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Robustifying multiple-set linear canonical analysis with S-estimator

Robustification de l'analyse canonique linéaire généralisée avec un S-estimateur

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Abstract. In this paper, we consider a robust version of multiple-set linear canonical analysis obtained by using a S-estimator of covariance operator. The related influence functions are derived. Asymptotic properties of this robust method are obtained and a robust test for mutual non-correlation is introduced.

Résumé. Dans cet article, nous considérons une version robuste de l'analyse canonique linéaire généralisée obtenue en utilisant un S-estimateur de l'opérateur de covariance. Les fonctions d'influence correspondantes sont déterminées. Les propriétés asymptotiques de cette méthode robuste sont obtenues, et un test robuste de non-corrélation mutuelle est introduit.

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L'analyse canonique linéaire généralisée (ACLG) de variables aléatoires X_1, \dots, X_K , avec $K \geq 2$, à valeurs dans des espaces Euclidiens $\mathcal{X}_1, \dots, \mathcal{X}_K$ permet d'analyser les liaisons entre ces variables; elle consiste en la recherche d'une solution au problème de maximisation (1) obtenue par à partir de l'analyse spectrale d'un opérateur s'écrivant sous la forme $T = \Phi^{-1/2} \Psi \Phi^{-1/2}$, où Φ et Ψ sont des opérateurs définis à partir de l'opérateur de covariance V de la variable aléatoire $X = (X_1, \dots, X_K)$ à valeurs dans $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K$. Cet opérateur est généralement estimé en remplaçant V par l'opérateur de covariance empirique qui est, malheureusement, un estimateur non robuste. Dans cet article, nous introduisons une estimation robuste de l'ACLG, obtenue en remplaçant V par un S-estimateur \tilde{V}_n , ce qui conduit à son estimation à partir de l'analyse spectrale de l'opérateur $\tilde{T}_n = \tilde{\Phi}_n^{-1/2} \tilde{\Psi}_n \tilde{\Phi}_n^{-1/2}$ obtenu à partir de \tilde{V}_n de la même façon que T l'est

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à partir de V . On obtient alors les fonctions d'influence des fonctionnelles correspondant à \tilde{T}_n et aux coefficients et directions canoniques dans les théorèmes suivants où λ , λ_j et H sont définis en (2), (4) et (3) respectivement.

Théorème 1. *On suppose vérifiées les hypothèses (\mathcal{A}_1) à (\mathcal{A}_4) . Alors :*

$$\text{IF}(x; T_s, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \lambda(x, V).$$

Théorème 2. *On suppose vérifiées les hypothèses (\mathcal{A}_1) à (\mathcal{A}_4) . Alors pour tout $j \in \{1, \dots, q\}$, on a :*

$$(i) \quad \text{IF}(x; \rho_{s,j}, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \sum_{k=1}^K \sum_{\substack{\ell=1 \\ \ell \neq k}}^K \langle \beta_k^{(j)}, x_k \rangle_k \langle x_\ell - V_{\ell k} x_k, \beta_\ell^{(j)} \rangle_\ell.$$

(ii) *Si $\rho_1 > \dots > \rho_q$, alors*

$$\text{IF}(x; \alpha_s^{(j)}, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \lambda_j(x, V) - H(\xi, \psi, V, x) \beta^{(j)}.$$

On vérifie facilement que ces fonctions d'influence sont bornées, ce qui établit le caractère robuste de l'estimateur proposé. La normalité asymptotique de \tilde{T}_n est ensuite obtenue :

Théorème 3. *On suppose que les hypothèses (\mathcal{A}_1) à (\mathcal{A}_4) sont vérifiées et que $\mathbb{E}(\|X\|_{\mathcal{X}}^4) < +\infty$. Alors, $\sqrt{n}(\tilde{T}_n - T)$ converge en loi, lorsque $n \rightarrow +\infty$, vers une variable aléatoire U_s ayant une loi normale dans $\mathcal{L}(\mathcal{X})$ centrée et d'opérateur de covariance égale à celle de la variable aléatoire Z_s donnée en (5).*

Ce résultat permet d'introduire un test robuste de non-correlation mutuelle, dont l'hypothèse nulle \mathcal{H}_0 est la nullité des opérateurs de covariance croisés. On prend comme statistique de test la variable aléatoire $\tilde{S}_n = \sum_{k=2}^K \sum_{\ell=1}^{k-1} \text{tr}(\pi_{k\ell}(\tilde{T}_n) \pi_{k\ell}(\tilde{T}_n)^*)$, où $\pi_{k\ell}$ est l'opérateur $A \mapsto \tau_k A \tau_\ell^*$.

Théorème 4. *On suppose que les hypothèses (\mathcal{A}_1) à (\mathcal{A}_4) sont vérifiées et que $\mathbb{E}(\|X\|_{\mathcal{X}}^4) < +\infty$. Alors, sous \mathcal{H}_0 , $(\kappa_0)^{-1} n \tilde{S}_n$ converge en loi, lorsque $n \rightarrow +\infty$, vers χ_d^2 , où $d = \sum_{k=1}^K \sum_{l=1}^{k-1} p_k p_l$ avec $p_k = \dim(\mathcal{X}_k)$.*

1. Introduction

Let (Ω, \mathcal{A}, P) be a probability space, and K an integer such that $K \geq 2$. Considering random variables X_1, \dots, X_K , with values in Euclidean vector spaces $\mathcal{X}_1, \dots, \mathcal{X}_K$ respectively, we assume that for any $k \in \{1, \dots, K\}$, one has $\mathbb{E}(X_k) = 0$ and $\mathbb{E}(\|X_k\|_k^2) < +\infty$, where $\|\cdot\|_k$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_k$ of \mathcal{X}_k . The multiple-set linear canonical analysis (MSLCA) of $X := (X_1, \dots, X_K)$ is a statistical method that allows us to analyse the relationships among these variables. It has been known for many years (e.g., [6]) and has been extensively studied (e.g., [5, 8, 11, 12, 14]). Formally, MSLCA is the search of a sequence $(\alpha^{(j)})_{1 \leq j \leq q}$ of vectors of $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K$, where $q = \dim(\mathcal{X})$, satisfying:

$$\alpha^{(j)} = \underset{\alpha \in \mathcal{C}_j}{\text{argmax}} \mathbb{E}(\langle \alpha, X \rangle_{\mathcal{X}}^2), \quad (1)$$

where $\mathcal{C}_1 = \{\alpha \in \mathcal{X} / \sum_{k=1}^K \text{var}(\langle \alpha_k, X_k \rangle_k) = 1\}$ and, for $j \geq 2$:

$$\mathcal{C}_j = \left\{ \alpha \in \mathcal{C}_1 / \sum_{k=1}^K \text{cov}(\langle \alpha_k^{(r)}, X_k \rangle_k, \langle \alpha_k, X_k \rangle_k) = 0, \forall r \in \{1, \dots, j-1\} \right\}.$$

A solution of the above maximization problem is obtained from the spectral analysis of a given operator T . For $(k, \ell) \in \{1, \dots, K\}^2$, considering the covariance operators $V_{k\ell} = \mathbb{E}(X_\ell \otimes X_k) = V_{\ell k}^*$ and $V_k := V_{kk}$, where \otimes denotes the tensor product such that $x \otimes y$ is the linear map : $h \mapsto \langle x, h \rangle y$, denoting by τ_k the canonical projection $\tau_k : \alpha = (\alpha_1, \dots, \alpha_K) \in \mathcal{X} \mapsto \alpha_k \in \mathcal{X}_k$ and assuming that each V_k is invertible, we have $T = \Phi^{-1/2} \Psi \Phi^{-1/2}$, where $\Phi = \sum_{k=1}^K \tau_k^* V_k \tau_k$

and $\Psi = \sum_{k=1}^K \sum_{\ell=1, \ell \neq k}^K \tau_k^* V_{k\ell} \tau_\ell$. If $\{\beta^{(1)}, \dots, \beta^{(q)}\}$ is an orthonormal basis of \mathcal{X} such that $\beta^{(j)}$ is an eigenvector of T associated with the j -th largest eigenvalue ρ_j , then we obtain a solution of (1) by taking $\alpha^{(j)} = \Phi^{-1/2} \beta^{(j)}$. Classical estimation of MSLCA is based on empirical covariance operators that are known to be very sensitive to outliers. This makes this method a non robust one and highlights the interest of providing a robust estimation of MSLCA as it was done for other multivariate statistical methods including principal component analysis, discriminant analysis, and canonical correlation analysis (cf. [1–3, 13]). A natural way for doing that is to replace the covariance operator of X by a robust estimator. Among such robust estimators, the S-estimator has been extensively studied (cf. [4, 9, 10]) and is known to have good robustness and efficiency properties. In this paper, we propose a robust version of MSLCA based on S-estimator of the covariance operator. This estimation procedure is introduced in Section 2. The related influence functions are derived in Section 3, and Section 4 is devoted to asymptotic properties and to robust test for mutual non-correlation.

2. Estimation of MSLCA based on S-estimator

We assume that the following condition holds:

- (A₁) X has an elliptical contoured distribution with density $f_X(x) = (\det(V))^{-1/2} h(\langle x, V^{-1}x \rangle_{\mathcal{X}})$, where $h : [0, +\infty[\rightarrow [0, +\infty[$ is a function having a strictly negative derivative h' .

Let $\{X^{(1)}, \dots, X^{(n)}\}$ be an i.i.d. sample of X , we consider a fixed real b_0 and a function $\xi : \mathbb{R} \rightarrow \mathbb{R}$. We denote by $\mathcal{P}(\mathcal{X})$ the set of positive definite symmetric operators from \mathcal{X} to itself. The S-estimators $\tilde{\mu}_n$ and \tilde{V}_n of the mean and the covariance operator of X respectively are given by the pair $(\tilde{\mu}_n, \tilde{V}_n)$ that minimizes the determinant $\det(G)$ over all $(\mu, G) \in \mathcal{X} \times \mathcal{P}(\mathcal{X})$ that satisfy $\frac{1}{n} \sum_{i=1}^n \xi(\|G^{-1/2}(X^{(i)} - \mu)\|_{\mathcal{X}}) \leq b_0$. It is well known that these estimators are robust and have high breakdown points (see, e.g., [4]). From this, we can introduce an estimator of MSLCA which is expected to be also robust. Indeed, putting $\tilde{\Phi}_n = \sum_{k=1}^K \tau_k^* \tilde{V}_{k.n} \tau_k$ and $\tilde{\Psi}_n = \sum_{k=1}^K \sum_{\ell=1, \ell \neq k}^K \tau_k^* \tilde{V}_{k\ell.n} \tau_\ell$, where $\tilde{V}_{k.n} = \tau_k \tilde{V}_n \tau_k^*$ and $\tilde{V}_{k\ell.n} = \tau_k \tilde{V}_n \tau_\ell^*$, we estimate T by $\tilde{T}_n = \tilde{\Phi}_n^{-1/2} \tilde{\Psi}_n \tilde{\Phi}_n^{-1/2}$. Considering the eigenvalues $\tilde{\rho}_{1.n} \geq \tilde{\rho}_{2.n} \geq \dots \geq \tilde{\rho}_{q.n}$ of \tilde{T}_n and $\{\tilde{\beta}_n^{(1)}, \dots, \tilde{\beta}_n^{(q)}\}$ an orthonormal basis of \mathcal{X} such that $\tilde{\beta}_n^{(j)}$ is an eigenvector of \tilde{T}_n associated with $\tilde{\rho}_{j.n}$, we estimate ρ_j by $\tilde{\rho}_{j.n}$, $\beta^{(j)}$ by $\tilde{\beta}_n^{(j)}$ and $\alpha^{(j)}$ by $\tilde{\alpha}_n^{(j)} = \tilde{\Phi}_n^{-1/2} \tilde{\beta}_n^{(j)}$.

3. Influence functions

In order to study the effect of a small amount of contamination at a given point on MSLCA it is important to use influence function, as it is usually done in robustness literature (see [7]). More specifically, we have to derive expressions of the influence functions related to the functionals that give T , ρ_j and $\alpha^{(j)}$ (for $1 \leq j \leq q$) at the distribution \mathbb{P}_X of X . Recall that the influence function of a functional S at \mathbb{P} is defined as

$$\text{IF}(x; S, \mathbb{P}) = \lim_{\varepsilon \downarrow 0} \frac{S((1-\varepsilon)\mathbb{P} + \varepsilon\delta_x) - S(\mathbb{P})}{\varepsilon},$$

where δ_x is the Dirac measure putting all its mass on x . In order to derive the influence functions related to the above estimator of MSLCA, we have to specify the functional that corresponds to it. We impose the following properties on the loss function ξ :

- (A₂) ξ is symmetric, has a continuous derivative ψ and is such that $\xi(0) = 0$;
(A₃) there exists $c_0 > 0$ such that ξ is strictly increasing on $[0, c_0]$ and constant on $[c_0, +\infty[$;
(A₄) the function $t \mapsto \psi(t)t^{-1}$ is continuous and bounded.

For example, the function $\xi(t) = \frac{c^2}{6} \left(1 - (1 - \frac{t^2}{c^2})^3\right) \mathbb{1}_{[-c,c]}(t) + \frac{c^2}{6} \mathbb{1}_{\mathbb{R} \setminus [-c,c]}(t)$, where $c > 0$, satisfies the above conditions. Its derivative is the Tukey's biweight function $\psi(t) = \left(1 - \frac{t^2}{c^2}\right)^2 \mathbb{1}_{[-c,c]}(t)$. The functional \mathbb{V}_s related to the aforementioned S-estimator of V is defined in [9] (see also [10]); it is such that $\mathbb{V}_s(\mathbb{P})$ is the solution to the problem of minimizing the determinant $\det(G)$ over all $\mu \in \mathcal{X}$ and $G \in \mathcal{P}(\mathcal{X})$ that satisfy

$$\int_{\mathcal{X}} \xi(\|G^{-1/2}(x - \mu)\|_{\mathcal{X}}) d\mathbb{P}(x) \leq b_0.$$

It is known that at elliptical distribution $\mathbb{V}_s(\mathbb{P}_X) = V$ (see [10, p. 222]). Therefore, the functional \mathbb{T}_s defined as $\mathbb{T}_s(\mathbb{P}) = f(\mathbb{V}_s(\mathbb{P}))^{-1/2} g(\mathbb{V}_s(\mathbb{P})) f(\mathbb{V}_s(\mathbb{P}))^{-1/2}$, where $f(A) = \sum_{k=1}^K \tau_k^* \tau_k A \tau_k^* \tau_k$ and $g(A) = \sum_{k=1}^K \sum_{\ell=1, \ell \neq k}^K \tau_k^* \tau_k A \tau_\ell^* \tau_\ell$, is such that $\mathbb{T}_s(\mathbb{P}_X) = T$. Putting $T_s = \mathbb{T}_s(\mathbb{P}_X)$,

$$\lambda(x, V) = \sum_{k=1}^K \sum_{\substack{\ell=1 \\ \ell \neq k}}^K -\frac{1}{2} \tau_k^*(x_k \otimes x_k) V_{k\ell} \tau_\ell - \frac{1}{2} \tau_\ell^* V_{\ell k} (x_k \otimes x_k) \tau_k + \tau_k^*(x_\ell \otimes x_k) \tau_\ell \quad (2)$$

$\gamma_1 = \frac{2\pi^{q/2}}{\Gamma(q/2)(q+2)} \int_0^{+\infty} (\psi'(r)r^2 + (q+1)\psi(r)r) r^{q-1} h(r^2) dr$ and $\gamma_2 = \frac{2\pi^{q/2}}{\Gamma(q/2)} \int_0^{+\infty} \psi(r)r^q h(r^2) dr$, Γ being the usual gamma function, we have:

Theorem 1. *We assume (\mathcal{A}_1) to (\mathcal{A}_4) . Then*

$$\text{IF}(x; T_s, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \lambda(x, V).$$

From the properties of ψ , it is easily seen that $\text{IF}(x; T_s, \mathbb{P}_X)$ equals 0 if $\|V^{-1/2}x\|_{\mathcal{X}} > c_0$. Otherwise, we have $\|x\|_{\mathcal{X}} \leq c_0 \|V^{1/2}\|_{\infty}$, where $\|\cdot\|_{\infty}$ denotes the usual operators norm defined by $\|A\|_{\infty} = \sup_{x \neq 0} (\|Ax\| / \|x\|)$. Then, it is easy to check the inequality

$$\sup_{x \in \mathcal{X}} \|\text{IF}(x; T_s, \mathbb{P}_X)\|_{\infty} \leq \sup_{t \in \mathbb{R}_+^*} \left(\frac{\psi(t)}{t} \right) \frac{Kq}{|\gamma_1|} (K-1) (\|V\|_{\infty} + 1) c_0^2 \|V^{1/2}\|_{\infty}^2$$

that shows that the influence function is bounded and, therefore, that the estimation procedure is robust. Now, we give the influence functions related to the canonical coefficients and the canonical directions obtained from the robust MSLCA introduced above. For $j \in \{1, \dots, q\}$, denoting by $\mathbb{R}_{s,j}$ (resp. $\mathbb{B}_{s,j}$; resp. $\mathbb{A}_{s,j}$) the functional such that $\mathbb{R}_{s,j}(\mathbb{P})$ is the j -th largest eigenvalue of $\mathbb{T}_s(\mathbb{P})$ (resp. the associated eigenvector; resp. $\mathbb{A}_{s,j}(\mathbb{P}) = f(\mathbb{V}_s(\mathbb{P}))^{-1/2} \mathbb{B}_{s,j}(\mathbb{P})$), we put $\rho_{s,j} = \mathbb{R}_{s,j}(\mathbb{P}_X)$, $\beta_s^{(j)} = \mathbb{B}_j(\mathbb{P}_X)$ and $\alpha_s^{(j)} = \mathbb{A}_{s,j}(\mathbb{P}_X)$. Putting

$$H(\xi, \psi, V, x) = \frac{2}{\gamma_2} (\xi(\|V^{-1/2}x\|_{\mathcal{X}}) - b_0) \mathbb{I} + \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}} \left(\frac{1}{\|V^{-1/2}x\|_{\mathcal{X}}^2} - \frac{1}{q} \right) \mathbb{I} \quad (3)$$

and

$$\begin{aligned} \lambda_j(x, V) &= \sum_{k=1}^K \sum_{\substack{\ell=1 \\ \ell \neq k}}^K \sum_{m=1}^q \frac{1}{\rho_j - \rho_m} \left(\langle \beta_k^{(m)}, x_k \rangle_k \langle x_\ell, \beta_\ell^{(j)} \rangle_\ell - \frac{1}{2} \langle \beta_k^{(m)}, x_k \rangle_k \langle x_k, V_{k\ell} \beta_\ell^{(j)} \rangle_k \right. \\ &\quad \left. - \frac{1}{2} \langle x_k, V_{k\ell} \beta_\ell^{(m)} \rangle_\ell \langle x_k, \beta_k^{(j)} \rangle_k \right) \beta^{(m)} \quad (4) \\ &\quad - \frac{1}{2} \left(\sum_{k=1}^K \left[\tau_k^*(x_k \otimes x_k) \tau_k + \langle \beta_k^{(j)}, x_k \rangle_k^2 \mathbb{I} \right] - 2\mathbb{I} \right) \beta^{(j)}, \end{aligned}$$

where \mathbb{I} denotes the identity operator of \mathcal{X} , we have:

Theorem 2. We assume (\mathcal{A}_1) to (\mathcal{A}_4) . Then, for any $j \in \{1, \dots, q\}$, we have:

$$(i) \text{ IF}(x; \rho_{s,j}, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \sum_{k=1}^K \sum_{\substack{\ell=1 \\ \ell \neq k}}^K \langle \beta_k^{(j)}, x_k \rangle_k \langle x_\ell - V_{\ell k} x_k, \beta_\ell^{(j)} \rangle_\ell.$$

(ii) If $\rho_1 > \dots > \rho_q$, then

$$\text{IF}(x; \alpha_s^{(j)}, \mathbb{P}_X) = \frac{q}{\gamma_1} \psi(\|V^{-1/2}x\|_{\mathcal{X}}) \|V^{-1/2}x\|_{\mathcal{X}}^{-1} \lambda_j(x, V) - H(\xi, \psi, V, x) \beta^{(j)}.$$

4. Asymptotic distributions

We first establish asymptotic normality for \tilde{T}_n . Putting $\beta_3 = \frac{2\pi^{q/2}}{\Gamma(q/2)} \int_0^{+\infty} \frac{4}{q+2} \psi(r) r^{q+2} h'(r^2) dr$, we have:

Theorem 3. We assume (\mathcal{A}_1) to (\mathcal{A}_4) and that $\mathbb{E}(\|X\|_{\mathcal{X}}^4) < +\infty$. Then, $\sqrt{n}(\tilde{T}_n - T)$ converges in distribution, as $n \rightarrow +\infty$, to a random variable U_s having a normal distribution in $\mathcal{L}(\mathcal{X})$, with mean 0 and covariance operator equal to that of the random operator

$$Z_s = -2q\beta_3^{-1} \psi(\|V^{-1/2}X\|_{\mathcal{X}}) \|V^{-1/2}X\|_{\mathcal{X}}^{-1} \lambda(X, V). \quad (5)$$

This result allows us to consider a robust test for mutual non-correlation, that is the test for the hypothesis $\mathcal{H}_0 : \forall (k, \ell) \in \{1, \dots, K\}^2, k \neq \ell, V_{k\ell} = 0$ against the alternative $\mathcal{H}_1 : \exists (k, \ell) \in \{1, \dots, K\}^2, k \neq \ell, V_{k\ell} \neq 0$. We take as test statistic the random variable $\tilde{S}_n = \sum_{k=2}^K \sum_{\ell=1}^{k-1} \text{tr}(\pi_{k\ell}(\tilde{T}_n)\pi_{k\ell}(\tilde{T}_n)^*)$, where $\pi_{k\ell}$ is the operator $A \mapsto \tau_k A \tau_\ell^*$. Then, putting

$$\kappa_0 = \frac{-2\beta_3^{-1}}{(q+1)} \mathbb{E}\left(\psi(\|X\|_{\mathcal{X}}) \|X\|_{\mathcal{X}}^3\right), \quad (6)$$

we have:

Theorem 4. We assume (\mathcal{A}_1) to (\mathcal{A}_4) and that $\mathbb{E}(\|X\|_{\mathcal{X}}^4) < +\infty$. Then, under \mathcal{H}_0 , the sequence $(\kappa_0)^{-1} n \tilde{S}_n$ converges in distribution, as $n \rightarrow +\infty$, to χ_d^2 , where $d = \sum_{k=1}^K \sum_{l=1}^{k-1} p_k p_l$ with $p_k = \dim(\mathcal{X}_k)$.

In practice, κ_0 is replaced by a consistent estimator; for example by

$$\hat{\kappa}_0 = \frac{-2\beta_3^{-1}}{n(q+1)} \sum_{i=1}^n \psi\left(\left\|X^{(i)}\right\|_{\mathcal{X}}\right) \left\|X^{(i)}\right\|_{\mathcal{X}}^3.$$

References

- [1] C. Croux, C. Dehon, “Analyse canonique basée sur des estimateurs robustes de la matrice des covariances”, *Rev. Stat. Appl.* **50** (2002), p. 5-26.
- [2] C. Croux, P. Filzmoser, K. Joossens, “Classification efficiencies for robust linear discriminant analysis”, *Stat. Sin.* **18** (2008), no. 2, p. 581-599.
- [3] C. Croux, G. Haesbroeck, “Principal component analysis based on robust estimators of the covariance or correlation matrix: Influence function and efficiencies”, *Biometrika* **87** (2000), no. 3, p. 603-618.
- [4] P. L. Davies, “Asymptotic behaviour of S-estimates of multivariate location and parameters and dispersion matrices”, *Ann. Stat.* **15** (1987), p. 1269-1292.
- [5] S. Gardner, J. C. Gower, N. J. Le Roux, “A synthesis of canonical variate analysis, generalised canonical correlation and Procrustes analysis”, *Comput. Stat. Data Anal.* **50** (2006), no. 1, p. 107-134.
- [6] A. Gifi, *Nonlinear multivariate analysis*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1990.
- [7] F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, W. A. Stahel, *Robust Statistics: The Approach based on influence functions*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1986.
- [8] H. Hwang, K. Jung, Y. Takane, T. S. Woodward, “Functional multiple-set canonical correlation analysis”, *Psychometrika* **77** (2012), p. 753-775.
- [9] H. P. Lopuhaä, “On the relation between S-estimator and M-estimator of multivariate location and covariance”, *Ann. Stat.* **17** (1989), no. 4, p. 1662-1683.

- [10] ———, “Asymptotic expansion of S-estimators of location and covariance”, *Stat. Neerl.* **51** (1997), no. 2, p. 220-237.
- [11] G. M. Nkiet, “Asymptotic theory of multiple-set linear canonical analysis”, *Math. Methods Stat.* **26** (2017), no. 3, p. 196-211.
- [12] Y. Takane, H. Hwang, H. Abdi, “Regularized multiple-set canonical correlation analysis”, *Psychometrika* **73** (2008), no. 4, p. 753-775.
- [13] S. Taskinen, I. Koch, H. Oja, “Robustifying principal component analysis with spatial sign vectors”, *Stat. Probab. Lett.* **82** (2012), no. 4, p. 765-774.
- [14] A. Tenenhaus, M. Tenenhaus, “Regularized generalized canonical correlation analysis”, *Psychometrika* **76** (2011), no. 2, p. 257-284.