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Low regularity solutions to the stochastic geometric wave equation driven by a fractional Brownian sheet

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Abstract. We announce a result on the existence of a unique local solution to a stochastic geometric wave equation on the one dimensional Minkowski space \mathbb{R}^{1+1} with values in an arbitrary compact Riemannian manifold. We consider a rough initial data in the sense that its regularity is lower than the energy critical.

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1. Introduction

Recently, the existence and the uniqueness of a global solution, in the strong and weak sense, for the stochastic geometric wave equations (SGWEs) on the Minkowski space \mathbb{R}^{1+m} , $m \geq 1$, with the target manifold (N, g) being a suitable n -dimensional Riemannian manifold, e.g. a sphere, has been established under various sets of assumptions by the first named author and M. Ondreját, see [1–3] for details. To the best of our knowledge, the most general result in the case $m = 1$, is a construction of a global $H_{loc}^1(N) \times L_{loc}^2(TN)$ -valued weakly continuous solution of SGWE, where TN denotes the tangent bundle of N , see [2].

The purpose of this note is to present a method by which we can prove the existence of a unique local solution to SGWE with $m = 1$ in the case of the initial data belonging to $H_{loc}^s(N) \times H_{loc}^{s-1}(TN)$ for $s \in (\frac{3}{4}, 1)$. In particular, we generalize the corresponding deterministic theory result of [8] to the stochastic setting, as well as the results of [1–3] to the wave maps equation with low regularity initial data (i.e. $s < 1$) and fractional (both in time and space) Gaussian noise.

A more detailed account of this work and the global theory, with complete proofs, will be presented in forthcoming papers.

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2. Problem formulation

We are interested in solutions having continuous paths and hence, motivated by [8] and [11], we find that it is suitable to formulate the Cauchy problem for the SGWE using local coordinates on the target manifold N . To be precise, given a sufficiently smooth function σ from \mathbb{R}^n to \mathbb{R}^n , for the wave map $z : \mathbb{R}^{1+1} \rightarrow N$ composed with a given local chart ϕ of N we consider the following Cauchy problem

$$\square u = N_0(u) + \sigma(u)\dot{\xi}, \quad u(0, x) = u_0(x), \quad \text{and} \quad \partial_t u(0, x) = u_1(x), \tag{1}$$

where $\phi \circ z := u : \mathbb{R}^{1+1} \rightarrow \mathbb{R}^n; \square := \partial_t^2 - \Delta_x;$

$$\partial_0 = \partial_t, \quad \partial^0 = -\partial_t, \quad \partial_1 = \partial^1 = \partial_x;$$

$N_0(u) := -\sum_{a,b=1}^n \sum_{\mu=0}^1 \Gamma_{ab}^k(u) (\partial_\mu u^a \partial^\mu u^b)$ with Γ_{ab}^k denoting the Christoffel symbols on N in the chosen local coordinate system and ξ is a suitable random field. The necessary assumptions will be given later in a precise manner.

An efficient way to simplify the computations of the required a priori estimates for (1) is to switch the coordinate-axis of (t, x) -variables to the null coordinates, see for instance [8, 9], and respectively [13], for the deterministic and the stochastic problem. Our approach is in line with these references. By performing the following transformation, which can be made rigorous for sufficiently regular case,

$$u^*(\alpha, \beta) := u\left(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}\right) = u(t, x) \text{ and } u(t, x) = u^*(t + x, t - x), \tag{2}$$

the problem (1) can be re-written as

$$\diamond u^* = \mathcal{N}(u^*) + \sigma(u^*)\Xi_{\alpha\beta}, \tag{3}$$

where $\Xi_{\alpha\beta} := \frac{\partial^2 \Xi}{\partial \alpha \partial \beta}$, subject to the following boundary conditions

$$u^*(\alpha, -\alpha) = u_0(\alpha), \quad \partial_\alpha u^*(\alpha, -\alpha) + \partial_\beta u^*(\alpha, -\alpha) = u_1(\alpha). \tag{4}$$

Here Ξ is a fractional Brownian sheet (fBs) on \mathbb{R}^2 with Hurst indices $H_1, H_2 \in (0, 1)$, i.e. Ξ is a centered Gaussian process such that

$$\mathbb{E}[\Xi(\alpha_1, \beta_1)\Xi(\alpha_2, \beta_2)] = R_{H_1}(|\alpha_1|, |\alpha_2|)R_{H_2}(|\beta_1|, |\beta_2|), \quad (\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{R}^2,$$

where $R_H(a, b) = \frac{1}{2} (a^{2H} + b^{2H} - |a - b|^{2H}), a, b \in \mathbb{R}$ and

$$\diamond u^* := 4 \frac{\partial^2 u^*}{\partial \alpha \partial \beta}, \quad \mathcal{N}(u^*) := 4 \sum_{a,b=1}^n \Gamma_{ab}(u^*) \frac{\partial u^{*a}}{\partial \alpha} \frac{\partial u^{*b}}{\partial \beta}.$$

From now on we will only work in the (α, β) -coordinates and hence, we will write u instead of u^* in the sequel. As usual in the SPDE theory, we understand the SGWE (3) in the following integral/mild form

$$u = S(u_0, u_1) + \diamond^{-1} \mathcal{N}(u) + \diamond^{-1} [\sigma(u)\Xi_{\alpha\beta}], \tag{5}$$

where, for $(\alpha, \beta) \in \mathbb{R}^2$,

$$[S(u_0, u_1)](\alpha, \beta) := \frac{1}{2} [u_0(\alpha) + u_0(-\beta)] + \frac{1}{2} \int_{-\beta}^\alpha u_1(r) dr; \tag{6}$$

$$[\diamond^{-1} \mathcal{N}(u)](\alpha, \beta) := \frac{1}{4} \int_{-\beta}^\alpha \int_{-a}^\beta \mathcal{N}(u(a, b)) db da; \tag{7}$$

and

$$[\diamond^{-1} (\sigma(u)\Xi_{\alpha\beta})](\alpha, \beta) := \frac{1}{4} \int_{-\beta}^\alpha \int_{-a}^\beta \sigma(u(a, b)) \Xi_{\alpha\beta}(da, db). \tag{8}$$

The integral on the right hand side of (8) is well-defined pathwise, see Proposition 4.

3. Relevant notation and function spaces

If x and y are two quantities (typically non-negative), we will write $x \lesssim y$ or $y \gtrsim x$ to denote the statement that $x \leq Cy$ for some positive constant $C > 0$.

By $L^p(\mathbb{R}^d)$, for $p \in [1, \infty)$, we denote the classical real Banach space of all (equivalence classes of) \mathbb{R} -valued p -integrable functions on \mathbb{R}^d . For $s \in \mathbb{R}$, we set

$$H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |[\mathcal{F}(f)](\xi)|^2 d\xi < \infty\},$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the set of all tempered distributions on \mathbb{R}^d , i.e. the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of all rapidly decreasing infinitely differentiable functions on \mathbb{R}^d , and $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^d$ and $\mathcal{F}(f)$ is the d -dimensional Fourier transform of f .

Definition 1. Let $s, \delta \in \mathbb{R}$. The hyperbolic $H^{s,\delta}$ and the product $H_t^s H_x^\delta$ Sobolev spaces are the sets of all $u \in \mathcal{S}'(\mathbb{R}^2)$ for which, the appropriate norm is finite, where, with $\mathcal{F}(u)$ being the space-time Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^2)$,

$$\begin{aligned} \|u\|_{H^{s,\delta}} &:= \left(\int_{\mathbb{R}^2} \langle |\tau| + |\xi| \rangle^{2s} \langle |\tau| - |\xi| \rangle^{2\delta} |[\mathcal{F}(u)](\tau, \xi)|^2 d\xi d\tau \right)^{1/2}, \\ \|u\|_{H_t^s H_x^\delta} &:= \left(\int_{\mathbb{R}^2} \langle \tau \rangle^{2s} \langle \xi \rangle^{2\delta} |[\mathcal{F}(u)](\tau, \xi)|^2 d\tau d\xi \right)^{1/2}. \end{aligned}$$

Let $\Phi(\mathbb{R}^d)$ be the set of all systems $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}'(\mathbb{R}^d)$ such that

- (1) $\text{supp } \varphi_0 \subset \{x : |x| \leq 2\}$, $\text{supp } \varphi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$, if $j \in \mathbb{N} \setminus \{0\}$.
- (2) For every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|} D^\alpha \varphi_j(x) \leq c_\alpha \quad \text{for all } j \in \mathbb{N} \text{ and all } x \in \mathbb{R}^d.$$

- (3) $\sum_{j=0}^\infty \varphi_j(x) = 1$ for every $x \in \mathbb{R}^d$.

It is known, see [12, Remark 2.3.1/1], that the system $\Phi(\mathbb{R}^d)$ is not empty. Given a dyadic partition of unity $\varphi := \{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^d)$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$, the Littlewood-Paley blocks of f are defined as $\Delta_{j,k} f := 0$, $j, k \leq -1$, and

$$\Delta_{j,k} f := \mathcal{F}^{-1}(\varphi_j(\tau)\varphi_k(\xi)[\mathcal{F}(f)](\tau, \xi)), \quad j, k \geq 0,$$

where \mathcal{F}^{-1} stands for the inverse Fourier transform on $\mathcal{S}'(\mathbb{R}^2)$. Next, for $(s_1, s_2) \in \mathbb{R}^2$, $p, q \in (1, \infty)$, we define the following Banach space

$$S_{p,q}^{s_1, s_2} B(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{S_{p,q}^{s_1, s_2} B(\mathbb{R}^2)}^q < \infty\},$$

where

$$\|f\|_{S_{p,q}^{s_1, s_2} B(\mathbb{R}^2)}^q := \left(\sum_{k=0}^\infty \sum_{j=0}^\infty 2^{q(s_1 j + s_2 k)} \|\Delta_{j,k} f\|_{L^p(\mathbb{R}^2)}^q \right)^{1/q}.$$

One can prove that the space $S_{p,q}^{s_1, s_2} B(\mathbb{R}^2)$ does not depend on the chosen system $\varphi \in \Phi(\mathbb{R}^d)$, see [12, Proposition 2.3.2/1], and the norms are pairwise equivalent.

It is known that $S_{2,2}^{s,\delta} B(\mathbb{R}^2) = H_t^s H_x^\delta(\mathbb{R}^2)$, for $s, \delta \in \mathbb{R}$, with equivalent norms.

The next proposition justifies the coordinate transformation (2) from the computation perspective, since in (α, β) -coordinate the knowledge of product Sobolev spaces is enough to have the local theory.

Proposition 2. *If $s \geq \delta \in \mathbb{R}$, then the map*

$$H^{s,\delta} \ni u(t, x) \mapsto u^*(\alpha, \beta) \in H_\alpha^s H_\beta^\delta \cap H_\beta^s H_\alpha^\delta =: \mathbb{H}^{s,\delta}, \quad (9)$$

is an isomorphism, where as usual the space $\mathbb{H}^{s,\delta}$ is equipped with the norm

$$\|u^*\|_{\mathbb{H}^{s,\delta}} := \sqrt{\|u^*\|_{H_\alpha^s H_\beta^\delta}^2 + \|u^*\|_{H_\beta^s H_\alpha^\delta}^2}.$$

In particular, we have

$$\|u^*\|_{\mathbb{H}^{s,\delta}} \lesssim \|u\|_{H^{s,\delta}} \lesssim \|u^*\|_{\mathbb{H}^{s,\delta}}.$$

With the spaces \mathbb{H}^{H_1, H_2} defined above introduced in the previous Proposition 2, we have the following result.

Proposition 3. *Assume that $H_1, H_2 \in (0, 1)$ and $H'_i \in (0, H_1 \wedge H_2)$, $i = 1, 2$. Then there exists a complete filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and a map,*

$$\Xi: \mathbb{R}_+^2 \times \Omega \rightarrow \mathbb{R},$$

such that \mathbb{P} -a.s. $\Xi(\cdot, \cdot, \omega) \in \mathbb{H}^{H'_1, H'_2}$ locally, i.e. for every bump function η ,

$$\eta(\alpha)\eta(\beta)\Xi(\alpha, \beta, \omega) \in \mathbb{H}^{H'_1, H'_2}.$$

Moreover, for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{R}^2$,

$$\mathbb{E}[\Xi(\alpha_1, \beta_1) \Xi(\alpha_2, \beta_2)] = R_{H_1}(|\alpha_1|, |\alpha_2|) R_{H_2}(|\beta_1|, |\beta_2|).$$

Here \mathbb{E} is the Expectation operator w.r.t. \mathbb{P} .

The above result, somehow related to a result proved in [11], can be proved by using Proposition 2 and a combination of results from [4] and [5].

4. The main result: the local well-posedness

Let us fix $s \geq \delta \in (\frac{3}{4}, 1)$ for the whole present section. To solve the SGWE problem (5) locally, which is sufficient to prove the local-wellposedness result we are aiming, let $\eta, \chi \in \mathcal{C}_0^\infty(\mathbb{R}; [0, 1])$ be even cut-off functions such that $\text{supp } \eta = \text{supp } \chi \subset [-4, 4]$ and $[-2, 2] \subset \eta^{-1}(\{1\}) = \chi^{-1}(\{1\})$. Let us put $\eta_T(x) := \eta(x/T)$, $x \in \mathbb{R}$, for any $T > 0$. Similarly, we define χ_T .

To simplify the exposition, without loss of generality, we restrict ourselves to the target manifold of dimension 2 and which can be covered by a family of charts such that the Christoffel symbols Γ_{ab}^k depend polynomially on u , that is, for every $k = 1, 2$, one can find $r \in \mathbb{N}$ and $A_{ab}^l \in \mathbb{R}^2$ such that $\Gamma_{ab}^k(u) = \sum_{|l| \leq r} A_{ab}^l u^l$, $u = (u^1, u^2) \in \mathbb{R}^2$, where, for $l = (l_1, l_2) \in \mathbb{N}^2$, $u^l = [u^1]^{l_1} [u^2]^{l_2}$.

Our first result in this section is a generalization of [10, Lemma 2.2].

Proposition 4. *Assume that $s, \delta \in (\frac{3}{4}, 1)$, such that $s \geq \delta$, and $f \in \mathbb{H}^{s-1, \delta-1}$ are given. Then $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F := H + I + J + G$ where, for $\alpha, \beta \in \mathbb{R}$,*

$$\begin{aligned} H(\alpha, \beta) &:= \int_{-\beta}^{\alpha} \int_{-\gamma}^{\beta} (\Delta_{0,0} f)(\gamma, \tau) \, d\tau \, d\gamma, \\ I(\alpha, \beta) &:= \int_{-\beta}^{\alpha} \sum_{n=1}^{\infty} \mathcal{F}^{-1} \left[\frac{1}{i\xi} \varphi_0(\tau) \varphi_n(\xi) (\mathcal{F} f)(\tau, \xi) \right] (\gamma, \beta) \, d\gamma, \\ &\quad - \int_{-\beta}^{\alpha} \sum_{n=1}^{\infty} \mathcal{F}^{-1} \left[\frac{1}{i\xi} \varphi_0(\tau) \varphi_n(\xi) (\mathcal{F} f)(\tau, \xi) \right] (\gamma, -\gamma) \, d\gamma, \\ J(\alpha, \beta) &:= \int_{-\alpha}^{\beta} \sum_{m=1}^{\infty} \mathcal{F}^{-1} \left[\frac{1}{i\tau} \varphi_0(\xi) \varphi_m(\tau) (\mathcal{F} f)(\tau, \xi) \right] (\alpha, \gamma) \, d\gamma \\ &\quad - \int_{-\alpha}^{\beta} \sum_{m=1}^{\infty} \mathcal{F}^{-1} \left[\frac{1}{i\tau} \varphi_0(\xi) \varphi_m(\tau) (\mathcal{F} f)(\tau, \xi) \right] (-\gamma, \gamma) \, d\gamma, \end{aligned}$$

and

$$\begin{aligned}
 G(\alpha, \beta) := & \sum_{j,k=1}^{\infty} \left[\mathcal{F}^{-1} \left[\frac{1}{(i\tau)(i\xi)} \varphi_j(\tau) \varphi_k(\xi) (\mathcal{F}\phi)(\tau, \xi) \right] \right] (\alpha, \beta) \\
 & - \frac{1}{2} \sum_{j,k=1}^{\infty} \left[\mathcal{F}^{-1} \left[\frac{1}{(i\tau)(i\xi)} \varphi_j(\tau) \varphi_k(\xi) (\mathcal{F}\phi)(\tau, \xi) \right] \right] (\alpha, -\alpha) \\
 & - \frac{1}{2} \sum_{j,k=1}^{\infty} \left[\mathcal{F}^{-1} \left[\frac{1}{(i\tau)(i\xi)} \varphi_j(\tau) \varphi_k(\xi) (\mathcal{F}\phi)(\tau, \xi) \right] \right] (-\beta, \beta) \\
 & - \frac{1}{2} \int_{-\beta}^{\alpha} \sum_{j,k=1}^{\infty} \left[\mathcal{F}^{-1} \left[\frac{1}{(i\xi)} \varphi_j(\tau) \varphi_k(\xi) (\mathcal{F}\phi)(\tau, \xi) \right] \right] (\gamma, -\gamma) \, d\gamma \\
 & - \frac{1}{2} \int_{-\beta}^{\alpha} \sum_{j,k=1}^{\infty} \left[\mathcal{F}^{-1} \left[\frac{1}{(i\tau)} \varphi_j(\tau) \varphi_k(\xi) (\mathcal{F}\phi)(\tau, \xi) \right] \right] (\gamma, -\gamma) \, d\gamma,
 \end{aligned}$$

is the unique tempered distribution such that $\frac{\partial^2 F}{\partial \alpha \partial \beta} = f$, and satisfy the following homogeneous boundary conditions

$$F(\alpha, -\alpha) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \alpha}(\alpha, -\alpha) + \frac{\partial F}{\partial \beta}(\alpha, -\alpha) = 0.$$

Moreover, for every η, χ and $T > 0$, there exists $C(\eta, \chi, T) > 0$ such that

$$\|\eta_T(\alpha) \chi_T(\beta) F(\alpha, \beta)\|_{\mathbb{H}^{s,\delta}} \leq C(\eta, \chi, T) \|f\|_{\mathbb{H}^{s-1,\delta-1}}.$$

We will use the following notation

$$F(\alpha, \beta) =: \int_{-\beta}^{\alpha} \int_{-a}^{\beta} f(da, db), \quad (\alpha, \beta) \in \mathbb{R}^2.$$

Proof. Using the properties of $S_{2,2}^{s,\delta} B(\mathbb{R}^2)$ spaces, we need to show that G, H, I, J are well-defined elements of $\mathbb{H}^{s,\delta}$. □

By following the approach of [7] we get the next required result.

Proposition 5. Assume that $\sigma \in \mathcal{C}_b^3(\mathbb{R}^2)$. Then $\sigma \circ u \in \mathbb{H}^{s,\delta}$ for every $u \in \mathbb{H}^{s,\delta}$ and there exist constants $C_i(\sigma) := C_i(\|\sigma\|_{\mathcal{C}_b^{i+1}})$, $i = 1, 2$ such that for $u, u_1, u_2 \in \mathbb{H}^{s,\delta}$,

$$\begin{aligned}
 \|\sigma \circ u\|_{\mathbb{H}^{s,\delta}}^2 & \leq C_1(\sigma) \|u\|_{\mathbb{H}^{s,\delta}}^2 \left[1 + \|u\|_{\mathbb{H}^{s,\delta}}^2 \right], \\
 \|\sigma \circ u_1 - \sigma \circ u_2\|_{\mathbb{H}^{s,\delta}}^2 & \leq C_2(\sigma) \|u_2 - u_1\|_{\mathbb{H}^{s,\delta}}^2 \left[1 + \sum_{i,k=1}^2 \|u_i\|_{\mathbb{H}^{s,\delta}}^{2k} \right].
 \end{aligned}$$

We now state and provide a sketch of proof of the main result of this note. Below we fix a realisation of the random field belonging to the space $\mathbb{H}^{s,\delta}$, see Proposition 3.

Theorem 6. Assume $s, \delta \in (\frac{3}{4}, 1)$ such that $\delta \leq s$ and $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. Let Ξ be a fractional Brownian sheet with Hurst indices $H_1, H_2 \in (s, 1)$. There exist a $R_0 \in (0, 1)$ and a $\lambda_0 := \lambda_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}, R_0) \gg 1$ such that for every $\lambda \geq \lambda_0$ there exists a unique $u := u(\lambda, R_0) \in \mathbb{B}_{R_0}$, where $\mathbb{B}_R := \{u \in \mathbb{H}^{s,\delta} : \|u\|_{\mathbb{H}^{s,\delta}} \leq R\}$, which satisfies the following integral equation

$$\begin{aligned}
 u(\alpha, \beta) = & \eta(\lambda\alpha)\eta(\lambda\beta) \left([S(\chi(\lambda)u_0, \chi(\lambda)u_1)](\alpha, \beta) + [\diamond^{-1}\mathcal{N}(u)](\alpha, \beta) + [\diamond^{-1}\sigma(u)\Xi_{\alpha\beta}] (\alpha, \beta) \right), \\
 & (\alpha, \beta) \in \mathbb{R}^2.
 \end{aligned}$$

Here the right hand side terms are, respectively, defined in (6), (7) and (8).

Sketch of proof of Theorem 6. Our proof is based on the Banach Fixed Point Theorem in the space $\mathbb{H}^{s,\delta}$. Note that all the constants below are positive and depend on η unless mentioned otherwise.

Step 1. Using the following well-known result, see e.g. [6],

$$\left\| \left\{ x \mapsto \chi(x) \int_0^x f(y) \, dy \right\} \right\|_{H^s} \lesssim \|f\|_{H^{s-1}},$$

we can estimate the localized homogeneous part of the solution as

$$\|\eta(\alpha)\chi(\beta)S(u_0, u_1)\|_{\mathbb{H}^{s,\delta}} \leq C_S (\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}).$$

Step 2. In view of the polynomiality of the Christoffel symbols, by using Proposition 4, we deduce the existence of a natural number $\gamma \geq 2$ such that

$$\|\eta(\alpha)\chi(\beta) \diamond^{-1}(\mathcal{N}(\phi) - \mathcal{N}(\psi))\|_{\mathbb{H}^{s,\delta}} \leq C_{\mathcal{N}} \|\phi - \psi\|_{\mathbb{H}^{s,\delta}} [\|\phi\|_{\mathbb{H}^{s,\delta}} + \|\psi\|_{\mathbb{H}^{s,\delta}}]^\gamma.$$

Step 3. By Propositions 4 and 5 followed by the continuity of the multiplication map

$$\mathbb{H}^{s,\delta} \times \mathbb{H}^{s-1,\delta-1} \rightarrow \mathbb{H}^{s-1,\delta-1},$$

see e.g. [8], we get

$$\|\eta(\alpha)\eta(\beta) \diamond^{-1}[(\sigma(u_1) - \sigma(u_2))\Xi_{\alpha\beta}]\|_{\mathbb{H}^{s,\delta}}^2 \leq C_\Xi C_2(\sigma) \|u_2 - u_1\|_{\mathbb{H}^{s,\delta}} \left[1 + \sum_{i,k=1}^2 \|u_i\|_{\mathbb{H}^{s,\delta}}^k \right] \|\Xi_{\alpha\beta}\|_{\mathbb{H}^{s-1,\delta-1}},$$

for some positive constants C_Ξ and $C_2(\sigma) := C_2(\|\sigma\|_{\mathcal{C}_b^3})$.

Step 4. We consider a map $\Theta^\lambda : \mathbb{H}^{s,\delta} \ni u \mapsto u_{\Theta^\lambda} \in \mathbb{H}^{s,\delta}$ defined by

$$u_{\Theta^\lambda} = \eta(\alpha)\eta(\beta) \left(S(u_0^\lambda, u_1^\lambda) + \diamond^{-1}[\mathcal{N}(u)] + \diamond^{-1}[\sigma(u)\Xi_{\alpha\beta}^\lambda] \right),$$

where

$$u_0^\lambda(\alpha) := \chi(\alpha) \left[u_0 \left(\frac{\alpha}{\lambda} \right) - \bar{u}_0^\lambda \right], \quad u_1^\lambda(\alpha) := \chi(\alpha) \lambda^{-1} u_1 \left(\frac{\alpha}{\lambda} \right), \quad \text{and} \quad \Xi_{\alpha\beta}^\lambda := \lambda^{-2} \Pi_\lambda \Xi_{\alpha\beta},$$

with

$$\bar{u}_0^\lambda := \int_{\mathbb{R}} u_0 \left(\frac{y}{\lambda} \right) \psi(y) \, dy.$$

Here ψ is any bump function which is non zero on the support of χ, η and $\int_{\mathbb{R}} \psi(x) \, dx = 1$. Then, in view of the assumption that $s + \delta > \frac{3}{2}$, by using the above estimates we obtain, for any $u, v \in \mathbb{B}_R$,

$$\|u_{\Theta^\lambda} - v_{\Theta^\lambda}\|_{\mathbb{H}^{s,\delta}} \lesssim \left[C_{\mathcal{N}} R + \lambda^{1-(s+\delta)} C_\Xi C_2(\sigma) (1 + R) \|\Xi_{\alpha\beta}\|_{\mathbb{H}^{s-1,\delta-1}} \right] \|u - v\|_{\mathbb{H}^{s,\delta}}.$$

Hence we can choose $R_0 \in (0, 1), \lambda_0 := \lambda_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}, R_0)$ in such a way that Θ^λ is $\frac{1}{2}$ -contraction as a map from \mathbb{B}_{R_0} into itself and, then by the Banach Fixed Point Theorem there exists a unique $u^\lambda \in \mathbb{B}_{R_0}$ such that $u^\lambda = \Theta^\lambda(u^\lambda)$.

Step 5. By working with another suitable translated coordinate chart on \mathcal{N} (which will remove the dependence on \bar{u}_0^λ) and by defining the inverse scaling $u(\alpha, \beta) := u^\lambda(\lambda\alpha, \lambda\beta)$ for the fixed point u^λ from Step 4, we deduce that

$$\begin{aligned} u(\alpha, \beta) &= \Theta^\lambda \left(u^\lambda \right) (\lambda\alpha, \lambda\beta) \\ &= \eta(\lambda\alpha)\eta(\lambda\beta) \left([S(\chi(\lambda)u_0, \chi(\lambda)u_1)] + [\diamond^{-1}\mathcal{N}(u)] + [\diamond^{-1}\sigma(u)\Xi_{\alpha\beta}] \right). \end{aligned}$$

Hence we conclude the proof of Theorem 6. □

We complete our study of local theory with the following Theorem.

Theorem 7. *Under the above mentioned assumptions, there exist a open set \mathcal{O} , containing the diagonal $\mathcal{D} := \{(\alpha, -\alpha) : \alpha \in \mathbb{R}\}$, and a function $u : \mathcal{O} \rightarrow \mathbb{R}^2$ such that for every $(\alpha_0, -\alpha_0) \in \mathcal{D}$, there exists $r > 0$ such that $u|_{B_r((\alpha_0, -\alpha_0))} \in \mathbb{H}^{s, \delta}$, where $B_r((\alpha, -\alpha))$ is open ball of radius r around $(\alpha, -\alpha)$, and u solves (3)-(4) uniquely in \mathcal{O} .*

To prove Theorem 7, for each fixed point $(\alpha_0, -\alpha_0) \in \mathcal{D}$, by Theorem 6 we find a unique solution u_{α_0} of a translated version of the problem (5) defined in some neighbourhood N_{α_0} of $(\alpha_0, -\alpha_0)$. By using the uniqueness we can glue “local” solutions to get a solution u as in the assertion.

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