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Plethysm and a character embedding problem
of Miller

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Abstract. We use a plethystic formula of Littlewood to answer a question of Miller on embeddings of symmetric group characters. We also reprove a result of Miller on character congruences.

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Given $d \geq 1$ and a partition $\lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots)$ of a positive integer $n$, let $\boxplus^d(\lambda)$ be the partition of $d^2 \cdot n$ given by $\boxplus^d(\lambda) := (d^{dm_1}(2d)^{dm_2}(3d)^{dm_3}\cdots)$. The Young diagram of $\boxplus^d(\lambda)$ is obtained from that of $\lambda$ by subdividing every box into a $d \times d$ grid, as suggested by the notation.

Let $S_n$ be the symmetric group on $n$ letters. For a partition $\lambda \vdash n$, let $V^\lambda$ be the corresponding $S_n$-irreducible with character $\chi^\lambda: S_n \rightarrow \mathbb{C}$. For $d \geq 1$, define a new class function $\boxplus^d(\chi^\lambda)$ on $S_n$ whose value on permutations of cycle type $\mu \vdash n$ is given by

$$\boxplus^d(\chi^\lambda)_\mu := \chi^\lambda_{\boxplus^d(\mu)}.$$  

Thus, the values of the class function $\boxplus^d(\chi^\lambda)$ on $S_n$ are embedded inside the character table of the larger symmetric group $S_{d^2 \cdot n}$. A. Miller conjectured [4] that the class functions $\boxplus^d(\chi^\lambda)$ are genuine characters of (rather than merely class functions on) $S_n$. We prove that this is so in Theorem 1 using plethysm of symmetric functions.

In the arguments that follow, we use standard material on symmetric functions; for details see [3]. For $\mu \vdash n$, let $m_i(\mu)$ be the multiplicity of $i$ as a part of $\mu$ and $z_\mu := 1^{m_1(\mu)}2^{m_2(\mu)}\cdots m_1(\mu)!m_2(\mu)\cdots$ be the size of the centralizer of a permutation $w \in S_n$ of cycle type $\mu$.

Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ be the ring of symmetric functions in an infinite variable set $(x_1,x_2,\ldots)$. Bases of $\Lambda$ are indexed by partitions; we use the Schur basis $\{s_\lambda\}$ and power sum basis $\{p_\lambda\}$. The basis $p_\lambda$ is multiplicative: if $\lambda = (1,\lambda_2,\ldots)$ then $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. The transition matrix from the Schur to the power sum basis encodes the character table of $S_n$; for $\lambda \vdash n$ we have

$$s_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda_\mu}{z_\mu} p_\mu.$$  

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Let \( \langle \cdot, \cdot \rangle \) be the Hall inner product on \( \Lambda \) with respect to which the Schur basis \( \{s_\lambda\} \) is orthonormal. The power sums are orthogonal with respect to this inner product. We have \( \langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \) where \( \delta \) is the Kronecker delta.

Write \( R = \oplus_{n \geq 0} R_n \) where \( R_n \) is the space of class functions \( \varphi : S_n \to \mathbb{C} \). The characteristic map \( \text{ch}_n : R_n \to \Lambda_n \) is given by \( \text{ch}_n(\varphi) = \frac{1}{n!} \sum_{w \in S_n} \varphi(w) \cdot \text{cyc}(w) \) where \( \text{cyc}(w) \) is the cycle type of \( w \in S_n \). The map \( \text{ch} = \bigoplus_{n \geq 0} \text{ch}_n \) is a linear isomorphism \( R \to \Lambda \). The space \( R \) has an induction product given by \( \varphi \otimes \psi := \text{Ind}_{S_n}^{S_m} (\varphi \otimes \psi) \) for all \( \varphi \in R_n \) and \( \psi \in R_m \). Under this product, the map \( \text{ch} : R \to \Lambda \) becomes a ring isomorphism. We record two properties of \( \text{ch} \).

- We have \( \text{ch}(x^A) = s_A \), so that \( \text{ch} \) sends the irreducible character basis of \( R \) to the Schur basis of \( \Lambda \).
- If \( \varphi : S_n \to \mathbb{C} \) is any class function and \( \mu \vdash n \), then
  \[ \langle \text{ch}(\varphi), p_\mu \rangle = \text{value of } \varphi \text{ on a permutation of cycle type } \mu. \]

Let \( \psi^d : \Lambda \to \Lambda \) be the map \( \psi^d : F(x_1, x_2, \ldots) \to F(x_1^d, x_2^d, \ldots) \) which replaces each variable \( x_i \) with its \( d \)-th power \( x_i^d \). The symmetric function \( \psi^d(F) \) is the plethysm \( p_d[F] \) of \( F \) into the power sum \( p_d \). Let \( \phi_d : \Lambda \to \Lambda \) be the adjoint of \( \psi^d \) characterized by \( \langle \psi^d(F), G \rangle = \langle F, \phi_d(G) \rangle \) for all \( F, G \in \Lambda \). In this note we apply the operators \( \psi^d \) and \( \phi_d \) to character theory; see [6] for an application to the cyclic sieving phenomenon of enumerative combinatorics.

**Theorem 1.** Let \( d \geq 1 \) and \( \mu \vdash n \). Consider the chain of subgroups \( \Delta(S_n) \subseteq S_n^d \subseteq S_{dn} \) where \( S_n^d = S_n \times \cdots \times S_n \) is the \( d \)-fold self-product of \( S_n \) and \( \Delta(S_n) \) is the diagonal \( \{(w, \ldots, w) : w \in S_n\} \) in \( S_n^d \). Then \( \boxplus^d(x^A) \) is the character of the \( \Delta(S_n) \) module
\[ \text{Res}^{S_{dn}}_{\Delta(S_n)} \left( V^A \otimes \cdots \otimes V^A \right) \]
obtained by restricting the \( d \)-fold induction product \( V^A \circ \cdots \circ V^A = \text{Ind}^{S_{dn}}_{S_n} (V^A \otimes \cdots \otimes V^A) \) to \( \Delta(S_n) \).

**Proof.** Let \( \lambda, \mu \vdash n \) be two partitions and let \( d \geq 1 \). By (2) we have the class function value
\[ \chi_{\boxplus^d(\lambda)}^{\boxplus^d(\mu)} = \langle s_{\boxplus^d(\lambda)}, p_{\boxplus^d(\mu)} \rangle = \langle s_{\boxplus^d(\lambda)}, \psi^d(p_\mu) \rangle = \langle \phi_d(s_{\boxplus^d(\lambda)}), p_\mu^d \rangle. \]
Littlewood [2, p. 340] proved (see also [1, Equation 13]) that for any partition \( \nu \vdash dm \), with empty \( d \)-core, the image \( \phi_d(s_\nu) \) is given by
\[ \phi_d(s_\nu) = \epsilon_d(\nu) \cdot s_{\nu^{(1)}} \cdots s_{\nu^{(d)}} \]
where \( \epsilon_d(\nu) \) is the \( d \)-sign of \( \nu \) and \( (\nu^{(1)}, \ldots, \nu^{(d)}) \) is the \( d \)-quotient of \( \nu \). We refer the reader to [1, 2] for definitions. In our context we have \( \epsilon_d(\boxplus^d(\lambda)) = +1 \) (since \( \boxplus^d(\lambda) \) admits a \( d \)-ribbon tiling with only horizontal ribbons) and the \( d \)-quotient of \( \boxplus^d(\lambda) \) is the constant \( d \)-tuple \( (\lambda, \ldots, \lambda) \). Equation (5) reads
\[ \phi_d(s_{\boxplus^d(\lambda)}) = s_\lambda^d. \]
Plugging (6) into (4) gives
\[ \chi_{\boxplus^d(\mu)}^{\boxplus^d(\lambda)} = \langle \phi_d(s_{\boxplus^d(\lambda)}), p_\mu^d \rangle = \langle s_\lambda^d, p_\mu^d \rangle \]
which (thanks to (2)) agrees with the trace of \( (w, \ldots, w) \in \Delta(S_n) \) on \( V^A \circ \cdots \circ V^A \) for \( w \in S_n \) of cycle type \( \mu \).

If \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a partition, let \( d \cdot \lambda = (d \lambda_1, d \lambda_2, \ldots) \) be the partition obtained by multiplying every part of \( \lambda \) by \( d \). The argument proving Theorem 1 applies to show that for \( \mu \vdash n \), the class function \( \chi_{\boxplus^d(\lambda)} : S_n \to \mathbb{C} \) given by \( (\chi_{\boxplus^d(\lambda)})_\mu := \chi_{d \cdot \lambda}^\mu \) is a genuine character (although its module does not have such a nice description). It may be interesting to find other ways to discover characters of \( S_n \) embedded inside characters of larger symmetric groups.

In closing, we use plethysm to give a quick proof of a character congruence result of Miller [5, Thm. 1]. Miller gave an interesting combinatorial proof of the following theorem by introducing objects called “cascades”.

**Theorem 2.** (Miller) Let \( d \geq 1 \). For any partitions \( \lambda \vdash n \) and \( \mu \vdash d n \), we have
\[
\chi_d^{\psi_d(\lambda)} \equiv 0 \mod d!.
\]  
Furthermore, suppose \( \lambda, \nu \vdash n \) with \( d \nmid n \). Then
\[
\chi_d^{\psi_d(\lambda)} = 0.
\]

**Proof.** Arguing as in the proof of Theorem 1, we have
\[
\chi_d^{\psi_d(\lambda)} = \left\langle \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho}, p_\mu \right\rangle = \left\langle \phi_d \left( \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho} \right), p_\mu \right\rangle
\]
where the last equality used Equation (6). We have \( s_\lambda = \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho} p_\rho \) so that
\[
\chi_d^{\psi_d(\lambda)} = \left\langle s_\lambda^d, p_\mu \right\rangle = \left\langle \left( \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho} \right)^d, p_\mu \right\rangle.
\]
We expand far right of (11) using the orthogonality of the \( p \)'s to obtain
\[
\left\langle \left( \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho} \right)^d, p_\mu \right\rangle = \sum_{(\mu_1, \ldots, \mu(d))} \frac{z_\mu}{z_{\mu_1} \cdots z_{\mu(d)}} \chi_{\mu_1}^\lambda \cdots \chi_{\mu(d)}^\lambda
\]
where the sum is over all \( d \)-tuples \( (\mu_1), \ldots, \mu(d) \) of partitions of \( n \) whose multiset of parts equals \( \mu \). In particular, (12) is zero unless every part of \( \mu \) is \( \leq n \); we assume this going forward. We want to show that (12) is divisible by \( d! \). To show this, we examine what happens when some of the entries in a tuple \( (\mu_1), \ldots, \mu(d) \) coincide.

Fix a \( d \)-tuple \( (\mu_1), \ldots, \mu(d)) \) of partitions of \( n \) whose multiset of parts is \( \mu \). The ratio of \( z \)'s in the corresponding term on the RHS of (12) is a product of multinomial coefficients
\[
\frac{z_\mu}{z_{\mu_1} \cdots z_{\mu(d)))} = \left( \frac{m_1(\mu)}{m_1(\mu_1)}, \ldots, \frac{m_n(\mu)}{m_n(\mu_1)}, \ldots, \frac{m_n(\mu)}{m_n(\mu_1)}, \ldots, \frac{m_n(\mu)}{m_n(\mu(d))} \right).
\]

Let \( \sigma = (\sigma_1, \ldots, \sigma_r) \vdash d \) be the partition of \( d \) obtained by writing the entry multiplicities in the \( d \)-tuple \( (\mu_1), \ldots, \mu(d)) \) in weakly decreasing order. For example, if \( n = 3, d = 5 \), and our \( d \)-tuple of partitions of \( n \) is \( (\mu_1), \ldots, \mu(3)) = ((2, 1), (3), (1, 1, 1), (3), (2, 1)) \), then \( \sigma = (2, 2, 1) \). Each multinomial coefficient in (13) for which \( m_i(\mu) > 0 \) is divisible by \( \sigma_1 \cdots \sigma_r \). Since each part of \( \mu \) is \( \leq n \), at least one \( m_i(\mu) > 0 \) and the whole product \( (\mu_1), \ldots, \mu(d)) \) of multinomial coefficients is divisible by \( \sigma_1 \cdots \sigma_r \). Thus, the sum of the terms in (12) indexed by rearrangements of \( (\mu_1), \ldots, \mu(d)) \) is divisible by \( \left( \sigma_1, \ldots, \sigma_r \right) \), \( \sigma_1 \cdots \sigma_r = d! \), so that (12) itself is divisible by \( d! \). This proves the first part of the theorem.

For the second part of the theorem, let \( \lambda, \nu \vdash n \) where \( d \nmid n \). Arguing as above, we have
\[
\chi_d^{\psi_d(\lambda)} = \left\langle \left( \sum_{\rho \vdash n} \frac{\chi_\rho^\lambda}{z_\rho} \right)^d, p_{d \cdot \nu} \right\rangle
\]
Since \( d \nmid n \), each partition \( \rho \vdash n \) appearing in the first argument of the inner product in (14) has at least one part not divisible by \( d \). Since the \( p \)'s are an orthogonal basis of \( \Lambda \), we see that (14) = 0, proving the second part of the theorem.

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