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On a congruence involving \( q \)-Catalan numbers

Sur une congruence impliquant des \( q \)-nombres de Catalan

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Abstract. Based on a \( q \)-congruence of the author and Petrov, we set up a \( q \)-analogue of Sun–Tauraso’s congruence for sums of Catalan numbers, which extends a \( q \)-congruence due to Tauraso.

Résumé. À partir d’une \( q \)-congruence de l’auteur et Petrov, nous établissons un \( q \)-analogue de la congruence de Sun–Tauraso pour des sommes de nombres de Catalan, qui étend la \( q \)-congruence due à Tauraso.

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1. Introduction

In combinatorics, the Catalan numbers are a sequence of natural numbers, which play an important role in various counting problems. The \( n \)th Catalan number is given by the following binomial coefficient:

\[
C_n = \binom{2n}{n} \frac{1}{n + 1} = \binom{n}{n} - \binom{2n}{2n + 1}.
\]

Closely related numbers are the central binomial coefficients \( \binom{2n}{n} \) for \( n \geq 0 \).

Both Catalan numbers and central binomial coefficients satisfy many interesting congruences (see, for instance, [7, 9–11]). In 2011, Sun and Tauraso [11] proved that for primes \( p \geq 5 \),

\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left( \frac{p}{3} \right) \pmod{p^2},
\]

\[
\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left( \frac{p}{3} \right) - \frac{1}{2} \pmod{p^2},
\]

(1)

(2)
where \( \left\lfloor \frac{n}{k} \right\rfloor \) denotes the Legendre symbol.

In the past few years, \( q \)-analogues of congruences (\( q \)-congruences) for indefinite sums of binomial coefficients as well as hypergeometric series attracted many experts’ attention (see, for example, [2–6,8,12,13]). It is worth mentioning that Guo and Zudilin [6] developed an interesting microscoping method to prove many \( q \)-congruences.

In order to discuss \( q \)-congruences, we first recall some \( q \)-series notation. The \( q \)-binomial coefficients are defined as

\[
\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},
\]

where \( |q| < 1 \) and \( (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \) for \( n \geq 1 \) and \( (a;q)_0 = 1 \). Moreover, the \( q \)-integers are defined by \( [n]_q = (1-q^n)/(1-q) \), and the \( n \)th cyclotomic polynomial is given by

\[
\Phi_n(q) = \prod_{1 \leq k \leq n \atop (n,k)=1} (q-\bar{e}^{2\pi i/n}).
\]

Recently, the author and Petrov [8] established a \( q \)-analogue for (1) as follows:

\[
\sum_{k=0}^{n-1} q^k \binom{2k}{k}_q \equiv \frac{n}{3} q^{\frac{n^2-1}{3}} \pmod{\Phi_n(q)^2},
\]

which was originally conjectured by Guo [2] and generalises a \( q \)-congruence of Tauraso [12]. There are several natural \( q \)-analogues of Catalan numbers (see [1]). Here and throughout the paper, we consider the following \( q \)-analogue of Catalan numbers:

\[
C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q = \frac{2n}{n} - q \left\lfloor \frac{2n}{n+1} \right\rfloor.
\]

In 2012, Tauraso [12] obtained a weak \( q \)-version of (2) as follows:

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} 
q^{-\frac{n^2-1}{3}} & \text{if } n \equiv 0,1 \pmod{3} \\
1 - q^{-\frac{n^2-1}{3}} & \text{if } n \equiv 2 \pmod{3}
\end{cases} \pmod{\Phi_n(q)},
\]

where \( [x] \) denotes the integral part of real \( x \). In this note, we aim to set up a \( q \)-analogue of (2) as well as another related \( q \)-congruence for sums of binomial coefficients.

**Theorem 1.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} 
-q^{-\frac{n^2-1}{3}} & \text{if } n \equiv 0,1 \pmod{3} \\
q^{-\frac{n^2-1}{3}} - q^{-\frac{n^2}{3}} & \text{if } n \equiv 2 \pmod{3}
\end{cases} \pmod{\Phi_n(q)}.
\]

In order to prove (5), we shall establish the following \( q \)-congruence.

**Theorem 2.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^k \frac{2k}{k+1} \equiv \begin{cases} 
q^{-\frac{n^2}{3}} & \text{if } n \equiv 0,1 \pmod{3} \\
q^{-\frac{n^2}{3}} - q^{-\frac{n^2}{3}} & \text{if } n \equiv 2 \pmod{3}
\end{cases} \pmod{\Phi_n(q)}.
\]

It is clear that (5) can be directly deduced from (3), (4) and (6). The remainder of the paper is organized as follows. We first set up a preliminary result in the next section, and prove Theorem 2 in Section 3.
2. An auxiliary result

**Lemma 3.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q) \):

\[
\sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (-1)^k q^{\frac{1}{2}(2k^2-k\left(\frac{k-1}{3}\right))} \frac{k^{k-1}}{1-q^k} \equiv \begin{cases} 0 & \text{if } n \equiv 2 \pmod{3}, \\ n^{-1} & \text{if } n \equiv 1 \pmod{3}. \end{cases}
\]  

(7)

**Proof.** Note that

\[
\sum_{k=1}^{n-1} (-1)^k \left(\frac{k-1}{3}\right) q^{\frac{1}{2}(2k^2-k\left(\frac{k-1}{3}\right))} \frac{k^{k-1}}{1-q^k} = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k q^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{\lfloor n/3 \rfloor} (-1)^k q^{\frac{k(3k+5)}{2}}.
\]

We shall distinguish two cases to prove (7).

**Case 1.** \( n \equiv 2 \pmod{3} \). This case is equivalent to

\[
\sum_{k=0}^{n-1} (-1)^k q^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{n} (-1)^k q^{\frac{k(3k+5)}{2}} \equiv 0 \pmod{\Phi_{3n+2}(q)}.
\]

(8)

Let \( \omega \) be a primitive \((3n+2)\)th root of unity. Letting \( k \to n-k \) in the following sum gives

\[
\sum_{k=0}^{n-1} (-1)^k \omega^{\frac{k(3k+5)}{2}} = n \sum_{k=1}^{n-1} (-1)^{n-k} \omega^{\frac{n^2-k(3n-3k+2)}{2}} = \sum_{k=1}^{n} (-1)^{n-k} \omega^{\frac{k(3k-1)}{2}} \cdot \frac{3(n+1)(n+2)-3(n+2)k}{2} = \sum_{k=1}^{n} (-1)^k \omega^{\frac{k(3k+5)}{2}},
\]

where we have used the fact that \( \omega^{\frac{(3n+2)(n+1)}{2}} = (-1)^{n+1} \). Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \omega^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{n} (-1)^k \omega^{\frac{k(3k+5)}{2}} = 0,
\]

which is equivalent to (8).

**Case 2.** \( n \equiv 1 \pmod{3} \). Let \( \zeta \) be a primitive \((3n+1)\)th root of unity. It suffices to show that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{n} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = \frac{n}{2}.
\]

(9)

Note that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = \sum_{k=n+1}^{2n} (-1)^{2n-k} \zeta^{\frac{(2n-k)(6n-3k+2)}{2}} = \sum_{k=n+1}^{2n} (-1)^k \zeta^{\frac{k(3k-1)}{2}} \cdot \frac{(3n+1)(2n-2k+1)}{2} = \sum_{k=n+1}^{2n} (-1)^k \zeta^{\frac{k(3k+5)}{2}} - \sum_{k=n+1}^{2n} (-1)^k \zeta^{\frac{k(3k+5)}{2}}.
\]

where we replace \( k \) by \( 2n-k \) in the first step. Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{n} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = - \sum_{k=1}^{n} (-1)^k \zeta^{\frac{k(3k+5)}{2}}.
\]

(10)
Furthermore, letting $k \to 2n + 1 - k$ on the right-hand side of (10) gives

$$
\sum_{k=0}^{n-1} \frac{(-1)^k \zeta^{(k+1)(3k+2)}}{1 - \zeta^{3k+2}} - \sum_{k=1}^{n} \frac{(-1)^k \zeta^{(3k+5)}}{1 - \zeta^{3k}} = - \sum_{k=1}^{2n} \frac{(-1)^{2n+1-k} \zeta^{(2n+1-k)(6n-3k+8)}}{1 - \zeta^{3(2n+1-k)}} \\
= - \sum_{k=1}^{2n} \frac{(-1)^{1-k} \zeta^{(k-1)(3k-2)}}{1 - \zeta^{1-3k}} \\
= - \sum_{k=1}^{2n} \frac{(-1)^k k^{(3k-1)}}{1 - \zeta^{3k-1}}. \tag{11}
$$

An identity due to the author and Petrov [8, (2.4)] says

$$
\sum_{k=1}^{2n} \frac{(-1)^k \zeta^{k(3k-1)}}{1 - \zeta^{3k-1}} = \frac{n}{2}. \tag{12}
$$

Then the proof of (9) follows from (11) and (12). \qquad \square

3. Proof of Theorem 2

Now we are in a position to prove Theorem 2. We recall the following identity:

$$
\sum_{k=0}^{n-1} q^k \left[ \frac{2k}{k+1} \right] = \sum_{k=0}^{n-1} \left( \frac{n-k-1}{3} \right) q^{\frac{1}{2}(2(n-k)^2-(n-k)(\frac{n-k-1}{2})-3) \left[ \frac{2n}{k} \right], \tag{13}
$$

which was proved by Tauraso in a more general form (see [12, Theorem 4.2]). Since $1 - q^n \equiv 0 \pmod{\Phi_n(q)}$, we have

$$
1 - q^{2n} = (1 + q^n)(1 - q^n) \equiv 2(1 - q^n) \pmod{\Phi_n(q)^2}.
$$

It follows that for $1 \leq k \leq n - 1$,

$$
\left[ \frac{2n}{k} \right] = \frac{(1 - q^{2n})(1 - q^{2n-1}) \cdots (1 - q^{2n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)} \\
\equiv 2(1 - q^n) \frac{(1 - q^{-1}) \cdots (1 - q^{-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)} \pmod{\Phi_n(q)^2} \\
= 2(q^n - 1) \frac{(-1)^k q^{-\frac{k(k-1)}{2}}}{1 - q^k}. \tag{14}
$$

Multiplying both sides of (13) by $q$ and substituting (14) into the right-hand side of (13), we arrive at

$$
\sum_{k=0}^{n-1} q^{k+1} \left[ \frac{2k}{k+1} \right] \\
= \left( -\frac{n-1}{3} \right) q^{\frac{1}{2}(2n^2-n(\frac{n-1}{2}))} + \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) q^{\frac{1}{2}(2(n-k)^2-(n-k)(\frac{n-k-1}{2})-3)} \left[ \frac{2n}{k} \right] \\
\equiv \left( -\frac{n-1}{3} \right) q^{\frac{1}{2}(2n^2-n(\frac{n-1}{2}))} \\
+ 2(q^n - 1) \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) (-1)^k q^{\frac{1}{2}(2(n-k)^2-(n-k)(\frac{n-k-1}{2})-\frac{k(k-1)}{2})} \pmod{\Phi_n(q)^2}. \tag{15}
$$
Furthermore,
\[
\sum_{k=1}^{n-1} \frac{(-1)^k q^{\frac{1}{3} \left(2(n-k)^3-(n-k)(n-k-1)(n-k-2)/2\right)}}{1-q^k} = \sum_{k=1}^{n-1} \frac{(-1)^{n-k} q^{\frac{1}{3} \left(2k^3-k\left(k^2+1\right)/3\right)-\frac{1}{2}k(k+1)(n-k)}}{1-q^{n-k}} = \sum_{k=1}^{n-1} \frac{(-1)^{n-k} q^{\frac{1}{3} \left(2k^3-k\left(k^2+1\right)/3\right)-\frac{1}{2}k(k+1)+nk}}{1-q^{n-k}} = \sum_{k=1}^{n-1} \frac{(-1)^{k+1} q^{\frac{1}{3} \left(2k^3-k\left(k^2+1\right)/3\right)-\frac{1}{2}k(k+1)}}{1-q^k} \pmod{\Phi_n(q)},
\]
where we set \(k \rightarrow n-k\) in the first step. Thus,
\[
\sum_{k=0}^{n-1} q^{k+1} \left[ \frac{2k}{k+1} \right] = \left( \frac{n-1}{3} \right) q^{\frac{1}{3} \left(2n^2-n\left(n+1\right)/3\right)} + 2(q^n-1) \sum_{k=1}^{n-1} \frac{(-1)^k q^{\frac{1}{3} \left(2k^3-k\left(k^2+1\right)/3\right)-\frac{1}{2}k(k+1)}}{1-q^k} \pmod{\Phi_n(q)^2}. \tag{16}
\]
We complete the proof of (6) by combining (7) and (16).

References