-classical analysis, they called μ a *Wigner measure*. They deal with an arbitrary sequence of states in Fock space and, therefore, obtain in the limit a measure μ which can live over the whole one-particle Hilbert space $\mathfrak{K}$, instead of the unit ball as in our situation.

In [40], we provided two different proofs of Theorem 3, which are both based on Theorem 2. The first proof uses the *finite-dimensional* de Finetti Theorem and the geometric techniques introduced in [39]. It has the merit of clarifying how the measure μ arises in case the density matrices $\gamma^{(k)}_{\Psi_N}$ do not converge strongly. This is particularly important to understand unconfined quantum systems. The second proof provided in the Appendix of [40] follows arguments similar to those of Hudson and Moody in [36].

That the quantum de Finetti theorem is useful to study the occurrence of Bose-Einstein condensation was known for a long time, see, e.g., [25, 54, 56]. The weak version and its importance to deal with unconfined systems seem to have been discovered only recently.

2.2. Confined case

With the (strong) de Finetti theorem at hand, it is very easy to write the proof of Theorem 1 in the confined case and, even, to describe the behavior of the sequence Ψ_N, in terms of its density matrices. The precise result is the following.

Theorem 4 (**Validity of Hartree and BEC, confined case** [40])**.** *Under the previous assumptions on V and w, and if $V_+ \to \infty$ at ∞, we have*

$$\lim_{N\to\infty} \frac{E(N)}{N} = e_{\rm H}.$$

If (Ψ_N) is any sequence such that $\langle \Psi_N, H_N \Psi_N\rangle = E(N) + o(N)$, then there exists a subsequence and a probability measure μ on the set $\mathcal{M}$ of minimizers of $e_{\rm H}$ (modulo a phase), such that

$$\lim_{j\to\infty} \gamma^{(k)}_{\Psi_{N_j}} = \int_{\mathcal{M}} d\mu(u)\, |u^{\otimes k}\rangle\langle u^{\otimes k}|$$

strongly in the trace-class for any fixed k. In particular, if $e_{\rm H}$ admits a unique minimizer u_0, then there is complete Bose-Einstein condensation (BEC) *on u_0:*

$$\lim_{N\to\infty} \gamma^{(k)}_{\Psi_N} = |u_0^{\otimes k}\rangle\langle u_0^{\otimes k}| \tag{2.7}$$

for all $k \geq 1$.

The minimizers of the Hartree functional arise naturally as limits of the k-particle density matrices of any sequence (Ψ_N) of approximate ground states. Since physically measurable quantities can usually be expressed in terms of the density matrices, we deduce that, if the Hartree functional has a unique minimizer u_0, they will all be given in terms of this minimizer u_0. For instance the density of particles in the system will converge to $|u_0(x)|^2$ and the momentum density to $|\widehat{u_0}(p)|^2$.

Because the effective model is nonlinear, it can happen that the system has some spacial invariance but that the minimizers of the Hartree energy do not (then only the set of minimizers is invariant). In this case there cannot be a unique Hartree minimizer, of course. Our theorem gives convergence to any possible convex combination of these minimizers and it is easy to check that, indeed, any such combination can be reached by an approximate sequence Ψ_N with $\langle \Psi_N, H_N \Psi_N \rangle = E(N) + o(N)$. By adding a small perturbation, one can force the system to converge to one of the Hartree minimizers.

In the rest of this section we give a sketch of the proof of Theorem 4, which is only based on extraction of weak limits in the trace class, Fatou's lemma for operators, and the strong quantum de Finetti theorem.

Sketch of the proof of Theorem 4. First, as before we extract subsequences such as to have $\gamma^{(k)}_{\Psi_{N_j}} \rightharpoonup \gamma^{(k)}$ weakly-$*$. We recall that

$$\frac{\langle \Psi_N, H_N \Psi_N \rangle}{N} = \frac{1}{2} \mathrm{Tr}_{L^2_s((\mathbb{R}^d)^2)} \Big(H_2 \gamma^{(2)}_{\Psi_N} \Big)$$

which tells us that $\mathrm{Tr}_{L^2_s((\mathbb{R}^d)^2)} \Big(H_2 \gamma^{(2)}_{\Psi_N} \Big)$ is bounded. Since when $V \to \infty$ the operator H_2 has a compact resolvent, it is easy to verify that this implies $\gamma^{(2)}_{\Psi_{N_j}} \to \gamma^{(2)}$ *strongly* in the trace-class. The same argument can be used to prove that $\gamma^{(k)}_{\Psi_{N_j}} \to \gamma^{(k)}$ for every fixed $k \geq 1$. By the continuity of the partial trace, we deduce that the sequence is consistent, that is, we have $\mathrm{Tr}_{k+1} \gamma^{(k+1)} = \gamma^{(k)}$ for all k.

Applying now the quantum de Finetti theorem, we obtain a Borel probability measure μ on the unit sphere of $L^2(\mathbb{R}^d)$ such that

$$\gamma^{(2)} = \int_{\|u\|_{L^2(\mathbb{R}^d)}=1} d\mu(u) \; |u^{\otimes 2}\rangle\langle u^{\otimes 2}|. \tag{2.8}$$

By Fatou's lemma for trace-class operators (using that $H_2 \geq 0$ except on a space of finite dimension) and inserting the de Finetti integral representation for the two-particle density matrix, we get

$$\begin{aligned}
\lim_{j\to\infty} \frac{E(N_j)}{N_j} = \lim_{j\to\infty} \frac{1}{2} \mathrm{Tr}_{\mathfrak{H}^2} \Big(H_2 \gamma^{(2)}_{\Psi_{N_j}} \Big) &\geq \frac{1}{2} \mathrm{Tr}_{\mathfrak{H}^2} \Big(H_2 \gamma^{(2)} \Big) \\
&= \int_{\|u\|_{L^2(\mathbb{R}^d)}=1} d\mu(u) \; \frac{\langle u^{\otimes 2}, H_2 u^{\otimes 2} \rangle}{2} \\
&= \int_{\|u\|_{L^2(\mathbb{R}^d)}=1} d\mu(u) \; \mathcal{E}_{\mathrm{H}}(u) \\
&\geq \int_{\|u\|_{L^2(\mathbb{R}^d)}=1} d\mu(u) \; e_{\mathrm{H}} = e_{\mathrm{H}}.
\end{aligned}$$

Note that in the last equality we have used that μ is a probability measure. Since $E(N) \leq e_{\mathrm{H}} N$, we have equality everywhere and, finally, we find that μ has its support on the set of minimizers for e_{H}. $\square$

The previous theorem can be generalized to several situations including rotating Bose gases and positive temperature [40].

2.3. Unconfined case

When the potential V does not tend to infinity at infinity, the system is not fully confined anymore and particles can escape to infinity. In order to describe this situation, we introduce the Hartree minimum energy corresponding to having $(1-\lambda)N$ of the particles at infinity:

$$e_{\rm H}^V(\lambda) := \inf_{\substack{u\in\mathfrak{H}\\ \int_{\mathbb{R}^d}|u|^2=\lambda}} \mathcal{E}_{\rm H}^V(u).$$

As usual, by sending a mass $1-\lambda$ to infinity, it is easy to verify the large inequality

$$e_{\rm H}^V(1) \leq e_{\rm H}^V(\lambda) + e_{\rm H}^0(1-\lambda).$$

Since the particles at infinity will not see the local potential V anymore, we need to emphasize the local potential V in our notation. In particular, $e_{\rm H}^0(1-\lambda)$ is the infimum of the Hartree energy with a mass $1-\lambda$ and with the term involving V removed. The particles however always interact with each other, and the nonlinear term involving w stays the same at infinity when $V=0$.

By using the concentration-compactness method of Lions [51, 52], one can easily prove that the strict binding inequalities

$$e_{\rm H}^V(1) < e_{\rm H}^V(\lambda) + e_{\rm H}^0(1-\lambda), \qquad \forall 0\leq\lambda<1,$$

guarantee the existence of a minimizer for $e_{\rm H}$, as well as the compactness of all the minimizing sequences in $H^1(\mathbb{R}^d)$. If $V\equiv 0$ then the system is translation-invariant, and the strict inequality

$$e_{\rm H}^0(1) < e_{\rm H}^0(\lambda) + e_{\rm H}^0(1-\lambda), \qquad \forall 0<\lambda<1$$

give the existence of at least one minimizer, and the compactness of all the minimizing sequences *up to translations.*

Our theorem concerning the validity of Hartree's theory does not rely on the validity of the binding inequalities. But the statement is stronger when they are satisfied.

Theorem 5 (**Validity of Hartree and BEC, unconfined case** [40]). *Under the previous assumptions on V and w, with $V_+\equiv 0$, we have*

$$\lim_{N\to\infty}\frac{E(N)}{N} = e_{\rm H}.$$

If (Ψ_N) is any sequence such that $\langle\Psi_N, H_N\Psi_N\rangle = E(N)+o(N)$, then there exists a subsequence and a probability measure μ on the unit ball $B\mathfrak{H}=\{u\in\mathfrak{H}\ :\ \|u\|_{L^2(\mathbb{R}^d)}\leq 1\}$, supported on the set

$$\mathcal{M}^V = \left\{u\in B\mathfrak{H}\ :\ \mathcal{E}_{\rm H}^V(u) = e_H^V(\|u\|^2) = e_H^V(1) - e_H^0(1-\|u\|^2)\right\}, \tag{2.9}$$

such that

$$\gamma_{N_j}^{(k)} \rightharpoonup_* \int_{\mathcal{M}^V} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\mu(u) \tag{2.10}$$

weakly-$*$ in the trace-class, for every $k\geq 1$.

If the strict binding inequality

$$e_{\rm H}^V(1) < e_{\rm H}^V(\lambda) + e_{\rm H}^0(1-\lambda) \tag{2.11}$$

is satisfied for all $0 \leq \lambda < 1$*, then* μ *is supported on the unit sphere of* $L^2(\mathbb{R}^d)$ *and the limit* (2.10) *for* $\gamma_{N_j}^{(k)}$ *is strong in the trace-class. In particular, if* $e_{\rm H}^V(1)$ *admits a unique minimizer* u_0*, up to a phase, then there is* complete Bose-Einstein condensation *on it:*

$$\gamma_{N_j}^{(k)} \to |u_0^{\otimes k}\rangle\langle u_0^{\otimes k}| \tag{2.12}$$

strongly in the trace class for any fixed $k \geq 1$*.*

We see that there is not always convergence of the density matrices of Ψ_N to the ones of minimizers of the Hartree energy, simply because these minimizers do not necessarily exist. However, there is always weak convergence if we allow the system to have less particles. The set $\mathcal{M}^V$ contains all the minimizers for $e_{\rm H}$, with a mass which may be less than 1. More precisely, $\mathcal{M}^V$ contains all the functions u for which $\mathcal{E}_{\rm H}^V(u) = e_{\rm H}^V(1)$ and, furthermore, $e_{\rm H}^V(1) = e_{\rm H}^V(\lambda) + e_{\rm H}^0(1-\lambda)$ where $\lambda := \int_{\mathbb{R}^d} |u|^2$.

When the potential w is non-negative (or, even more generally, when $H_2 \geq 0$ in the sense of quadratic forms), then $e^0(\lambda) = 0$ for all $0 \leq \lambda \leq 1$, and the proof of Theorem 5 goes along the same lines as the ones of Theorem 4. One can use the weak de Finetti theorem and Fatou's lemma, since $H_2 \geq 0$.

However, when w has no particular sign, then the particles escaping to infinity can have a non trivial behavior (that is, they can bind), and a much more delicate analysis is needed. The proof of [40] is based on the geometric localization techniques which have been developed by the author in [39], as well as on some ideas of Lieb, Thirring and Yau in [49, 50] in order to deal with the problem at infinity. It would be too long to explain this here and we refer the reader to [40] for details.

3. The next order: Bogoliubov's theory

We have said that the approximate minimizer Ψ_N is not necessarily close to a Hartree state $u^{\otimes N}$ in norm in $\mathfrak{H}^N$. In this section we give the exact behavior of the wave function Ψ_N, under the additional assumption that there is a unique, non degenerate Hartree minimizer. This requires to expand the energy to the next order in N, which we can do for any eigenvalue, not only for the lowest one.

Theorem 6 (Validity of Bogoliubov's theory [42])**.** *We work under the same assumptions on* V *and* w *as before, and assume furthermore that* $e_{\rm H}$ *possesses a unique, non-degenerate minimizer* u_0*, and that*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)^2\, |u_0(x)|^2 |u_0(y)|^2\, dx\, dy < \infty. \tag{3.1}$$

Let us denote by $\lambda_j(H_N)$ *the* j*th min-max level of the operator* H_N*. Then*

$$\lambda_j(H_N) = N\, e_{\rm H} + \lambda_j(\mathbb{H}) + o(1) \tag{3.2}$$

where $\mathbb{H}$ *is the Bogoliubov Hamiltonian on the Fock space*

$$\mathcal{F}_+ := \mathbb{C} \oplus \bigoplus_{n\geq 1} \bigotimes_{\rm sym}^n \mathfrak{H}_+ := \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathfrak{H}_+^n, \qquad \mathfrak{H}^+ = \{u_0\}^\perp,$$

that is, the second quantization of $(1/2)\mathrm{Hess}\ \mathcal{E}_{\rm H}(u_0)$*, the Hessian of the Hartree functional* $\mathcal{E}_{\rm H}$ *at the point* u_0 *on the tangent space* $\mathfrak{H}_+ = \{u_0\}^\perp$*.*

Furthermore, if $\lambda_j(\mathbb{H})$ lies below the bottom of the essential spectrum of $\mathbb{H}$, then so does $\lambda_j(H_N)$ for N large enough and, if $\Psi_N^{(j)}$ is a corresponding sequence of eigenvectors, we have, up to extraction of a subsequence,

$$\lim_{N\to\infty}\left\|\Psi_N^{(j)}-\sum_{k=0}^{N}(u_0)^{\otimes N-k}\otimes_s\varphi_k^{(j)}\right\|_{\mathfrak{H}^N}=0 \tag{3.3}$$

where $\Phi^{(j)}=(\varphi_k^{(j)})_{k\geq 0}\in\mathcal{F}_+$ is a corresponding normalized eigenvector for $\mathbb{H}$:

$$\mathbb{H}\Phi^{(j)}=\lambda_j\Phi^{(j)}.$$

Let us recall what "non-degenerate" means. First, since u_0 minimizes $\mathcal{E}_{\rm H}$, then $|u_0|$ as well and, by uniqueness, we get that u_0 is real and >0. The first order condition reads

$$h\,u_0=0,\qquad\text{with}\quad h=-\Delta+V+|u_0|^2*w-\mu_{\rm H}, \tag{3.4}$$

where $\mu_{\rm H}$ is the Lagrange multiplier associated with the constraint $\|u\|=1$. The second order condition is that the Hessian is non-negative on the tangent plane at u_0. A simple calculation gives

$$\begin{aligned}\frac12\text{Hess }\mathcal{E}_{\rm H}(u_0)(v,v)&=\langle v,hv\rangle+\frac12\int_\Omega\int_\Omega w(x-y)u_0(x)u_0(y)\Big(\overline{v(x)}v(y)+v(x)\overline{v(y)}\\&\qquad\qquad+\overline{v(x)v(y)}+v(x)v(y)\Big)\,dx\,dy\\&=\frac12\left\langle\begin{pmatrix}v\\ \overline{v}\end{pmatrix},\begin{pmatrix}h+K&K\\K&h+K\end{pmatrix}\begin{pmatrix}v\\ \overline{v}\end{pmatrix}\right\rangle_{\mathfrak{H}_+\oplus\mathfrak{H}_+}\end{aligned} \tag{3.5}$$

for all v in the tangent plane $\mathfrak{H}_+:=\{u_0\}^\perp$. Here K is the restriction to $\mathfrak{H}_+$ of the operator with kernel $k(x,y)=w(x-y)u_0(x)\,u_0(y)$. More precisely, the kernel of K can be obtained by projecting the symmetric function k onto $\mathfrak{H}_+\otimes_s\mathfrak{H}_+$. The operator h leaves $\mathfrak{H}_+$ invariant and we do not use any specific notation for its restriction. Now, that the minimizer u_0 is non-degenerate means that there exists a positive constant $\eta_{\rm H}>0$ such that

$$\begin{pmatrix}h+K&K\\K&h+K\end{pmatrix}\geqslant\eta_{\rm H}\qquad\text{on}\quad\mathfrak{H}_+\oplus\mathfrak{H}_+. \tag{3.6}$$

Note that since $u_0>0$, it is necessarily the first eigenfunction of h and we then know that it is non-degenerate. In particular, $h\geq\eta'_{\rm H}>0$ on $\mathfrak{H}_+$. If $K\geq 0$ (for instance when $\widehat{w}\geq 0$), we deduce that

$$\begin{pmatrix}h+K&K\\K&h+K\end{pmatrix}\geq\begin{pmatrix}h&0\\0&h\end{pmatrix}\geq\eta'_{\rm H}.$$

The minimizer u_0 is therefore always non-degenerate when $\widehat{w}\geq 0$ (and it is also always unique). For more general potentials w, we need the non-degeneracy assumption, however.

The theorem says that the exact behavior of the sequence $\Psi_N^{(j)}$ (for instance $\Psi_N^{(1)}$ for the lowest eigenvalue $\lambda_1(H_N)=E(N)$) can be described through a series which involves the functions u_0 and $\varphi_k^{(j)}$ which are *independent of N*. The eigenfunction

$\Psi_N^{(j)}$ is not close to the Hartree state $(u_0)^{\otimes N}$ and we have, indeed,

$$\begin{aligned}\left\|\Psi_N^{(j)}-(u_0)^{\otimes N}\right\|_{\mathfrak{H}^N}^2 &= \left\|(\varphi_0^{(j)}-1)(u_0)^{\otimes N}+\sum_{k=1}^N (u_0)^{\otimes N-k}\otimes_s\varphi_k^{(j)}\right\|_{\mathfrak{H}^N}^2\\ &= |\varphi_0^{(j)}-1|^2+\sum_{k=1}^N\left\|\varphi_k^{(j)}\right\|_{\mathfrak{H}^k}^2\\ &\underset{N\to\infty}{\longrightarrow} |\varphi_0^{(j)}-1|^2+\sum_{k=1}^\infty\left\|\varphi_k^{(j)}\right\|_{\mathfrak{H}^k}^2.\end{aligned}$$

Except in the very exceptional situation where the particles do not interact, $w\equiv 0$, then we have $\varphi_k^{(j)}\neq 0$ for an infinite number of ks. Therefore $\|\Psi_N^{(j)}-(u_0)^{\otimes N}\|\nrightarrow 0$ and all the terms in the series must be taken into account in order to reach convergence.

The term $(u_0)^{\otimes N-k}\otimes_s\varphi_k^{(j)}$ describes a situation where $N-k$ particles are in the condensate in the state u_0, whereas k are excited outside of the condensate, in the state $\varphi_k^{(j)}\in\mathfrak{H}_+^k$. Having a finite number of excited particles furnishes the same energy $Ne_{\rm H}$ to first order, and this is why $\Psi_N^{(j)}$ is a superposition of such states. Note, however, that since $\Phi^{(j)}$ is normalized in the Fock space $\mathcal{F}_+$,

$$|\varphi_0^{(j)}|^2+\sum_{k=1}^\infty\left\|\varphi_k^{(j)}\right\|_{\mathfrak{H}^k}=1,$$

the higher excitations give a small (but N-independent) contribution.

So far we have not defined the Bogoliubov Hamiltonian $\mathbb{H}$ explicitly. Second quantization means that we must replace $v(x)$ by an annihilation operator $a(x)$ and $\overline{v(x)}$ by a creation operator $a^\dagger(x)$ in the formula of the Hessian of $\mathcal{E}_{\rm H}$, with $v\in\mathfrak{H}_+$. We arrive at the following expression

$$\begin{aligned}\mathbb{H}:=\int_\Omega a^*(x)\Big((h+K)\,a\Big)(x)\,dx\\ +\frac12\int_\Omega\int_\Omega K(x,y)\Big(a^*(x)a^*(y)+a(x)a(y)\Big)dx\,dy.\end{aligned}\tag{3.7}$$

For the reader unacquainted to second quantization, we describe this Hamiltonian in a different fashion. The space on which $\mathbb{H}$ acts is the Fock space

$$\mathcal{F}_+:=\mathbb{C}\oplus\mathfrak{H}_+\oplus\underbrace{\Big(\mathfrak{H}_+\otimes_s\mathfrak{H}_+\Big)}_{:=\mathfrak{H}_+^2}\oplus\underbrace{\Big(\mathfrak{H}_+\otimes_s\mathfrak{H}_+\otimes_s\mathfrak{H}_+\Big)}_{:=\mathfrak{H}_+^3}\oplus\cdots$$

which is the direct sum of all the spaces with an arbitrary number of excited particles. This Hilbert space is endowed with its usual Hilbert scalar product. Now, $\mathbb{H}$ is the sum of three terms,

$$\mathbb{H}=\mathbb{H}_1+\mathbb{H}_2+(\mathbb{H}_2)^*$$

where $\mathbb{H}_1$ corresponds to the (particle-conserving) terms with $a^\dagger(x)a(x)$, whereas $\mathbb{H}_2$ corresponds to the other (non particle-conserving) terms involving $a(x)a(y)$ and its adjoint. The operator $\mathbb{H}_1$ is diagonal with respect to the decomposition of $\mathcal{F}_+$ in a direct sum:

$$\mathbb{H}_1 = 0 \oplus \Big(h+K\Big) \oplus \Big((h+K)_1 + (h+K)_2\Big) \oplus \cdots \oplus \underbrace{\left(\sum_{j=1}^{k}(h+K)_j\right)}_{\text{acts on } \mathfrak{H}_+^k} \oplus \cdots.$$

The operator $\mathbb{H}_2$ is off-diagonal and it maps a vector of $\mathfrak{H}_+^k$ into $\mathfrak{H}_+^{k+2}$:

$$\mathbb{H}_2\Big(0 \oplus \cdots \oplus \underbrace{\varphi_k}_{\in\mathfrak{H}_+^k} \oplus 0 \oplus 0 \oplus \cdots\Big) = 0 \oplus \cdots \oplus 0 \oplus 0 \oplus \underbrace{\left(\frac{K \otimes_s \varphi_k}{\sqrt{2}}\right)}_{\in\mathfrak{H}_+^{k+2}} \oplus \cdots.$$

Here $K(x,y)$ is the kernel of the operator K defined above. We see the importance of our assumption (3.1) that $K \ni L^2(\mathbb{R}^d \times \mathbb{R}^d)$, which is necessary to properly define $\mathbb{H}_2$.

Under (3.1) and the non-degeneracy assumption, it may be proved that the Bogoliubov Hamiltonian $\mathbb{H}$ is a bounded below self-adjoint operator on $\mathcal{F}_+$. We refer to [42] for more properties of $\mathbb{H}$, and for some examples.

A result similar to (3.2) has recently been obtained for weakly interacting Bose gases by Seiringer [58] and Grech-Seiringer [31]. They assumed that w is bounded, decays fast enough and has non-negative Fourier transform. In [58] the system is restricted to a box with periodic boundary conditions and in [31] only the confined case is considered. Our method is different from that of [58, 31] and it applies to a larger class of models. Recently, the expansion (3.2) was considered for a system in a box which slowly grows with N in [18], leading to the famous cusp at the origin, for the Bogoliubov spectrum $\sigma(\mathbb{H})$. The latter is the main explanation of the superfluid behavior of such systems, which was originally predicted by Bogoliubov [10].

Since Bogoliubov's work, there have been several attempts to formulate Bogoliubov's theory in a mathematically rigorous way. This was especially successful for completely integrable 1D systems [28, 44, 43, 12, 11, 64, 65], for the ground state energy of one and two-component Bose gases [47, 48, 61], and for the Lee-Huang-Yang formula of dilute gases [23, 29, 68]. See, e.g., [69] for a recent review on the subject and [15] for a discussion of translation-invariant systems.

We will now quickly explain how the Bogoliubov Hamiltonian $\mathbb{H}$ and the Fock space $\mathcal{F}_+$ arise in this theory. The main crucial observation of [42] is that any function Ψ of the bosonic N-particle space $\mathfrak{H}^N$ can be written as

$$\Psi := \varphi_0\, u_0^{\otimes N} + u_0^{\otimes(N-1)} \otimes_s \varphi_1 + u_0^{\otimes(N-2)} \otimes_s \varphi_2 + \cdots + \varphi_N$$

where $\varphi_k \in \mathfrak{H}_+^k$ and u_0 is our Hartree minimizer (but so far it could be any fixed reference function). Saying differently, we have

$$\mathfrak{H}^N = \mathfrak{K}_0^N \oplus \cdots \oplus \mathfrak{K}_N^N$$

where $\mathfrak{K}_0^N = \text{span}(u_0 \otimes \cdots \otimes u_0)$ and

$$\mathfrak{K}_k^N = \underbrace{u_0 \otimes \cdots \otimes u_0}_{N-k} \otimes_s \bigotimes_{\text{sym}}^{k} \mathfrak{H}_+ = u_0^{\otimes(N-k)} \otimes_s \mathfrak{H}_+^k.$$

So, there is a natural isometry

$$\begin{array}{rccl} U_N: & \mathfrak{H}^N & \to & \mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^{N} \mathfrak{H}_+^n \\ & \Psi & \mapsto & \varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_N \end{array}$$

from $\mathfrak{H}^N$ onto the truncated Fock space $\mathcal{F}_+^{\leq N}$, which is itself a subspace of the full Fock space $\mathcal{F}_+$. The unitary U_N is adapted to the description of the fluctuations around the Hartree minimizer u_0. In particular, the Hartree state $(u_0)^{\otimes N}$ is simply mapped onto the vacuum $1 \in \mathbb{C}$ in the Fock space $\mathcal{F}_+$.

After applying the unitary U_N (which does not change the spectrum of H_N), we settle the eigenvalue problem for H_N in the truncated Fock space $\mathcal{F}_+^{\leq N}$. In the limit $N \to \infty$, we obtain this way a problem posed on the Fock space $\mathcal{F}_+$. Now, the Bogoliubov Hamiltonian is simply the weak limit of $H_N - N\, e_{\rm H}$, as stated in the following lemma.

Lemma 1 (Weak limit towards $\mathbb{H}$ [42])**.** *Under the previous assumptions, $U_N(H_N - N\, e_{\rm H})U_N^*$ converges weakly to $\mathbb{H}$, in the sense that*

$$\lim_{N\to\infty} \left\langle \Phi', U_N\Big(H_N - N\, e_{\rm H}\Big)U_N^*\, \Phi\right\rangle_{\mathcal{F}_+} = \langle \Phi', \mathbb{H}\, \Phi\rangle_{\mathcal{F}_+} \tag{3.8}$$

for every fixed Φ, Φ' in the quadratic form domain of $\mathbb{H}$, and where U_N^* is by convention extended by 0 outside of $\mathcal{F}_+^{\leq N}$.

Verifying the above convergence is not difficult, but requires some tedious calculations. These are much simpler to carry out using the formalism of second-quantization. With Lemma 1 it is not hard to prove an upper bound on the eigenvalues of H_N in the limit $N \to \infty$. The proof of the lower bound in [42] is based on a localization procedure in Fock space, inspired of [47].

Conclusion

In this article we have reviewed recent results obtained in [40, 42] concerning the behavior of Bose gases in the mean-field regime.

We have mainly discussed the emergence of Hartree's theory, which is a consequence of the special structure of the set of bosonic density matrices in the limit $N \to \infty$, as described by the quantum de Finetti theorem. For unconfined system, a careful analysis of the lack of compactness of the infinitely many particles is necessary and a new weak de Finetti theorem is then useful to describe the particles which have not escaped to infinity. In all cases, the first eigenvalue of the many-particle Hamiltonian is, to first order in N, given by the *nonlinear* Hartree functional which is obtained by restricting to wave functions of the form $u^{\otimes N}$.

If the Hartree minimizer u_0 is unique and non degenerate, it is possible to expand the energy to the next order in N. The next term in the expansion is described by a *linear* model posed in Fock space, based on the so-called Bogoliubov Hamiltonian $\mathbb{H}$, the second-quantization of the Hessian of the Hartree energy at the minimizer u_0. The eigenvectors of $\mathbb{H}$ give the precise behavior of the N-particle wave function Ψ_N in the limit $N \to \infty$.

References

[1] A. Aftalion, *Vortices in Bose–Einstein Condensates*, vol. 67 of Progress in nonlinear differential equations and their applications, Springer, 2006.

[2] A. Aftalion, X. Blanc, and J. Dalibard, *Vortex patterns in a fast rotating Bose-Einstein condensate*, Phys. Rev. A, 71 (2005), p. 023611.

[3] A. Aftalion, X. Blanc, and F. Nier, *Lowest Landau level functional and Bargmann spaces for Bose-Einstein condensates*, J. Funct. Anal., 241 (2006), pp. 661–702.

[4] Z. Ammari and F. Nier, *Mean field limit for bosons and infinite dimensional phase-space analysis*, Annales Henri Poincaré, 9 (2008), pp. 1503–1574. 10.1007/s00023-008-0393-5.

[5] ——, *Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states*, J. Math. Pures Appl., 95 (2011), pp. 585–626.

[6] V. Bach, *Ionization energies of bosonic Coulomb systems*, Lett. Math. Phys., 21 (1991), pp. 139–149.

[7] V. Bach, R. Lewis, E. H. Lieb, and H. Siedentop, *On the number of bound states of a bosonic N-particle Coulomb system*, Math. Z., 214 (1993), pp. 441–459.

[8] C. Bardos, F. Golse, and N. J. Mauser, *Weak coupling limit of the N-particle Schrödinger equation*, Methods Appl. Anal., 7 (2000), pp. 275–293. Cathleen Morawetz: a great mathematician.

[9] R. Benguria and E. H. Lieb, *Proof of the Stability of Highly Negative Ions in the Absence of the Pauli Principle*, Physical Review Letters, 50 (1983), pp. 1771–1774.

[10] N. N. Bogoliubov, *On the theory of superfluidity*, J. Phys. (USSR), 11 (1947), p. 23.

[11] F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Mathematical Phys., 12 (1971), pp. 419–436.

[12] F. Calogero and C. Marchioro, *Lower bounds to the ground-state energy of systems containing identical particles*, J. Mathematical Phys., 10 (1969), pp. 562–569.

[13] G. Choquet, *Lectures on analysis. Vol 2. Representation theory*, Mathematics lecture note series, W.A. Benjamin, Inc, New York, 1969.

[14] M. Christandl, R. König, G. Mitchison, and R. Renner, *One-and-a-half quantum de Finetti theorems*, Comm. Math. Phys., 273 (2007), pp. 473–498.

[15] H. D. Cornean, J. Derezinski, and P. Zin, *On the infimum of the energy-momentum spectrum of a homogeneous bose gas*, J. Math. Phys., 50 (2009), p. 062103.

[16] B. De Finetti, *Funzione caratteristica di un fenomeno aleatorio.* Atti della R. Accademia Nazionale dei Lincei, 1931. Ser. 6, Memorie, Classe di Scienze Fisiche, Matematiche e Naturali.

[17] B. de Finetti, *La prévision : ses lois logiques, ses sources subjectives*, Ann. Inst. H. Poincaré, 7 (1937), pp. 1–68.

[18] J. Dereziński and M. Napiórkowski, *Excitation spectrum of interacting bosons in the mean-field infinite-volume limit*, Annales Henri Poincaré, (2014), pp. 1–31.

[19] P. Diaconis and D. Freedman, *Finite exchangeable sequences*, Ann. Probab., 8 (1980), pp. 745–764.

[20] E. B. Dynkin, *Classes of equivalent random quantities*, Uspehi Matem. Nauk (N.S.), 8 (1953), pp. 125–130.

[21] A. Elgart, L. Erdős, B. Schlein, and H.-T. Yau, *Gross-Pitaevskii equation as the mean field limit of weakly coupled bosons*, Arch. Ration. Mech. Anal., 179 (2006), pp. 265–283.

[22] A. Elgart and B. Schlein, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math., 60 (2007), pp. 500–545.

[23] L. Erdös, B. Schlein, and H.-T. Yau, *Ground-state energy of a low-density Bose gas: A second-order upper bound*, Phys. Rev. A, 78 (2008), p. 053627.

[24] L. Erdős, B. Schlein, and H.-T. Yau, *Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential*, J. Amer. Math. Soc., 22 (2009), pp. 1099–1156.

[25] M. Fannes, H. Spohn, and A. Verbeure, *Equilibrium states for mean field models*, J. Math. Phys., 21 (1980), pp. 355–358.

[26] J. Fröhlich, A. Knowles, and S. Schwarz, *On the mean-field limit of bosons with Coulomb two-body interaction*, Commun. Math. Phys., 288 (2009), pp. 1023–1059.

[27] J. Ginibre and G. Velo, *The classical field limit of scattering theory for non-relativistic many-boson systems. I*, Commun. Math. Phys., 66 (1979), pp. 37–76.

[28] M. Girardeau, *Relationship between systems of impenetrable bosons and fermions in one dimension*, J. Mathematical Phys., 1 (1960), pp. 516–523.

[29] A. Giuliani and R. Seiringer, *The ground state energy of the weakly interacting Bose gas at high density*, J. Stat. Phys., 135 (2009), pp. 915–934.

[30] A. D. Gottlieb, *Examples of bosonic de Finetti states over finite dimensional Hilbert spaces*, J. Stat. Phys., 121 (2005), pp. 497–509.

[31] P. Grech and R. Seiringer, *The excitation spectrum for weakly interacting bosons in a trap*, Comm. Math. Phys., 322 (2013), pp. 559–591.

[32] D. R. Hartree, *The wave-mechanics of an atom with a non-Coulomb central field. Part I. Theory and methods.*, Proc. Camb. Phil. Soc., 24 (1928), pp. 89–312.

[33] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Comm. Math. Phys., 35 (1974), pp. 265–277.

[34] E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc., 80 (1955), pp. 470–501.

[35] M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, *Schrödinger inequalities and asymptotic behavior of the electron density of atoms and molecules*, Phys. Rev. A, 16 (1977), pp. 1782–1785.

[36] R. L. Hudson and G. R. Moody, *Locally normal symmetric states and an analogue of de Finetti's theorem*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 33 (1975/76), pp. 343–351.

[37] M. K.-H. Kiessling, *The Hartree limit of Born's ensemble for the ground state of a bosonic atom or ion*, J. Math. Phys., 53 (2012), p. 095223.

[38] A. Knowles and P. Pickl, *Mean-field dynamics: singular potentials and rate of convergence*, Commun. Math. Phys., 298 (2010), pp. 101–138.

[39] M. Lewin, *Geometric methods for nonlinear many-body quantum systems*, J. Funct. Anal., 260 (2011), pp. 3535–3595.

[40] M. Lewin, P. T. Nam, and N. Rougerie, *Derivation of Hartree's theory for generic mean-field Bose gases*, Adv. Math., 254 (2014), pp. 570–621.

[41] M. Lewin, P. T. Nam, and B. Schlein, *Fluctuations around Hartree states in the mean-field regime*. arXiv eprint, 2013.

[42] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, *Bogoliubov spectrum of interacting Bose gases*, Comm. Pure Appl. Math., in press (2013).

[43] E. H. Lieb, *Exact analysis of an interacting Bose gas. II. The excitation spectrum*, Phys. Rev. (2), 130 (1963), pp. 1616–1624.

[44] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Phys. Rev. (2), 130 (1963), pp. 1605–1616.

[45] E. H. Lieb and R. Seiringer, *Derivation of the Gross-Pitaevskii equation for rotating Bose gases*, Commun. Math. Phys., 264 (2006), pp. 505–537.

[46] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The mathematics of the Bose gas and its condensation*, Oberwolfach Seminars, Birkhäuser, 2005.

[47] E. H. Lieb and J. P. Solovej, *Ground state energy of the one-component charged Bose gas*, Commun. Math. Phys., 217 (2001), pp. 127–163.

[48] E. H. Lieb and J. P. Solovej, *Ground state energy of the two-component charged Bose gas.*, Commun. Math. Phys., 252 (2004), pp. 485–534.

[49] E. H. Lieb and W. E. Thirring, *Gravitational collapse in quantum mechanics with relativistic kinetic energy*, Ann. Physics, 155 (1984), pp. 494–512.

[50] E. H. Lieb and H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), pp. 147–174.

[51] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–149.

[52] ——, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 223–283.

[53] P.-L. Lions, *Mean-field games and applications*. Lectures at the Collège de France, unpublished, Nov 2007.

[54] D. Petz, G. A. Raggio, and A. Verbeure, *Asymptotics of Varadhan-type and the Gibbs variational principle*, Comm. Math. Phys., 121 (1989), pp. 271–282.

[55] P. Pickl, *A simple derivation of mean-field limits for quantum systems*, Lett. Math. Phys., 97 (2011), pp. 151–164.

[56] G. A. Raggio and R. F. Werner, *Quantum statistical mechanics of general mean field systems*, Helv. Phys. Acta, 62 (1989), pp. 980–1003.

[57] I. Rodnianski and B. Schlein, *Quantum fluctuations and rate of convergence towards mean field dynamics*, Commun. Math. Phys., 291 (2009), pp. 31–61.

[58] R. Seiringer, *The excitation spectrum for weakly interacting bosons*, Commun. Math. Phys., 306 (2011), pp. 565–578.

[59] R. Seiringer, J. Yngvason, and V. A. Zagrebnov, *Disordered Bose-Einstein condensates with interaction in one dimension*, J. Stat. Mech., 2012 (2012), p. P11007.

[60] J. P. Solovej, *Asymptotics for bosonic atoms*, Lett. Math. Phys., 20 (1990), pp. 165–172.

[61] J. P. Solovej, *Upper bounds to the ground state energies of the one- and two-component charged Bose gases*, Commun. Math. Phys., 266 (2006), pp. 797–818.

[62] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Modern Phys., 52 (1980), pp. 569–615.

[63] E. Størmer, *Symmetric states of infinite tensor products of C^*-algebras*, J. Functional Analysis, 3 (1969), pp. 48–68.

[64] B. Sutherland, *Quantum Many-Body Problem in One Dimension: Ground State*, J. Mathematical Phys., 12 (1971), pp. 246–250.

[65] ——, *Quantum Many-Body Problem in One Dimension: Thermodynamics*, J. Mathematical Phys., 12 (1971), pp. 251–256.

[66] M. van den Berg, J. T. Lewis, and J. V. Pulé, *The large deviation principle and some models of an interacting boson gas*, Comm. Math. Phys., 118 (1988), pp. 61–85.

[67] R. F. Werner, *Large deviations and mean-field quantum systems*, in Quantum probability & related topics, QP-PQ, VII, World Sci. Publ., River Edge, NJ, 1992, pp. 349–381.

[68] H.-T. Yau and J. Yin, *The second order upper bound for the ground energy of a Bose gas*, J. Stat. Phys., 136 (2009), pp. 453–503.

[69] J. Yngvason, *The interacting Bose gas: A continuing challenge*, Phys. Particles Nuclei, 41 (2010), pp. 880–884.

CNRS & Université de Cergy-Pontoise (UMR 8088)
95000 Cergy-Pontoise, France.
`mathieu.lewin@math.cnrs.fr`