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Uniqueness theorem for a coupled system of wave equations with incomplete internal observation and application to approximate controllability

Théorème d'unicité pour un système d'équations des ondes avec observation interne incomplète et application à la contrôlabilité approachée

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Abstract. We show that a Kalman rank condition is necessary and sufficient for the uniqueness of solution to a system of wave equations associated with incomplete internal observation without any restriction neither on the controlled subregion nor on the coupling matrices. The obtained result can be applied to the approximate internal controllability of the corresponding system.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ of class C^1 and $\omega \subset \Omega$ be a subregion of Ω. Let A be a matrix of order N and D be a full column-rank matrix of order $N \times M$. The goal of this paper is to study the uniqueness of solution to a coupled system of wave equations for the variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$:

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } (0,T)\times\Omega, \\ \Phi = 0 & \text{on } (0,T)\times\Gamma \end{cases} \tag{1}$$

associated with the initial data

$$t = 0: \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1 \quad \text{in } \Omega \tag{2}$$

and the internal observation

$$D^T\chi_\omega\Phi \equiv 0 \quad \text{in } (0,T)\times\Omega, \tag{3}$$

where "T" stands for the transpose of a matrix, and χ_ω is the characteristic function of the subregion $\omega \subset \Omega$.

By classic theory of semigroups [24], system (1) forms a C^0-semigroup in the space $(L^2(\Omega) \times H^{-1}(\Omega))^N$.

We will show in Proposition 5 that the following Kalman rank condition:

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N \tag{4}$$

is necessary for the uniqueness of solution to problem (1)-(2) associated with the internal D-observation (3). This kind of uniqueness could be regarded as a weak observability of system (1) by means of the incomplete internal D-observation (3). However, since the matrix D is not invertible in general, the incomplete internal D-observation (3) can not imply the nullity of all the components:

$$\chi_\omega\Phi = 0 \quad \text{in } (0,T)\times\Omega. \tag{5}$$

Therefore, the uniqueness of solution to the previous problem does not correspond to a standard unique continuation of Holmgren's type. Fortunately, because of the commutation of the internal observation with the d'Alembert operator:

$$D^T\chi_\omega\Box\Phi = \Box D^T\chi_\omega\Phi \quad \text{in } \mathscr{D}'((0,T)\times\omega), \tag{6}$$

Kalman rank condition (4) formally plays the same role as in the case of ODEs [11]. More precisely, we have

Theorem 1. *Assume that* (A, D) *satisfies Kalman rank condition* (4). *Then for any given initial data* $(\Phi_0, \Phi_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, *problem* (1)-(2) *associated with the internal D-observation* (3) *has only the trivial solution, provided that* $T > 2d(\Omega)$, *where* $d(\Omega)$ *is the geodesic diameter of* Ω.

Now let us recall some known results on the uniqueness of solution to problem (1)-(2) associated with the boundary D-observation

$$D^T\partial_\nu\Phi \equiv 0 \quad \text{on } (0,T)\times\Gamma. \tag{7}$$

Kalman rank condition (4) is still necessary for the uniqueness of solution to problem (1)-(2) associated with the boundary D-observation (7). Similarly to the internal case, the matrix D is not invertible in general and the incomplete boundary D-observation (7) can not imply the nullity of all the components:

$$\partial_\nu\Phi \equiv 0 \quad \text{in } (0,T)\times\Gamma. \tag{8}$$

Therefore, only Kalman rank condition (4) is not sufficient for the uniqueness of solution (see [14] and [16, Theorem 8.11]). In order to obtain the desired uniqueness of solution, our basic idea is to combine the uniform observability of a scalar wave system in (1):

$$\begin{cases} \phi'' - \Delta\phi = 0 & \text{in } (0,T)\times\Omega, \\ \phi = 0 & \text{on } (0,T)\times\Gamma \end{cases} \tag{9}$$

and Kalman rank condition (4). The first attempts for realizing this idea were carried out in [14–16]. More precisely, we have

Theorem 2 ([3] and [15]). *Assume that Ω satisfies the usual geometrical control condition, and that A is a cascade, or more generally, a nilpotent matrix. Assume furthermore that the pair (A, D) satisfies Kalman rank condition* (4)*. Then, for any given initial data $(\Phi_0, \Phi_1) \in (H_0^1(\Omega)\times L^2(\Omega))^N$, problem* (1)-(2) *associated with the boundary D-observation* (7) *has only the trivial solution, provided that the time of observation T is large enough.*

In order to broaden the scope of application, we extended in [17] the observation (7) from a finite interval $(0,T)$ to the infinite horizon $(0,+\infty)$. By duality (see [17] for details), we are led to consider the uniqueness of solution to the following elliptic system with a real parameter β for the state variable $\Psi = (\psi^{(1)},\dots,\psi^{(N)})^T$:

$$\begin{cases} \Delta\Psi + A^T\Psi = \beta^2\Psi & \text{in } \Omega, \\ \Psi = 0 & \text{on } \Gamma \end{cases} \tag{10}$$

associated with the boundary D-observation

$$D^T\partial_\nu\Psi \equiv 0 \quad \text{on } \Gamma. \tag{11}$$

Theorem 3 ([17, Theorem 5.3]). *Assume that Ω satisfies the geometrical control condition and that there exists a positive constant a such that $\|A - aI\|$ is small enough. Assume furthermore that the pair (A, D) satisfies Kalman rank condition* (4)*. Then, system* (10) *associated with the boundary D-observation* (11) *has only the trivial solution.*

We point out that Theorem 1 does not need any condition neither on Ω nor on the matrix A, only Kalman rank condition (4) is sufficient for the uniqueness of solution. Moreover, the time of observation is uniquely determined by the geodesic diameter of Ω. This is essentially different from the case of boundary observation discussed in Theorem 2, where the usual geometrical control condition is required for establishing the uniform observability for the scalar system (9). Only Kalman rank condition (4) can not provide sufficient information on the connection of the equations in (1) with the boundary. This is why some additional spectral condition, for example, the matrix A is nilpotent, should be required. This condition is not necessary, but technically indispensable in the proof. Moreover, the time of observation T depends on the spectral density of $-\Delta + A$, the rank of D and the number N of equations. When Ω is an interval [14] or a spherical domain [28], it can be explicitly given by

$$T > (N - rank(D) + 1)\pi. \tag{12}$$

This is the main weakness in the case of boundary observation.

The usual geometrical control condition and the smallness of $\|A - aI\|$ are still required in Theorem 3 because of the perturbation argument in the proof.

Now we comment the related literature. Unlike the hyperbolic system, which was less studied and the obtained results are of different natures (see [2, 8, 23, 25]), the parabolic problem has been abundantly investigated in the literature. We only quote [4, 5] and the references therein for the internal control of coupled systems of heat equations with the same diffusion coefficients and constant or time-dependent coupling terms by means of Carleman estimates. We also mention [21] for the internal controllability of a system of heat equations with analytic nonlocal coupling terms and [22] for the internal observability of some parabolic equations with constant or time-dependent coupling terms using Lebeau–Robbiano strategy. The optimal control for the exact synchronization of parabolic system was recently investigated in [27]. We quote [1, 9] for the synchronization of distributed parameter systems on networks.

In what follows, we will further develop Theorem 1 and give its application to the corresponding approximate internal controllability.

We refer to the complete version [13] for the related approximate internal synchronization.

2. Uniqueness theorem under Kalman rank condition

We first recall the following fundamental result.

Lemma 4 ([14, Lemma 3.1]).

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N - d \tag{13}$$

where d is the largest dimension of the subspaces which are contained in $\operatorname{Ker}(D^T)$ *and invariant for A^T.*

Proposition 5. *For any given initial data $(\Phi_0, \Phi_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, if problem* (1)-(2) *associated with the internal D-observation* (3) *has only the trivial solution, then the pair (A, D) should satisfy Kalman rank condition* (4).

Proof. Assume that Kalman rank condition (4) fails. Then, by Lemma 4, there exist a unit vector E and $a \in \mathbb{C}$, such that

$$D^T E = 0, \quad A^T E = aE.$$

Let ϕ be a solution to the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \mu^2\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases}$$

Let w satisfy

$$w'' + (\mu^2 + a)w = 0, \quad -\infty < t < \infty.$$

Setting $\Phi = w\phi E$, we easily check that

$$\Phi'' - \Delta\Phi + A^T\Phi = (w'' + (\mu^2 + a)w)\phi E = 0$$

and

$$D^T\Phi = w\phi D^T E = 0 \quad \text{in } \mathbb{R} \times \Omega.$$

We find thus a non-trivial solution to the overdetermined problem (1)–(3), which leads to a contradiction. □

Inversely, we have

Theorem 6. *Assume that (A, D) satisfies Kalman rank condition* (4). *Then for any given initial data $(\Phi_0, \Phi_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, problem* (1)–(2) *associated with the internal D-observation* (3) *has only the trivial solution, provided that $T > 2d(\Omega)$, where $d(\Omega)$ is the geodesic diameter of Ω.*

Proof. Let $\Phi \in C^0(\mathbb{R}; L^2(\Omega)) \cap C^1(\mathbb{R}; H^{-1}(\Omega))$ be a solution to the overdetermined problem (1)–(3). Since $\chi_\omega \equiv 1$ in ω, the d'Alembert operator $\Box = \partial_{tt} - \Delta$ commutes with the internal D-observation:

$$D^T\chi_\omega\Box\Phi = \Box D^T\chi_\omega\Phi \quad \text{in } \mathscr{D}'((0,T)\times\omega). \tag{14}$$

Then, applying $D^T\chi_\omega$ to (1), we get

$$\Box D^T\chi_\omega\Phi + D^T A^T\chi_\omega\Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega).$$

Noting (3), we get

$$D^T A^T \chi_\omega \Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega). \tag{15}$$

Similarly to (14), we have

$$D^T A^T \chi_\omega \Box \Phi = \Box D^T A^T \chi_\omega \Phi \quad \text{in } \mathscr{D}'((0,T)\times\omega). \tag{16}$$

Then, applying $D^T A^T \chi_\omega$ to (1), we get

$$\Box D^T A^T \chi_\omega \Phi + D^T (A^2)^T \chi_\omega \Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega).$$

Noting (15), we get

$$D^T (A^2)^T \chi_\omega \Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega).$$

Repeating the above procedure, we successively get

$$D^T \chi_\omega \Phi = D^T A^T \chi_\omega \Phi = D^T (A^2)^T \chi_\omega \Phi = \cdots = 0,$$

namely,

$$\chi_\omega \Phi^T (D, AD, \ldots, A^{N-1}D) = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega).$$

Since Kalman matrix $(D, AD, \ldots, A^{N-1}D)$ is of full row-rank, it follows that

$$\Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\omega).$$

Finally, by Holmgren's uniqueness theorem, we get

$$\Phi = 0 \quad \text{in } \mathscr{D}'((0,T)\times\Omega),$$

provided that $T > 2d(\Omega)$, where $d(\Omega)$ denotes the geodesic diameter of Ω (see [10, Theorem 0.1] or [20, Theorem 8.2]). □

Remark 7. Because of relations (14) and (16) etc, what happens for the internal D-observation is almost as in the case of ordinary differential equations (see [11]). This is why Kalman rank condition is also sufficient for the uniqueness without any restriction neither on the matrix A nor on the damping subregion ω.

Remark 8. The observation time $T > 2d(\Omega)$ is independent of rank(D) and of the order N of A. It is exactly the same as for a scalar equation. This is essentially different from the case of boundary observation, where because of the spectral density, the observation time T depends on the rank of D and also on the number N of equations (see (12)).

3. Approximate internal controllability

Now we consider the following problem for the variable $U = (u^{(1)}, \ldots, u^{(N)})^T$:

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{17}$$

with the initial data

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{18}$$

where χ_ω is the characteristic function of a subregion $\omega \subset \Omega$.

Let us first recall the following standard well-posedness result (see [6, 7, 24]).

Proposition 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ of class C^1. For any given $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$ and any given $H \in (L^1_{loc}(\mathbb{R}^+; L^2(\Omega)))^N$, problem* (17)–(18) *admits a unique weak solution U such that*

$$(C^1_{loc}(\mathbb{R}^+; L^2(\Omega)))^N \cap (C^0_{loc}(\mathbb{R}^+; H_0^1(\Omega)))^N. \tag{19}$$

Moreover, the linear mapping

$$(\widehat{U}_0, \widehat{U}_1, H) \to (U, U') \tag{20}$$

is continuous with respect to the corresponding topologies.

The exact internal controllability of wave equations was abundantly studied in the literature. We quote [10, 12, 26] for a single 1-D wave equation with locally distributed control in any fixed non-empty subinterval of a bounded interval. In higher dimensional case, the exact internal controllability was established by HUM method in [20] with control distributed in an ϵ-neighbourhood ω of the boundary Γ satisfying the usual geometrical control condition.

We first show the non-exact controllability in the case of fewer internal controls, namely, when $\operatorname{rank}(D) < N$. This is the motivation to consider the approximate internal controllability under weaker rank condition on D.

Theorem 10. *Assume that* $\operatorname{rank}(D) < N$. *Then no matter how large $T > 0$ is, system* (17) *is not exactly controllable in the space $(H_0^1(\Omega) \times L^2(\Omega))^N$.*

Proof. Let $E \in \mathbb{R}^N$ be a unit vector such that $D^T E = 0$. For any given $\theta \in L^2(\Omega)$, we choose the special initial data as

$$t = 0: \quad U = 0, \quad U' = \theta E. \tag{21}$$

Assume that system (17) is exactly controllable. Let H_0 be the internal control which realizes the exact null controllability with the minimal norm. There exists a positive constant c_1 independent of θ, such that

$$\|H_0\|_{(L^2(0,T;L^2(\Omega)))^M} \leqslant c_1 \|\theta\|_{L^2(\Omega)}. \tag{22}$$

By Proposition 9, there exists a positive constant c_2 independent of θ, such that

$$\|U\|_{(C^0(0,T;(H_0^1(\Omega)))^N} + \|U'\|_{(C^0(0,T;(L^2(\Omega)))^N} \leqslant c_2 \|\theta\|_{L^2(\Omega)}. \tag{23}$$

Now, applying E to (17) and noting $w = (E, U)$, we get the following backward problem:

$$\begin{cases} w'' - \Delta w = -(E, AU) & \text{in } (0,T) \times \Omega, \\ w = 0 & \text{on } (0,T) \times \Gamma, \\ t = T: \quad w = 0, \quad w' = 0 & \text{in } \Omega. \end{cases} \tag{24}$$

First, by Proposition 9, the mapping

$$-(E, AU) \to w \tag{25}$$

is continuous from $L^1(0,T; L^2(\Omega))$ to $C^1(0,T; L^2(\Omega)) \cap C^0(0,T; H_0^1(\Omega))$. On the other hand, using (23) and Theorem 5.1 in [19], the mapping

$$\theta \to -(E, AU) \tag{26}$$

is compact from $L^2(\Omega)$ into $L^2(0,T; L^2(\Omega))$. Then the mapping

$$\theta \to w \tag{27}$$

is compact from $L^2(\Omega)$ into $C^1(0,T; L^2(\Omega)) \cap C^0(0,T; H_0^1(\Omega))$. In particular, noting the initial condition:

$$t = 0: \quad w = 0, \quad w' = \theta \quad \text{in } \Omega, \tag{28}$$

the mapping of identity

$$\theta \to w'(0) = \theta \tag{29}$$

would be compact in $L^2(\Omega)$. We then get a contradiction. □

Definition 11. *System* (17) *is approximately null controllable at the time* $T > 0$ *if for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, *there exists a sequence* $\{H_n\}$ *of internal controls in* $(L^2(\mathbb{R}^+; L^2(\Omega)))^M$ *with compact support in* $[0, T]$, *such that the sequence* $\{U_n\}$ *of corresponding solutions satisfies the following condition*

$$(U_n, U_n') \to (0,0) \quad in\ (C^0_{loc}([T, +\infty); H_0^1(\Omega) \times L^2(\Omega)))^N \tag{30}$$

as $n \to +\infty$.

By duality [13], the approximate null controllability of a coupled system of wave equations can be transformed into the uniqueness of solution to the corresponding adjoint system (1). More precisely, we have

Proposition 12. *System* (17) *is approximately null controllable at the time* $T > 0$ *in the space* $(H_0^1(\Omega) \times L^2(\Omega))^N$ *if and only if for any given initial data* $(\Phi_0, \Phi_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, *system* (1) *associated with the internal observation* (3) *has only the trivial solution.*

As a direct consequence of Theorem 6 and Proposition 12, we have the following result.

Theorem 13. *If system* (17) *is approximately null controllable, then* (A, D) *satisfies Kalman rank condition* (4). *Inversely, under Kalman rank condition* (4), *system* (17) *is approximately null controllable, provided that* $T > 2d(\Omega)$, *where* $d(\Omega)$ *denotes the geodesic diameter of* Ω.

Noting that the rank of D in (4) may be much smaller than N, this is the advantage to consider the approximate internal null controllability. However, the following result shows that the corresponding sequence of internal controls $\{H_n\}$ is unbounded in general.

Proposition 14. *System* (17) *is exactly null controllable at the time* T *in the space* $(H_0^1(\Omega) \times L^2(\Omega))^N$ *by means of an internal control* $H \in (L^2(0, T; L^2(\Omega)))^N$, *if and only if it is approximately null controllable at the time* T *in the space* $(H_0^1(\Omega) \times L^2(\Omega))^N$ *by means of a sequence* $\{H_n\}$ *of internal controls which is bounded in* $(L^2(0, T; L^2(\Omega)))^N$.

Proof. For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, let $\{H_n\}$ be a sequence in $(L^2(0, T; L^2(\Omega)))^M$, which realizes the approximate null controllability of system (17). Let $\{U_n\}$ be the corresponding solution to system (17) with $H = H_n$.

Assume that the sequence $\{H_n\}$ is bounded in $(L^2(0, T; L^2(\Omega)))^M$. Without loss of generality, we may assume that there exists a function H in $(L^2(0, T; L^2(\Omega)))^M$ such that

$$H_n \rightharpoonup H \quad \text{weakly in } (L^2(0, T; L^2(\Omega)))^M. \tag{31}$$

The linear map (20) is also continuous for the corresponding weak topologies, namely, we have

$$(U_n, U_n') \rightharpoonup (U, U') \quad \text{weakly in } (L^2(0, T; H_0^1(\Omega) \times L^2(\Omega)))^N. \tag{32}$$

Now for any given initial data $(\Phi_0, \Phi_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, let Φ be the corresponding solution to the adjoint problem (1)–(2). Multiplying system (17) by Φ, integrating by parts over $[0, t] \times \Omega$ for any given t with $0 < t < T$, we get

$$\langle (U_n(t), U_n'(t)), (\Phi'(t), -\Phi(t)) \rangle = \langle (\widehat{U}_0, \widehat{U}_1), (\Phi_1, -\Phi_0) \rangle + \int_0^t \int_\Omega (D\chi_\omega H_n, \Phi) \mathrm{d}x\mathrm{d}t, \tag{33}$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the duality between the spaces $(H_0^1(\Omega) \times L^2(\Omega))^N$ and $(H^{-1}(\Omega) \times L^2(\Omega))^N$. Passing to the limit in (33) as $n \to +\infty$, and noting (31)–(32), we get

$$\langle (U(t), U'(t)), (\Phi'(t), -\Phi(t)) \rangle = \langle (\widehat{U}_0, \widehat{U}_1), (\Phi_1, -\Phi_0) \rangle + \int_0^t \int_\Omega (D\chi_\omega H, \Phi) \mathrm{d}x\mathrm{d}t. \tag{34}$$

This means that U is the solution to system (17)-(18) associated with the initial data $(\widehat{U}_0, \widehat{U}_1)$ and the internal control H given by the weak limit of the sequence $\{H_n\}$ in (31). In particular, noting (30), we have $U(T) = U'(T) = 0$. System (17) is then exactly null controllable. The other side of the proposition is trivial. The proof is complete. □

4. Comments

Now let us summarize the results obtained in this work. The main result is Theorem 6 on the uniqueness of solution only under Kalman rank condition (4). For the application, Theorem 10 first gives the non-exact controllability when $\text{rank}(D) < N$, which motivates the study on the approximate controllability. Then by Theorem 13, we establish the approximate null controllability of system (17) only under Kalman rank condition (4). However, as shown by Proposition 14, the underlying sequence of controls is unbounded, which is a disadvantage of the approximate null controllability. Finally, we mention some questions to be developed in the future.

1. The study can be similarly carried up for a system of wave equations with Neumann boundary condition

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu U = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{35}$$

2. Consider a system of wave equations with mixed controls:

$$\begin{cases} U'' - \Delta U + AU = D_{in}\chi_\omega H_{in} & \text{in } \mathbb{R}^+ \times \Omega, \\ U = D_{bd} H_{bd} & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{36}$$

We easily check that Kalman rank condition (4) with $D = (D_{in}, D_{bd})$ is necessary for the approximate null controllability, and the sufficiency will be investigated in a forthcoming work.

3. Similarly to [18], we can consider a system of wave equations with Robin condition and mixed controls:

$$\begin{cases} U'' - \Delta U + AU = D_{in}\chi_\omega H_{in} & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu U + BU = D_{bd} H_{bd} & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{37}$$

Assume that there exists a subspace $V = \text{Span}\{E_1, \dots, E_d\}$, which is invariant for A^T and B^T, and contained in $\text{Ker}(D^T)$ with $D = (D_{in}, D_{bd})$. Then, there exist coefficients α_{rs} and β_{rs} such that

$$A^T E_r = \sum_{s=1}^{d} \alpha_{rs} E_s, \quad B^T E_r = \sum_{s=1}^{d} \beta_{rs} E_s, \quad r = 1, \dots, d. \tag{38}$$

Applying E_r to system (37) and setting $u_r = (E_r, U)$, we get a homogeneous system for $r = 1, \dots, d$:

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^{d} \alpha_{rs} u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu u_r + \sum_{s=1}^{d} \beta_{rs} u_s = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{39}$$

which is uncontrollable. But the sufficiency is largely open.

References

[1] L. Aguilar, Y. Orlov, A. Pisano, "Leader-follower synchronization and ISS analysis for a network of boundary-controlled wave PDEs", *IEEE Control Sys. Lett.* **5** (2021), no. 2, p. 683-688.

[2] F. Alabau-Boussouira, "A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems", *SIAM J. Control Optim.* **42** (2003), no. 3, p. 871-903.

[3] ———, "A hierarchic multi-level energy method for the control of bidiagonal and mixed n-coupled cascade systems of PDE's by a reduced number of controls", *Adv. Differ. Equ.* **18** (2013), no. 11-12, p. 1005-1073.

[4] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. González-Burgos, "A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems", *Differ. Equ. Appl.* **1** (2009), no. 3, p. 427-457.

[5] ———, "A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems", *J. Evol. Equ.* **9** (2009), no. 2, p. 267-291.

[6] A. V. Balakrishnan, *Applied Functional Analysis*, Applications of Mathematics, vol. 3, Springer, 1976.

[7] T. Cazenave, A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 13, Clarendon Press, 1998.

[8] B. Dehman, J. Le Rousseau, M. Léautaud, "Controllability of two coupled wave equations on a compact manifold", *Arch. Ration. Mech. Anal.* **211** (2014), no. 1, p. 113-187.

[9] M. A. Demetriou, F. Fahroo, "Optimisation and adaptation of synchronisation controllers for networked second-order infinite-dimensional systems", *Int. J. Control* **92** (2019), no. 1, p. 112-131.

[10] A. Haraux, "A generalized internal control for the wave equation in a rectangle", *J. Math. Anal. Appl.* **153** (1990), no. 1, p. 190-216.

[11] R. E. Kalman, "Contributions to the theory of optimal control", *Bol. Soc. Mat. Mex., II. Ser.* **5** (1960), p. 102-119.

[12] J. Lagnese, "Control of wave processes with distributed controls on a subregion", *SIAM J. Control Optim.* **21** (1983), p. 68-85.

[13] T. Li, B. Rao, "Approximate internal controllability and synchronization of a coupled system of wave equations", in course.

[14] ———, "Criteria of Kalman type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls", *SIAM J. Control Optim.* **54** (2016), no. 1, p. 49-73.

[15] ———, "Kalman criterion on the uniqueness of continuation for the nilpotent system of wave equations", *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 11-12, p. 1188-1192.

[16] ———, *Boundary synchronization for hyperbolic systems*, Progress in Nonlinear Differential Equations and their Applications, vol. 94, Birkhäuser, 2019.

[17] ———, "Uniqueness of solution to systems of elliptic operators and application to asymptotic synchronization of linear dissipative systems", *ESAIM, Control Optim. Calc. Var.* **26** (2020), article no. 117 (26 pages).

[18] ———, "Approximate boundary synchronization by groups for a couples system of wave equationswith coupled Robin boundary conditions", *ESAIM, Control Optim. Calc. Var.* **27** (2021), article no. 10 (30 pages).

[19] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Études mathématiques, Gauthier-Villars, 1969.

[20] ———, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1: Contrôlabilité exacte*, Recherches en Mathématiques Appliquées, vol. 8, Masson, 1988.

[21] P. Lissy, E. Zuazua, "Internal controllability for parabolic systems involving analytic non-local terms", *Chin. Ann. Math., Ser. B* **39** (2018), no. 2, p. 281-296.

[22] ———, "Internal observability for coupled systems of linear partial differential equations", *SIAM J. Control Optim.* **57** (2019), no. 2, p. 832-853.

[23] Z. Liu, B. Rao, "A spectral approach to the indirect boundary control of a system of weakly coupled wave equations", *Discrete Contin. Dyn. Syst.* **23** (2009), no. 1-2, p. 399-413.

[24] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer, 1982.

[25] L. Rosier, L. de Teresa, "Exact controllability of a cascade system of conservative equations", *C. R. Math. Acad. Sci. Paris* **349** (2011), no. 5-6, p. 291-295.

[26] D. L. Russell, "Nonharmonic Fourier series in the control theory of distributed parameter systems", *J. Math. Anal. Appl.* **18** (1967), p. 542-560.

[27] L. Wang, Q. Yan, "Optimal control problem for exact synchronization of parabolic system", *Math. Control Relat. Fields* **9** (2019), no. 3, p. 411-424.

[28] C. Zu, T. Li, B. Rao, "Sufficiency of Kalman rank condition for the approximate boundary controllability on spherical domain", *Math. Methods Appl. Sci.* **44** (2021), no. 17, p. 13509-13525.