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Rational Groups whose character degree graphs are disconnected

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Abstract. A finite group all of whose complex character values are rational is called a rational group. In this paper, we classify all rational groups whose character degree graphs are disconnected.

Résumé. Un groupe fini dont toutes les valeurs de caractères complexes sont rationnelles est appelé un groupe rationnel. Dans cet article, nous classifions tous les groupes rationnels dont les graphes de degrés de caractère sont déconnectés.

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1. Introduction

Throughout this paper, G is a finite group and $\mathrm{Irr}(G)$ is the set of all irreducible complex characters of G. The set of all irreducible complex character degrees of G is denoted by $\mathrm{cd}(G)$ so that $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. Let $\rho(G)$ be the set of all primes that divide some irreducible character degrees in $\mathrm{cd}(G)$. The character degree graph of G, denoted by $\triangle(G)$, is the graph whose vertex set is $\rho(G)$. Two distinct vertices p, q in $\rho(G)$ are connected by an edge if and only if there exists at least one degree $a \in \mathrm{cd}(G)$ such that pq divides a. Note that if N is a normal subgroup of G, then $\triangle(G/N)$ and $\triangle(N)$ are subgraphs of $\triangle(G)$.

A finite group G all of whose character values are rational is called a rational group. Equivalently, G is a rational group if and only if all generators of the cyclic group $\langle x\rangle$ are conjugate in G for every $x \in$ G. Thus, G is a rational group if and only if $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong \mathrm{Aut}(\langle x\rangle)$ for every $x \in$ G. For example, all symmetric groups, extra special 2-groups and elementary abelian 2-groups are

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rational groups. A quotient group of a rational group and the direct product of a finite number of rational groups are also rational groups. The center Z(G) of a rational group G is an elementary abelian 2-group and so the abelian rational groups are precisely the elementary abelian 2-groups. Kletzing has presented a detailed investigation into the structure of the rational groups by [8]. Gow has shown in [4] that if G is a solvable rational group then $\pi(G) \subseteq \{2,3,5\}$ where $\pi(G)$ is the set of all primes that divide $|G|$. We use E(2) to denote an elementary abelian 2-group.

In this paper, we determine all rational groups whose character degree graphs are disconnected. The main theorems are the following:

Theorem 1. *Let G be a solvable rational group. The character degree graph $\triangle(G)$ of G is disconnected if and only if $G \cong \mathrm{S}_4 \times \mathrm{E}(2)$.*

Theorem 2. *Let G be a nonsolvable rational group. The character degree graph $\triangle(G)$ of G is disconnected if and only if one of the following holds:*

(i) $G \cong \mathrm{S}_5 \times \mathrm{E}(2)$.
(ii) $G \cong \mathrm{S}_6 \times \mathrm{E}(2)$.
(iii) $G \cong \mathrm{Aut}(A_6) \times \mathrm{E}(2)$.

2. The Proofs of the Main Theorems

Manz, Willems and Wolf proved in [11] that the character degree graph $\triangle(G)$ of a finite group G has at most three connected components and if G is solvable, then $\triangle(G)$ has at most two connected components. Palfy showed in [12] that each connected component of the character degree graph of a solvable group must be a complete graph. In [9], Lewis classified all solvable groups having disconnected character degree graphs. Later, Lewis and White classified in [10] nonsolvable groups whose character degree graphs are disconnected. Motivated by these results, we give the proofs of the main theorems.

Proof of Theorem 1. Let G be a solvable rational group whose character degree graph is disconnected. We know from [9] that there exist six types of solvable groups having disconnected character degree graphs. Then let's continue the proof by examining these types. We first assume that G is as in Example 2.1 of [9]. G has a normal nonabelian Sylow p-subgroup P and an abelian p-complement K for some prime p. That is $G = PK$. Since G is a rational group, $G/P \cong K$ is an abelian rational group. Thus, K is an elementary abelian 2-group. By Proposition 21 in [8], we obtain that $P \in \mathrm{Syl}_3(G)$. Since $\mathrm{C}_G(K) = \mathrm{Z}(K) = K$ from Corollary 16.A of [8], we see that C:= $\mathrm{C}_P(K) = 1$. This is a contradiction because $1 \neq P' \leq C$ in Example 2.1 of [9].

G cannot be either Example 2.2 or Example 2.3 of [9] since SL(2,3) and GL(2,3) are not rational groups.

Let G be a group as in Example 2.4 of [9]. Then G is the semi-direct product of V by a subgroup H where V is an elementary abelian p-group for some prime p. Let F and E/F be the Fitting subgroups of G and G/F respectively. We know that G/E and E/F are cyclic groups from Lemma 3.4 of [9]. Also, H/K is a cyclic group where K is Fitting subgroup of H. Since $G/V \cong H$ is a rational group, the cyclic group H/K is also a rational group and so $[H{:}K] = m = 2$. Thus we can observe from (vi) of Lemma 3.4 in [9] that $[E{:}F]$ is not divisible by 2. Therefore, we know that $G \neq E$ since G is rational and E/F is a non trivial cyclic group of odd order. By Lemma 3.4 in [9], we know that Z:= $\mathrm{C}_H(V) = \mathrm{Z}(G)$ and also know that H/K and K/Z are cyclic groups with coprime orders. Because of the fact that the Sylow 2-subgroup of the rational group H/Z is an abelian group, we get from Proposition 21 of [8] that the Sylow 3-subgroup K/Z of H/Z is a group of order 3. That is, $H/Z \cong \mathrm{S}_3$. Since $|V| = q^2$ where q is a prime power and $q+1 \mid 3$ from Example 2.4 of [9], we find that $q = 2$, and so $|V| = 4$. By considering the fact that the Sylow 2-subgroup of the rational group H is abelian group, we obtain from Proposition 21 of [8] that $H \cong S_3 \times E(2)$. Finally, we get that $G \cong S_4 \times E(2)$ as desired.

Now we assume that G is a group as in Example 2.5 of [9]. G has a normal non-abelian 2-subgroup Q and an abelian 2-complement K with the property that $[G : KQ] = 2$ and G/Q is not an abelian group. Also, $Z := C_K(Q) \le Z(G)$. Since G is a rational group, $Z(G)$ must be an elementary abelian 2-group, and so $Z = 1$. Now let $H := N_G(K)$. It follows from Lemma 3.5 of [9] that $H \cap Q = C_Q(K) =: C$ and $H \cap [Q,K] = Q'$. Also, $G = H.[Q,K]$. By considering the fact that $G/Q' \cong S_4 \times E(2)$ from (ii) of Lemma 3.5 in [9], we deduce that $|K| = 3$, and so Q/C is an elementary abelian 2-group of order 4 from Example 2.5 of [9]. Since $[G : KQ] = 2$ and $|K| = 3$, we have that $[G:Q] = 6$. Thus, $G/Q \cong S_3$ since G/Q is not an abelian group. Since $[H : KQ'] = 2$ from Lemma 3.5 in [9], we obtain that $Q' = C$ and $Q = [Q,K]$. Therefore, $G/Q' \cong S_4$. Now, we consider that $Q' > 1$. Let $T \le Q'$ be a chief factor of G. Since Q'/T is a minimal normal subgroup of G/T, we know that Q'/T is an elementary abelian 2-group. Observe that Q/T is the Fitting subgroup of G/T. We know that $Q'/T \le Z(Q/T)$ since Q/T is a 2-group and Q'/T is a minimal normal subgroup of G/T. Since Q/T is non-abelian and $[Q:Q'] = 4$, we conclude that $Z(Q/T) = Q'/T$. Thus, Q'/T is the unique minimal normal subgroup of G/T. Now let A/T be a maximal abelian subgroup of Q/T. Since $[Q:A] = 2$, we know that A/T is a normal subgroup of Q/T. By Lemma 12.12 in [7], we get $|A/T| = |Q'/T|.|Z(Q/T)|$. It follows that $2|Q'/T| = |Q'/T|^2$, and so $|Q'/T| = 2$. Therefore, G/T is a rational group of order 48 and $Z(G/T) = Q'/T$. We know that G/T has at least one faithful irreducible character since $Z(G/T) = Q'/T$ is the unique minimal normal subgroup of G/T. Because of the fact that $F(G/T) = Q/T$ is a non-abelian 2-group, all of the faithful irreducible character degrees of G/T are even numbers by Clifford Theorem. By considering the fact that the sum of squares of degrees of all faithful irreducible characters of G/T is 24, we say that G/T has at least one faithful irreducible character ψ of degree 2. It follows that $\psi_{PT/T} \in \mathrm{Irr}(PT/T)$ where $P \in \mathrm{Syl}_2(G)$ since $\psi_{Q/T} \in \mathrm{Irr}(Q/T)$. Thus, we deduce that $Z(PT/T) = Z(G/T) = Q'/T$ and $|(PT/T)'| = 4$. Therefore, PT/T is a maximal class 2-group of order 16. That is, one of the situations $PT/T \cong Q_{16}$, $PT/T \cong SD_{16}$ or $PT/T \cong D_8$ is true. This is a contradiction, because the faithful irreducible characters of Q_{16}, SD_{16} and D_8 are not rational.

We know from [5] that if G is a solvable rational group, then $\rho(G) = \{2,3\}$. Therefore, G can not be a group as in Example 2.6 because the character degree graph of such a group has at least 3 vertices.

All of this proves that the character degree graph of G is disconnected if and only if $G \cong S_4 \times E(2)$ as desired. □

Corollary 3. *Let G be a solvable rational group such that* $G \ncong S_4 \times E(2)$ *and* $|\rho(G)| > 1$. *Then there exists at least one irreducible character of G whose degree is divisible by* 6.

Proof. It is an easy consequence of [5] and Theorem 1. □

In 1988, Feit and Seitz [3, Theorem B], proved that a noncyclic finite simple group G is a composition factor of a rational group iff G is isomorphic to an alternating group or one of the groups $PSp_4(3)$, $Sp_6(2)$, $O_8^+(2)'$, $PSL(3,4)$, $PSU(4,3)$. Thus we can observe that the groups $PSL(2,q)$ for all prime powers $q \neq 2$ are not rational groups. We use frequently this fact in the nonsolvable case of main theorems. For the reader's convenience, we give an elementary proof of it in the following Lemma before the proof of Theorem 2.

Lemma 4. *The groups* $PSL(2,p^f)$ *are not rational groups, where p is prime, f is a positive integer and* $p^f \neq 2$.

Proof. Suppose that $G := PSL(2,p^f)$ is a rational group for a prime power p^f. We can assume that $p^f \ge 7$ since the groups $PSL(2,3) \cong A_4$, $PSL(2,4) \cong A_5 \cong PSL(2,5)$ are not rational. We know from Theorem 8.3 of [6] that $PSL(2,p^f)$ has a cyclic subgroup $D = \langle x \rangle$ of order $u := \frac{p^f-1}{k}$ where k

is the greatest common divisor of 2 and p^f-1. Also, $\mathrm{N}_G(D)$ is a dihedral group of order $2u$ from Theorem 8.3 of [6]. Thus we get that $|\mathrm{Aut}(\langle x\rangle)| = 2$ since $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong \mathrm{Aut}(\langle x\rangle)$. Therefore $\varphi(u) = 2$ and so $u \in \{3,4,6\}$, where φ is Euler's function. It follows that $p \neq 2$ because $G \ncong A_5$. Thus, we obtain that G is isomorphic to one of the groups $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,9)$ and $\mathrm{PSL}(2,13)$ since $p^f = 2u+1$. Assume that $G := \mathrm{PSL}(2,13)$ is rational. We know from Theorem 8.4 of [6] that G has a cyclic subgroup $\langle a\rangle$ of order 7, which $\langle a\rangle \in \mathrm{Syl}_7(G)$. Thus we have that all of the elements of order 7 in G are conjugate. By Theorem 8.4 [6], we know that $N_G(\langle a\rangle)$ is a dihedral group of order 14, and so the number of sylow 7-subgroups of G is $[G : N_G(\langle a\rangle)] = 6.13$. Therefore, the cardinality of the conjugacy class containing all elements of order 7 in G is 6.6.13. This is a contradiction since $|\mathrm{PSL}(2,13)| = 13.12.7$ is not divisible by 6.6.13. Then $\mathrm{PSL}(2,13)$ is not a rational group. Similarly, $\mathrm{PSL}(2,9)$ is not a rational group. Otherwise we get a contradiction that all of the elements of order 5 in $\mathrm{PSL}(2,9)$ are conjugate. We know that $|\mathrm{Syl}_7(\mathrm{PSL}(2,7))| = 8$ from Theorem 8.2 of [6]. This is why, all of the elements of order 7 in $\mathrm{PSL}(2,7)$ are not in the same conjugacy class. That is, $\mathrm{PSL}(2,7)$ is not a rational group. So we are done. □

Proof of Theorem 2. Let G be a nonsolvable rational group whose character degree graph is disconnected. By [11], we know that $\triangle(G)$ has two or three connected components. Suppose that $\triangle(G)$ has three connected components. Then $G \cong S \times A$, where $S \cong \mathrm{PSL}(2,2^n)$ for an integer $n \geq 2$ and A is an abelian group from Theorem 4.1 of [10]. Since G is a rational group, $G/A \cong S \cong \mathrm{PSL}(2,2^n)$ is also a rational group, which is a contradiction by Lemma 4. Therefore, $\triangle(G)$ must have two connected components. It follows that G has normal subgroups N and K that satisfy conditions (1) – (6) in Theorem 6.3 of [10]. Thus we know from (1) of Theorem 6.3 in [10] that $K/N \cong \mathrm{PSL}(2,q)$, where $q \geq 4$ is a power of a prime p. Since G is a rational group, we also know from (2) of Theorem 6.3 in [10] that G/K is an elementary abelian 2-group.

We first assume that $q = 2^f$, where $f \geq 2$. If $q > 5$, then we know from (3) of Theorem 6.3 in [10] that $2 \nmid [G : CK]$ where $C/N = \mathbf{C}_{G/N}(K/N)$. Thus we obtain that $G = CK$ since G is a rational group. Because of the fact that $G/N = C/N \times K/N$ is a rational group, we find that $K/N \cong \mathrm{PSL}(2,q)$ is also a rational group, which is a contradiction by Lemma 4. This implies that $q = 4$. Now let's consider the normal subgroup N of G. Assume that $N>1$. Then there exists an elementary abelian 2-subgroup L of order 16 such that $K/L \cong \mathrm{SL}(2,4)$ and K/L acts transitively on $\mathrm{Irr}(L)-\{1\}$ from (4) of Theorem 6.3 in [10]. Now, let $1 \neq \nu \in \mathrm{Irr}(L)$ and let $T := \mathrm{I}_G(\nu)$ be the inertia group of ν in G. Thus we find that $[K : (K \cap T)] = q^2 - 1 = 15$ since $|K/L| = (q^2-1)|K/L \cap T/L|$. By considering (6) of Theorem 6.3 in [10], we obtain that $q^2 - 1 = 15 \in \mathrm{cd}(K)$. Since K is a normal subgroup of G, there exists $\chi \in \mathrm{Irr}(G)$ such that $15 \mid \chi(1)$. Therefore the connected components of $\triangle(G)$ must be $\{2\}$ and $\{3,5\}$ since $\pi(G) = \{2,3,5\}$ where $\pi(G)$ is the set of all primes dividing the order of G. On the other hand, $K/N \times C/N = KC/N < G/N$ since G/N is a rational group and $K/N \cong \mathrm{A}_5$. Thus, we obtain that $G/C \cong \mathrm{S}_5$ since G/C is isomorphic to a subgroup of $\mathrm{Aut}(K/N) \cong S_5$ and $\mathrm{A}_5 \cong \mathrm{PSL}(2,4) \cong KC/C < G/C$. Therefore, $\triangle(\mathrm{S}_5)$ is a subgraph of $\triangle(G)$. But this is a contradiction since the connected components of $\triangle(\mathrm{S}_5)$ are $\{2,3\}$ and $\{5\}$. This contradiction gives that $N = 1$. Therefore, we know that $K \cong \mathrm{PSL}(2,4)$ and $C := \mathrm{C}_G(K) \leq \mathrm{Z}(G)$ from Theorem 6.3 in [10]. Since G is a rational group, we get that $K \times C < G$. Thus, we find that $G/C \cong \mathrm{S}_5$ since $\mathrm{A}_5 \cong KC/C < G/C$ and G/C is isomorphic to a subgroup of $\mathrm{Aut}(K) \cong \mathrm{S}_5$. Now, let $x \in G - KC$. Then we have that $G = K\langle x\rangle C$ since $[G : KC] = 2$. Also, $K\langle x\rangle \cap C = 1$ since G/K is an elementary abelian 2-subgroup and $K \cap C = 1$. Thus, $K\langle x\rangle \cong \mathrm{S}_5$. By considering the fact that $C = \mathrm{Z}(G)$ is an elementary abelian 2-group, we obtain that $G \cong \mathrm{S}_5 \times \mathrm{E}(2)$.

Now, suppose that $2 \nmid q$. Since $K/N \cong \mathrm{PSL}(2,q)$ is not a rational group by Lemma 4, we know that $KC < G$. Also, $KC/C \cong K/N \cong \mathrm{PSL}(2,q)$ since $K \cap C = N$. Thus we obtain that $\mathrm{PSL}(2,q) \cong KC/C < G/C$ and G/C is isomorphic to a subgroup of $\mathrm{Aut}(K/N)$. Therefore, G/C is an almost simple group. From Theorem 1.1 of [13], $\mathrm{PSL}(2,q)$ is isomorphic to an alternating group A_n, where

$n \geq 5$. Thus, $G/C \cong \mathrm{S}_n$ or $G/C \cong \mathrm{Aut}(\mathrm{A}_6)$. We know that if $q = 5$, then $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and so this case coincide with the first case. If $q > 5$, then $\triangle(G/C)$ has two connected components from Theorem 2.7 in [10]. Thus we obtain from [1] that G/C must be isomorphic to S_6 if $G/C \cong \mathrm{S}_n$ for some n. Therefore, $G/C \cong \mathrm{S}_6$ or $G/C \cong \mathrm{Aut}(\mathrm{A}_6)$. Moreover, $K/N \cong \mathrm{A}_6 \cong \mathrm{PSL}(2,9)$. If $N = 1$, then we obtain that $G \cong \mathrm{S}_6 \times \mathrm{E}(2)$ or $G \cong \mathrm{Aut}(\mathrm{A}_6) \times \mathrm{E}(2)$. Indeed these groups are rational groups whose character degree graphs are disconnected.

Now, assume that $N > 1$. Then there exists a normal subgroup L of G such that $K/L \cong \mathrm{SL}(2,9)$ from (4) of Theorem 6.3 in [10]. Since G/N is a rational group and $C/N = \mathbf{C}_{G/N}(K/N) \leq Z(G/N)$, we know that C/N is an elementary abelian 2-group. Thus the order of each element of C/L must be 2 or 4 since $|N/L| = 2$. Also, $\langle xL \rangle$ is a normal subgroup of G/L for every $xL \in C/L$. It follows that $[G/L : \mathbf{C}_{G/L}(\langle xL \rangle)] \leq 2$ since G/L is rational. Since $K/L = (K/L)' \leq (G/L)' \leq \mathbf{C}_{G/L}(\langle xL \rangle)$ for every $xL \in C/L$, we obtain that $C/L \leq \mathbf{C}_{G/L}(K/L)$. Suppose that $G/C \cong \mathrm{S}_6$. Then $[G : KC] = 2$ and so there exists an element $y \in G - KC$ such that $G = K\langle y \rangle C$ and $K\langle y \rangle \cap KC = K$. We know that $U := K\langle y \rangle / L$, a bicyclic extension of $\mathrm{PSL}(2,9)$, is not a rational group from ATLAS [2]. Now let $\psi \in \mathrm{Irr}(U)$ be a non-rational character. Then there exists an irreducible character θ of K/L such that $[\psi_{K/L}, \theta] \neq 0$. Let T/L be the inertia group of θ in G/L. Since $C/L \leq \mathbf{C}_{G/L}(K/L)$, we know that $KC/L \leq T/L$. U is not a subgroup of T/L. Otherwise, $T/L = G/L$ and so we know that from Theorem 6.11 in [7] that there exists an irreducible character χ of G/L such that$[\chi_U, \psi] \neq 0$ and $\chi_{K/L} = \alpha\theta$ for a positive integer α. Thus, $\chi_U = \alpha\psi$ by Corollary 6.20 in [7]. This is a contradiction since G/L is rational. Therefore, we get that $\theta^U = \psi$ and $T/L = KC/L$. It follows that there exists an irreducible character λ of T/L such that $\lambda^{G/L} \in \mathrm{Irr}(G/L)$ and $\lambda_{K/L} = n\theta$ for a positive interger n. Thus we get from Problem 5.2 of [7] that $(\lambda^{G/L})_U = (n\theta)^U = n\psi$. This is a contradiction because $\lambda^{G/L}$ is a rational character but $n\psi$ is not. This contradiction implies that $G/C \ncong \mathrm{S}_6$. On the other hand, we know from [2] that there is no group with structure $2.\mathrm{A}_6.2_3$ in the ATLAS notation. Therefore, we obtain that $G/C \ncong \mathrm{Aut}(\mathrm{A}_6)$. Finally, we get that N must be trivial. So the proof is complete. □

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