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Published online: 28 July 2020

https://doi.org/10.5802/crmath.68

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Les Comptes Rendus. Mathématique sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569
The mod 2 Margolis homology of the Dickson algebra

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Abstract. We completely compute the mod 2 Margolis homology of the Dickson algebra $D_n$, i.e. the homology of $D_n$ with the differential to be the Milnor operation $Q_j$, for every $n$ and $j$. The motivation for this problem is that, the Margolis homology of the Dickson algebra plays a key role in study of the Morava K-theory $K(j)^*(BS_m)$ of the symmetric group on $m$ letters $S_m$.

We show that Pengelley–Sinha' s conjecture on $H_*(D_n; Q_j)$ for $n \leq j$ is true if and only if $n = 1$ or 2. For $3 \leq n \leq j$, our result proves that this conjecture turns out to be false since the occurrence of some "critical elements" $h_{s_1, \ldots, s_k}$ of degree $(2^{j+1} - 2^n) + \sum_{i=1}^k (2^n - 2^i)$ in this homology for $0 < s_1 < \cdots < s_k < n$ and $k > 1$.

Résumé. Dans cette note on calcule entièrement l’homologie de Margolis modulo 2 de l’algèbre de Dickson $D_n$, i.e. l’homologie de $D_n$ en choisissant pour différentielles les opérations de Milnor $Q_j$, pour tous $n$ et $j$. La motivation pour cette étude est le rôle clé joué par cette homologie dans l’étude de la K-théorie de Morava $K(j)^*(BS_m)$ du groupe symétrique $S_m$ en $m$ lettres.

Nous montrons que la conjecture de Pengelley–Sinha sur $H_*(D_n; Q_j)$ pour $n \leq j$ est vraie si et seulement s'il est $1, 2$. Pour $3 \leq n \leq j$ notre résultat montre que la conjecture est fausse à cause de l’occurrence d’éléments « critiques » $h_{s_1, \ldots, s_k}$ de degré $(2^{j+1} - 2^n) + \sum_{i=1}^k (2^n - 2^i)$ dans cette homologie pour $0 < s_1 < \cdots < s_k < n$ et $k > 1$.


Funding. This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2019.300.

Manuscript received 24th February 2020, revised 2nd May 2020, accepted 4th May 2020.

Let $\mathcal{A}$ be the mod 2 Steenrod algebra, generated by the cohomology operations $Sq^j$ with $j \geq 0$ and subject to the Adem relation with $Sq^0 = 1$. Further $\mathcal{A}$ is a Hopf algebra, whose coproduct is given by the formula $\Delta(Sq^j) = \sum_{i=0}^j Sq^i \otimes Sq^{j-i}$.

Let $\mathcal{A}_*$ be the Hopf algebra, which is dual to $\mathcal{A}$. Let $\xi_j = (Sq^{2^j} \cdots Sq^2 Sq^1)^*$ be the Milnor element of degree $2^{j+1} - 1$ in $\mathcal{A}_*$, for $j \geq 0$, where the duality is taken with respect to the admissible basis of $\mathcal{A}$. According to Milnor [4], as an algebra, $\mathcal{A}_* \cong \mathbb{F}_2[\xi_0, \xi_1, \ldots, \xi_j, \ldots]$, the polynomial algebra in infinitely many generators $\xi_0, \xi_1, \ldots, \xi_j, \ldots$.

Let $Q_j$, for $j \geq 0$, be the Milnor operation (see [4]) of degree $(2^{j+1} - 1)$ in $\mathcal{A}$, which is dual to $\xi_j$ with respect to the basis of $\mathcal{A}_*$ consisting of all monomials in the generators $\xi_0, \xi_1, \ldots, \xi_j, \ldots$. 

In memory of Nguyệt Thị Thanh Bình
Remarkably, $Q_j$ is a differential, that is $Q_j^2 = 0$ for every $j$. In fact, $Q_0 = Sq^1$, $Q_j = [Q_{j-1}, Sq^{2j}]$, the commutator of $Q_{j-1}$ and $Sq^{2j}$ in the Steenrod algebra $\mathcal{A}$, for $j > 0$.

In the article, we compute the Margolis homology of the Dickson algebra $D_n$, i.e. the homology of $D_n$ with the differential to be the Milnor operation $Q_j$.

The real goal that we pursue is to compute the Morava $K$-theory $K(j)^*(BS_m)$ of the symmetric group $S_m$ on $m$ letters. It was well known that, the Milnor operation is the first non-zero differential, $Q_j = d_{2j+1}$, in the Atiyah–Hirzebruch spectral sequence for computing $K(j)^*(X)$, the Morava $K$-theory of a space $X$. So, the $Q_j$-homology of $H^*(X)$ is the $E_{2j+1}$-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(X)$. (See e.g. Yagita [10, §2], although the fact was well known before this article.)

A key step in the determination of the symmetric group’s cohomology is to apply the Quillen restriction from this cohomology to the cohomologies of all elementary abelian subgroups of the symmetric group. For $m = 2^n$ and the “generic” elementary abelian 2-subgroup $(\mathbb{Z}/2)^n$ of the symmetric group $S_{2^n}$, the image of the restriction $H^*(BS_{2^n}) \to H^*(B(\mathbb{Z}/2)^n)$ is exactly the Dickson algebra $D_n$ (see Mùi [5, Thm. II.6.2]). So, the $E_{2j+1}$-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(BS_{2^n})$ maps to the Margolis homology $H_*(D_n;Q_j)$. This is why the Margolis homology of the Dickson algebra is taken into account.

Let us study the range $n$ Dickson algebra of invariants

$$D_n = \mathbb{F}_2[x_1, \ldots, x_n]^{GL(n, \mathbb{F}_2)},$$

where each generator $x_i$ is of degree 1, and the general linear group $GL(n, \mathbb{F}_2)$ acts canonically on $\mathbb{F}_2[x_1, \ldots, x_n]$. Following Dickson [1], let us consider the determinant

$$[e_1, \ldots, e_n] = \det\left(\begin{array}{ccc} x_1^{e_1} & \cdots & x_n^{e_1} \\ \vdots & \ddots & \vdots \\ x_1^{e_n} & \cdots & x_n^{e_n} \end{array}\right)$$

for non-negative integers $e_1, \ldots, e_n$. Then $\omega[e_1, \ldots, e_n] = \det(\omega)[e_1, \ldots, e_n]$, for $\omega \in GL(n, \mathbb{F}_2)$ (see [1]). Set

$$L_{n,s} = [0, 1, \ldots, \hat{s}, \ldots, n], \quad (0 \leq s \leq n),$$

where $\hat{s}$ means $s$ being omitted, and $L_n = L_{n,n}$. The Dickson invariant $c_{n,s}$ of degree $2^n - 2^s$ is originally defined as follows:

$$c_{n,s} = L_{n,s}/L_n, \quad (0 \leq s < n).$$

Dickson proved in [1] that $D_n$ is a polynomial algebra on the Dickson invariants

$$D_n = \mathbb{F}_2[c_{n,0}, \ldots, c_{n,n-1}].$$

To be explicit, the Dickson invariant can be expressed as in Hùng–Peterson [3, §2]:

$$c_{n,s} = \sum_{i_1 + \cdots + i_n = 2^n - 2^s} x_1^{i_1} \cdots x_n^{i_n}, \quad (0 \leq s < n).$$

where the sum is over all sequences $i_1, \ldots, i_n$ with $i_k$ either 0 or a power of 2.

We are interested in the following element of the Dickson algebra $D_n$:

$$A_{n,s} = [0, \ldots, \hat{s}, \ldots, n-1, j]/L_n,$$

for $0 \leq s < n \leq j$. By convention, $A_{n,n-1} = 0$.

In this article, when $j$ and $n$ are fixed, the elements $c_{n,s}$ and $A_{j,n,s}$ will respectively be denoted by $c_s$ and $A_s$ for abbreviation.
Lemma 1. For $0 \leq j, 0 \leq s < n$,

$$Q_j(c_s) = \begin{cases} 
    c_0, & 0 \leq j < n - 1, j = s - 1, \\
    0, & 0 \leq j < n - 1, j \neq s - 1, \\
    c_0 c_s, & j = n - 1, \\
    c_0 (c_s A_{n-1}^2 + A_{s-1}^2), & 0 \leq s \leq j.
\end{cases}$$

The action of the Steenrod algebra on the Dickson one is basically computed in [2]. Related and partial results concerning the lemma can be seen in [7–9].

The next two theorems are stated in Sinha [6]. Their proofs are straightforward from Lemma 1.

Theorem 2. For $0 \leq j < n - 1$,

$$H_*(D_n, Q_j) \cong \mathbb{F}_2[c_{j+1}^2] \otimes \mathbb{F}_2[c_1, \ldots, \widehat{c_{j+1}}, \ldots, c_{n-1}],$$

where $\widehat{c_{j+1}}$ means $c_{j+1}$ being omitted.

Let $\mathbb{F}_2[c_1, \ldots, c_{n-1}]_{ev}$ be the $\mathbb{F}_2$-submodule of $\mathbb{F}_2[c_1, \ldots, c_{n-1}]$ generated by all the monomials $c_1^{i_1} \cdots c_{n-1}^{i_{n-1}}$ with $i_1 + \cdots + i_{n-1}$ even.

Theorem 3. $H_*(D_n; Q_{n-1}) \cong \mathbb{F}_2[c_1, \ldots, c_{n-1}]_{ev}$.

Proposition 4. For $0 \leq s_1, \ldots, s_k < n \leq j$,

$$Q_j(c_{s_1} \cdots c_{s_k}) = c_0 \left( k c_{s_1} \cdots c_{s_k} A_{n-1}^2 + \sum_{i=1}^{k} c_{s_1} \cdots \widehat{c_{s_i}} \cdots c_{s_k} A_{s_i-1}^2 \right),$$

where $\widehat{c_{s_i}}$ means $c_{s_i}$ being omitted.

Conjecture 5 (D. Pengelley – D. Sinha, see [6]). For $n \leq j$, $H_*(D_n; Q_j) \cong D_n^{2j} / \left( Q_j(c_0), Q_j(c_0 c_1), \ldots, Q_j(c_0 c_{n-1}) \right)$.

Let $D_n^{\text{odd}}$ be the $\mathbb{F}_2$-submodule of $D_n$ spanned by all monomials $c_0^{i_0} \cdots c_{n-1}^{i_{n-1}}$ with at least one of the exponents $i_0, \ldots, i_{n-1}$ odd. Note clearly that $D_n^{\text{odd}}$ is not a $Q_j$-submodule of $D_n$, but $\text{Im} Q_j \cap D_n^{\text{odd}}$ is, since $Q_j$ vanishes on this module.

Pengelley–Sinha’s conjecture is equivalent to the equality: $\ker Q_j = (\text{Im} Q_j \cap D_n^{\text{odd}}) \oplus D_n^{2j}$. In other words, there is no class in $H_*(D_n; Q_j)$ represented by an element in $D_n^{\text{odd}}$. The following two theorems show that Pengelley–Sinha’s conjecture is true for $n = 1$ or $2$ and every $j$.

Theorem 6. For $n = 1, 0 \leq j$,

$$H_*(D_1; Q_j) \cong \mathbb{F}_2[c_1^2] / \left( c_1^{2j+1} \right).$$

In particular, $H_*(D_1; Q_0) = \mathbb{F}_2$ (this is also a special case of Theorem 3), $H_*(D_1; Q_1) = \Lambda(c_0^2)$, where $\Lambda(c_0^2)$ denotes the $\mathbb{F}_2$-exterior algebra on $c_0$.

Theorem 7. Denote $\overline{c_0^2} = \Lambda(c_0^2) / (\mathbb{F}_2 \cdot 1)$. If $n = 2, 0 \leq j$,

$$H_*(D_2; Q_j) \cong \begin{cases} 
    \mathbb{F}_2[c_1^2], & j = 0, 1, \\
    \Lambda(c_0^2) \oplus \mathbb{F}_2[c_1^2], & j = 2, \\
    \mathbb{F}_2[c_0^2, c_1^2] / (c_0^2 A_0^2 c_0^2 A_1^2), & j > 2,
\end{cases}$$

where $A_0 = (x_1^2 x_2^2 + x_1^2 x_2^2) / (x_1 x_2 + x_1 x_2^2), A_1 = (x_1 x_2^2 + x_1 x_2^2) / (x_1 x_2 + x_1 x_2^2)$.

The cases $j = 0, 1$ in the previous theorem are special cases of Theorems 2 and 3.

Proposition 8. Pengelley–Sinha’s Conjecture for $n \leq j$ is true if and only if $1 \leq n \leq 2$. 

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How can we adjust Pengelley–Sinha’s conjecture to make a correct one in the problem for \( 3 \leq n \leq j \)? The critical elements \( h_{s_1, \ldots, s_k} \), defined below in the Margolis homology of the Dickson algebra \( D_n \), are the main ingredient in our correction of Pengelley–Sinha’s conjecture for \( 3 \leq n \leq j \).

Note that, \( c_0^2 \) divides \( Q_j(c_0 c_{s_1} \cdots c_{s_k}) \) in \( D_n \), if \( s_1, \ldots, s_k \) are pairwise distinct.

**Definition 9.** For \( n \leq j \), \( 0 \leq s_1, \ldots, s_k < n \), and \( s_1, \ldots, s_k \) pairwise distinct:

\[
h_{s_1, \ldots, s_k} = \frac{1}{c_0^2} Q_j(c_0 c_{s_1} \cdots c_{s_k}).
\]

To be more explicit, under the hypotheses of the definition:

\[
h_{s_1, \ldots, s_k} = (k + 1)c_1 \cdots c_{s_k} A^2_{n-1} + \sum_{i=1}^{k} (c_{s_1} \cdots \hat{c}_{s_i} \cdots c_{s_k}) A^2_{s_i-1}.
\]

Note that, \( h_{s_1, \ldots, s_k} \in D^\text{odd}_n \) if \( k > 1 \), and that \( h_{s_1, \ldots, s_k} \) depends also on \( n \) and \( j \).

**Lemma 10.**

(i) \( A_s = \begin{cases} 0 \mod (c_0, \ldots, c_r), & 0 \leq s \leq r < n, \\ \neq 0 \mod (c_0, \ldots, c_r), & 0 \leq r < s < n. \end{cases} \)

(ii) \( A_r \) and \( A_s \) are coprime in \( D_n \) for \( 0 \leq r \neq s < n \).

(iii) If \( n \leq j \), then \( h_{s_1, \ldots, s_k} \in \ker Q_j \) and \( [h_{s_1, \ldots, s_k}] \neq 0 \) in the Margolis homology \( H_*(D_n; Q_j) \) for \( 0 \leq s_1 < \cdots < s_k < n, 1 < k \).

The lemma is based on the following inductive formula, in which the complete notations \( A_{j,n,s} \) and \( c_{n,s} \) are used instead of the simplified ones \( A_s \) and \( c_s \):

\[
A_{j,n,s} = A^2_{j-1,n-1,s-1} + A^2_{j-1,n,n-1} c_{n-1,s} L_n L_{n-1},
\]

for \( 0 \leq s < n \leq j \). Here, by convention, \( A_{n-1,n,n-1} = 1, c_{n-1,n-1} = 1 \).

Note that \( Q_j \) is a (total) derivation, that is \( Q_j(ab) = Q_j(a)b + aQ_j(b) \). We study the \( s \)-th partial derivation for \( 0 < s \leq n \), and its “inverse”, the so-called integral on a direction. These notions will play key roles in the remaining part of the article.

**Definition 11.** Let \( s_1, \ldots, s_k \) be pairwise distinct, with \( 0 \leq s_1, \ldots, s_k < n \), and \( R \in D_n \). The \( s \)-th partial derivation is the morphism defined for \( 0 \leq s \leq n \) by

\[
\partial_s(c_{s_1} \cdots c_{s_k} R^2) = \begin{cases} c_0 c_{s_1} \cdots c_{s_k} A^2_{n-1} R^2, & k \text{ odd}, s = n, \\ c_0 c_{s_1} \cdots \hat{c}_{s_i} \cdots c_{s_k} A^2_{s_i-1} R^2, & s = s_i, \\ 0, & \text{otherwise}. \end{cases}
\]

Since \( A_{s-1} = 0 \), it yields \( \partial_0 = 0 \). If \( \partial_s(c_{s_1} \cdots c_{s_k}) \neq 0 \), then \( s \) should be one of the indices \( s_1, \ldots, s_k \) or \( n \). Obviously, \( \text{Im} \partial_s \subset c_0 A^2_{s-1} D_n \). Proposition 4 leads to:

**Lemma 12.** Let \( s_1, \ldots, s_k \) be pairwise distinct, with \( 0 \leq s_1, \ldots, s_k < n \leq j \), and \( R \in D_n \). Then

\[
Q_j(c_{s_1} \cdots c_{s_k} R^2) = \sum_{s=1}^{n} \partial_s(c_{s_1} \cdots c_{s_k} R^2).
\]

**Definition 13.** The integral on the \( r \)-th direction \( I_r : c_0 A^2_{r-1} D_n \to D^\text{odd}_n \), for \( 0 < r \leq n \), is the morphism given by:

\[
I_r(c_0 c_{s_1} \cdots c_{s_k} A^2_{r-1} R^2) = \begin{cases} c_{s_1} \cdots c_{s_k} R^2, & k \text{ odd}, r = n, \\ c_{s_1} \cdots c_{s_k} c_{r} R^2, & r \neq s_1, \ldots, s_k, n, \\ 0, & \text{otherwise}. \end{cases}
\]

where \( s_1, \ldots, s_k \) are pairwise distinct, \( 0 \leq s_1, \ldots, s_k < n, 0 \leq k \), and \( R \in D_n \).
Lemma 14. Let $s_1, \ldots, s_k$ be pairwise distinct, with $0 \leq s_1, \ldots, s_k < n$, $0 < s \leq n$, and $R \in D_n$. Then

(i) $I_0 \partial_x (c_1 \cdots c_k R^2) = \begin{cases} c_1 \cdots c_k R^2, & \text{either } k \text{ odd, } s = n, \text{ or } s \in \{s_1, \ldots, s_k\}, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $\partial_x I_0 (c_0 c_1 \cdots c_k A^2 s_{s-1} R^2) = \begin{cases} c_0 c_1 \cdots c_k A^2 s_{s-1} R^2, & \text{either } k \text{ odd, } s = n, \text{ or } s \neq s_1, \ldots, s_k, n, \\ 0, & \text{otherwise.} \end{cases}$

Let $hc_0^2 D^2_n$ and $hD^2_n$ be the submodules of $D_n$ generated by the generators $\{h_{s_1, \ldots, s_k} | 0 < s_1 < \cdots < s_k < n, 1 < k\}$ over $c_0^2 D_n$ and $D_n = \mathbb{F}_2[c^1_1, \ldots, c^1_{n-1}]$ respectively. Let $h_0 D^2_n$ be the submodule of $D_n$ generated by $\{h_{0, s_2, \ldots, s_k} | 0 = s_1 < s_2 < \cdots < s_k < n, 1 < k\}$ over $D^2_n$.

Theorem 15. For $3 \leq n \leq j$,

$\text{Ker } Q_j \cap D_n^2 = (\text{Im } Q_j \cap D_n^2) + hD^2_n$, where $\text{Im } Q_j \cap D_n^2 = h_0 D^2_n \oplus h c_0^2 D^2_n$, and $h_0 D^2_n \cap hD^2_n = \{0\}$.

The exponent of $c_0$ in each Dickson monomial of $h_0 D^2_n$ is odd, whereas the exponent of $c_0$ in every Dickson monomial of $hc_0^2 D^2_n$ or of $hD^2_n$ is even. It yields $h_0 D^2_n \cap hc_0^2 D^2_n = \{0\}$ and $h_0 D^2_n \cap hD^2_n = \{0\}$.

The smallest natural number $n$ such that there exists a sequence $0 < s_1 < \cdots < s_k < n$ with $k > 1$ is $n = 3$.

Remark 16. The sum in Theorem 15 is not a direct sum. This is a consequence of the fact that the critical elements are not linear independent over $D^2_n$.

Let $S = (s_1, \ldots, s_k)$ be a sequence with $0 < s_1 < \cdots < s_k < n$ and $k > 2$. It is remarkable that

$$H_S = kh_{s_1, \ldots, s_k} A^2_{s-1} n + \sum_{i=1}^{k} h_{s_1, \ldots, s_i, \ldots, s_k} A^2_{s_i-1} = 0.$$  

Let $\pi : D^2_n \to \mathbb{F}_2[c^1_1, \ldots, c^1_{n-1}]$ be the projection, whose kernel is $c_0^2 D^2_n$. We denote $\pi(Z^2)$ by $Z^2$ for abbreviation. So $Z^2 + Z^2 \in c_0^2 D^2_n$ for $Z^2 \in D^2_n$. The equality $H_S = 0$ implies

$$H_S + \overline{H}_S = kh_{s_1, \ldots, s_k} A^2_{s-1} n + \sum_{i=1}^{k} h_{s_1, \ldots, s_i, \ldots, s_k} A^2_{s_i-1} + k \sum_{i=1}^{k} h_{s_1, \ldots, s_i, \ldots, s_k} A^2_{s_i-1} = \overline{H}_S.$$  

The left hand side belongs to $hc_0^2 D^2_n \subset (\text{Im } Q_j \cap D_n^2)$ (it is in $D_n^2$ as $k - 1 > 1$), while the right hand side is in $hD^2_n$ with at most one “coefficient” $\overline{A}_{s_i-1}$ being zero. (The zero-coefficient occurs when $s_i = 1$, since $\overline{A}_{s_i-1} \neq 0$ for $s_i > 1$ by Lemma 10.) Therefore, $\text{Im } Q_j \cap D_n^2 \cap hD^2_n = h_0 c_0^2 D^2_n \cap hD^2_n \neq \{0\}$.

The following main result of the article is a consequence of the preceding one and the equalities: $Q_j(c_0) = c_0^2 A^2_{s-1} n$, $Q_j(c_0 c_j) = c_0^2 A^2_{s-1} n$ ($0 < s < n$).

Theorem 17. For $3 \leq n \leq j$,

$$H_n(D_n; Q_j) = \frac{D^2_n}{(c_0^2 A^2_{n-1}, c_0^2 A^2_{n-1})} \oplus \frac{hD^2_n}{h_0 c_0^2 D^2_n \cap hD^2_n}.$$  

Example 18. For $j = n \geq 3$, we have $A_s = c_s$ for $0 \leq s < n$. So the critical element, which also depends on $n$ and $j$, is explicitly given by

$$h_{s_1, \ldots, s_k} = (k + 1)c_{s_1} \cdots c_{s_k} c_{n-1} + \sum_{i=1}^{k} (c_{s_1} \cdots c_{s_i} \cdots c_{s_k}) c_{s_i-1}.$$  

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for $0 < s_1 < \cdots < s_k < n$, $1 < k$. Theorem 17 yields
\[
H_\ast(D_n; Q_n) = \frac{D_n^2}{c_0^2, c_1^2, c_2^2, \ldots, c_{n-1}^2} \oplus \frac{hD_n^2}{hc_0^2 D_n^2 \cap hD_n^2} = \Lambda(c_0^2) \oplus \mathbb{F}_2[c_1^2, \ldots, c_{n-1}^2] \oplus \frac{hD_n^2}{hc_0^2 D_n^2 \cap hD_n^2}.
\]

For $k > 2$, by Remark 16,
\[
h_{s_2, \ldots, s_k} c_0^2 = kh_{s_1, s_2, \ldots, s_k} c_{n-1}^2 + \sum_{i=2}^{k} h_{s_1, s_2, \ldots, s_k} c_1^2, \ldots, c_{s_i-1}^2
\]
is a nonzero element in $hc_0^2 D_n^2 \cap hD_n^2$ for $1 = s_1 < \cdots < s_k < n$, while
\[
kh_{s_1, s_2, \ldots, s_k} c_{n-1}^2 + \sum_{i=1}^{k} h_{s_1, \ldots, s_i, \ldots, s_k} c_{s_i-1}^2 = 0
\]
is a linear relationship of the critical elements over $D_n^2$ for $1 < s_1 < \cdots < s_k < n$.

The contents of this note will be published in detail elsewhere.

Acknowledgment

This research was carried out when the author visited the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, in the academic year 2019-2020. He would like to express his warmest thank to the VIASM for the hospitality and for the wonderful working condition.

The author hearty thanks Dev Sinha for an interesting discussion on the subject during the Vietnam-US Mathematical joint meeting, Quynhon June 10-13, 2019. He is deeply indebted to H. Miller, D. Ravenel, L. Schwartz, and S. Wilson for generous discussions on the goal of the problem and the references of this article.

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