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Limit spectral distribution for non-degenerate hypersurface singularities

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Abstract. We establish Kyoji Saito's continuous limit distribution for the spectrum of Newton non-degenerate hypersurface singularities. Investigating Saito's notion of dominant value in the case of irreducible plane curve singularities, we find that the log canonical threshold is strictly bounded below by the doubled inverse of the Milnor number. We show that this bound is asymptotically sharp.

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1. Introduction

Let $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with isolated critical point 0 and Milnor number μ . Its spectrum is a discrete invariant formed by μ rational *spectral numbers* (see [12, §II.8.1])

$$\alpha_1, \dots, \alpha_\mu \in \mathbb{Q} \cap (0, n+1).$$

They are certain logarithms of the eigenvalues of the monodromy on the middle cohomology of the Milnor fibre which correspond to the equivariant Hodge numbers of Steenbrink's mixed Hodge structure. In the context of Poincaré polynomials it is convenient to consider the spectrum as a polynomial

$$\mathrm{Sp}_f(t) := \sum_{i=1}^{\mu} t^{\alpha_i} \in \mathbb{Z}[\mathbb{Q}].$$

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K. Saito [16] was the first to study the asymptotic distribution of the spectrum. He considered the normalized spectrum of f ,

$$\chi_f(t) := \frac{\text{Sp}_f(T)}{\mu} = \frac{1}{\mu} \sum_{j=1}^{\mu} T^{\alpha_j}, \quad T = \exp(2\pi i t),$$

as the Fourier transform of the discrete probability density on the interval $(0, n+1)$,

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \delta(s - \alpha_i) ds,$$

where $\delta(s)$ is Dirac's delta function. In the case of Brieskorn–Pham singularities (see Example 9) he identified a continuous limit probability distribution N_{n+1} defined by

$$N_{n+1}(s) ds := \int_{x_0 + \dots + x_n = s} \varphi(x_0) \cdots \varphi(x_n) dx_0 \cdots dx_n,$$

where φ is the indicator function of the unit interval $[0, 1]$,

$$\varphi(x) := \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \notin [0, 1]. \end{cases}$$

Under the Fourier transform \mathcal{F} , N_{n+1} corresponds to the power

$$\mathcal{F}(N_{n+1}) = \mathcal{F}(\varphi)^{n+1}. \quad (1)$$

K. Saito [16, (2.5) i)] suggested to find singularities for which χ_f converges to N_{n+1} . Our main result establishes his limit spectral distribution for Newton non-degenerate singularities.

Theorem 1. *For a fixed Newton diagram Γ , consider the Newton diagrams $\omega\Gamma$ obtained from Γ by scaling with the factor ω . Then we have*

$$\lim_{\omega \rightarrow \infty} \chi_{f_\omega} = \mathcal{F}(N_{n+1}), \quad (2)$$

where f_ω is any Newton non-degenerate function germ of $n+1$ variables with Newton diagram $\omega\Gamma$.

The proof is given in Section 4. Combined with the following remark, our result generates new cases where Saito's limit distribution is valid for a suitably chosen limit as in (2). The general choice of limit is unclear.

Remark 2. K. Saito [16, (3.7), (3.9), (3.2.1)] proved the following facts.

- (a) For quasihomogeneous f of degree 1 with respect to weights w_0, \dots, w_n , (2) holds, even with $\lim_{\omega \rightarrow \infty}$ replaced by $\lim_{w_0, \dots, w_n \rightarrow 0}$.
- (b) For irreducible plane curve singularities f with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$,

$$\lim_{n_g \rightarrow \infty} \chi_f = \mathcal{F}(N_2).$$

- (c) The join $f + g$ of two functions in disjoint sets of variables satisfies

$$\chi_{f+g} = \chi_f \cdot \chi_g.$$

Therefore (2) is compatible with joins by (1).

K. Saito [16, (2.5) ii), (2.8) i)] further suggested to describe up to what extent the spectral distribution is bounded by N_{n+1} and introduced the notion of (*weakly*) *dominating values*. Consider the function

$$\Phi_f: [0, 1] \rightarrow \mathbb{R}, \quad r \mapsto \int_0^r N_{n+1}(s) - \frac{1}{\mu} \sum_{i=1}^{\mu} \delta(s - \alpha_i) ds$$

defined by the difference of the continuous and discrete spectral distributions. By definition $0 < r < \frac{n+1}{2}$ is a *dominating value* if $\Phi_f(r) > 0$ for all f in $n+1$ variables. A *weakly dominating value* is defined by replacing $<$ by \leq and \int_0^r by $\int_0^{r-\epsilon}$ for all $\epsilon > 0$. In particular, K. Saito [16, (2.8) iv)]

formulated the following question which by work of M. Saito [17] extends Durfee’s conjecture on the *geometric genus*

$$p_g = |\{i \mid \alpha_i \leq 1\}| \tag{3}$$

from the case of surface singularities.

Question 3. *Is 1 a dominating value for all $n \geq 2$? In other words, for f in $n + 1$ variables, is the geometric genus bounded by*

$$p_g < \frac{\mu}{(n + 1)!}?$$

Kerner and Nemethi [9] give a positive answer for Newton non-degenerate singularities with Newton diagram $\omega\Gamma$ for sufficiently large ω .

As opposed to Question 3, Hertling’s variance conjecture [6, Conj. 6.7] addresses the distribution of the spectrum in its full range.

Conjecture 4 (Hertling’s Variance Conjecture). *The variance of spectral numbers $\alpha_1 \leq \dots \leq \alpha_\mu$ is bounded by*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n}{2} \right)^2 \leq \frac{\alpha_\mu - \alpha_1}{12}.$$

It was confirmed by Brélivet [2, 3] for Newton non-degenerate singularities and for plane curves. We refer to Brélivet and Hertling [4] for refined investigations in this direction.

In Section 3, we investigate (the extremal) spectral numbers below 1 for their dominance in the case $n = 1$ of irreducible plane curve singularities $C = f^{-1}(0)$. For a single *Puiseux pair* (p, q) we describe these spectral values in terms of the *value semigroup* $S = \langle p, q \rangle$ of C (see (10)). This can be used to visualize the graph of Φ_f as a difference (see Figure 1).

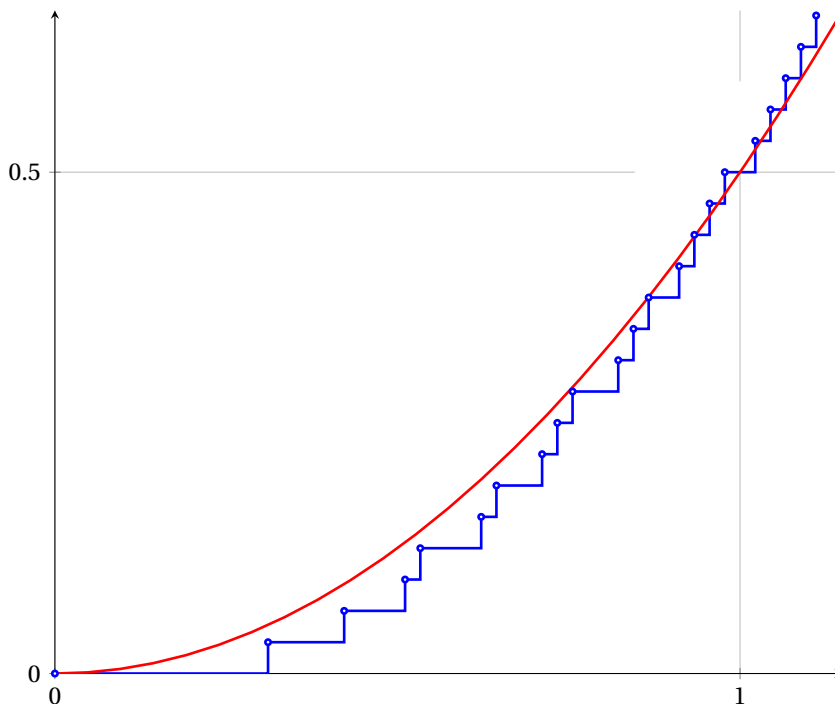


Figure 1. The function Φ_f as a difference for $S = \langle 5, 9 \rangle$.

One can write the smallest spectral number, the *log canonical threshold*, as $\frac{1}{p} + \frac{1}{q}$ and the largest below 1 as $1 - \frac{1}{pq}$. For these extremal spectral numbers we prove the following

Proposition 5. *If $f(z_0, z_1)$ has a single Puiseux pair (p, q) , then*

- (a) $\Phi_f\left(\frac{1}{p} + \frac{1}{q}\right) > 0$ unless $p = 2$ and $q \in \{3, 5\}$, with $\lim_{p \rightarrow \infty} \Phi_f\left(\frac{1}{p} + \frac{1}{q}\right) = 0$,
- (b) $\Phi_f\left(1 - \frac{1}{pq}\right) < 0$ with $\lim_{p \rightarrow \infty} \Phi_f\left(1 - \frac{1}{pq}\right) = 0$.

For general irreducible plane curve singularities, Igusa [8, Thm. 1] showed that the log canonical threshold depends only on multiplicity and the first Puiseux exponent (see also [14, Proof of Thm. 1.1]). It thus equals $\frac{1}{\beta_0} + \frac{1}{\beta_1}$ where β_0, β_1 are the two smallest minimal generators of the value semigroup. The statement of Proposition 5.(a) remains valid in this extended generality.

Theorem 6. *For any irreducible plane curve singularity $C = f^{-1}(0)$ with value semigroup different from $\langle 2, 3 \rangle$ and $\langle 2, 5 \rangle$, we have $\Phi_f\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) > 0$. In other words, the squared log canonical threshold is bounded by*

$$\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right)^2 > \frac{2}{\mu}.$$

Moreover, $\lim_{n_g \rightarrow \infty} \Phi_f\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) = 0$.

In particular, Theorem 6 provides a quite surprising constraint on the first Puiseux pair of an irreducible plane curve singularity with a given Milnor number.

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2. Spectrum of non-degenerate singularities

Suppose that $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is *Newton non-degenerate*. This means that there are local coordinates z_0, \dots, z_n such that

$$f = f(z_0, \dots, z_n) \in \mathbb{C}\{z_0, \dots, z_n\} = \mathcal{O}_{\mathbb{C}^{n+1}, 0} =: \mathcal{O}$$

is a *Newton non-degenerate convenient* power series (see [11, 1.19 Def.] and [12, §II.8.5]). Let Γ denote the Newton diagram of f . We write $\sigma \in \Gamma$ to indicate that σ is a face of Γ . For $\sigma, \tau \in \Gamma$, write $\tau \leq \sigma$ if τ is a face of σ . By g_σ we denote the polynomial obtained from the power series $g \in \mathcal{O}$ by restricting the monomial support to the cone of σ .

There is a (decreasing) Newton filtration \mathcal{N} defined by Γ on \mathcal{O} . Following Steenbrink [19, (5.6)], denote the Newton graded ring associated to \mathcal{O} by

$$A := \text{gr}_{\mathcal{N}} \mathcal{O}$$

and the principal parts of $z_0 \frac{\partial f}{\partial z_0}, \dots, z_n \frac{\partial f}{\partial z_n}$ with respect to \mathcal{N} by F_0, \dots, F_n . For $\sigma \in \Gamma$ let A_σ be the corresponding graded subring of A and denote by

$$d(\sigma) := \dim A_\sigma = \dim \mathbb{Q}\sigma = \dim \sigma + 1$$

its dimension.

The *Brieskorn module* (see [15, 18])

$$\Omega_f := \Omega_{(\mathbb{C}^{n+1}, 0)}^{n+1} / df \wedge \Omega_{(\mathbb{C}^{n+1}, 0)}^n$$

also carries a Newton filtration which is induced by the inclusion

$$\begin{CD}
 \Omega_{(\mathbb{C}^{n+1},0)}^{n+1} = \mathcal{O} dz_0 \wedge \cdots \wedge dz_n @< \frac{z_0 \cdots z_n}{dz_0 \wedge \cdots \wedge dz_n} << \mathcal{O} \\
 @VV \Downarrow V @VV \Downarrow V \\
 \Omega_f @<< \hookrightarrow \mathcal{O} / \langle z_0 \frac{\partial f}{\partial z_0}, \dots, z_n \frac{\partial f}{\partial z_n} \rangle.
 \end{CD} \tag{4}$$

M. Saito [18] and Varchenko–Khovanskii [20] identified the Poincaré series of Ω_f with the singularity spectrum of f defined by Steenbrink [19].

Theorem 7 (M. Saito, Varchenko–Khovanskii). *For Newton non-degenerate f , the Poincaré series of the Newton filtered vector space Ω_f reads*

$$p_{\Omega_f}(t) = t^{\alpha_1} + \cdots + t^{\alpha_\mu} =: \text{Sp}_f(t)$$

where $\alpha_1, \dots, \alpha_\mu$ are the spectral numbers of f . □

The inclusion (4) identifies

$$\begin{aligned}
 \Omega_f &\cong z_0 \cdots z_n \mathcal{O} / \left\langle z_0 \frac{\partial f}{\partial z_0}, \dots, z_n \frac{\partial f}{\partial z_n} \right\rangle, \\
 \text{gr}_{\mathcal{N}} \Omega_f &\cong A / \langle F_0, \dots, F_n \rangle =: H_f.
 \end{aligned} \tag{5}$$

Based on results of Kouchnirenko [11] (and Hochster [7]) Steenbrink [19, (5.7)] gave a formula for Newton non-degenerate f decomposing $p_{H_f} = \text{Sp}_f$ with respect to faces of the Newton diagram: For a face $\sigma \in \Gamma$ he first writes the Poincaré series of the subspace of $A_\sigma / \langle F_{0,\sigma}, \dots, F_{n,\sigma} \rangle$ corresponding to the interior of the cone $\mathbb{Q}_{\geq 0}\sigma$ of σ as

$$q_\sigma(t) = \sum_{\tau \leq \sigma} (-1)^{d(\sigma)-d(\tau)} (1-t)^{d(\tau)} p_{A_\sigma}(t).$$

Denote the minimal dimension of a coordinate space containing $\sigma \in \Gamma$ by

$$k(\sigma) := \min\{k \in \mathbb{Z} \mid \exists i_1, \dots, i_k \in \{0, \dots, n\} : \sigma \subset \mathbb{Q}e_{i_1} + \cdots + \mathbb{Q}e_{i_k}\}.$$

Then Steenbrink's formula is given by

Theorem 8 (Steenbrink). *For Newton non-degenerate f in $n + 1$ variables, the Poincaré series of H_f can be written as*

$$\begin{aligned}
 p_{H_f}(t) &= \sum_{\sigma \in \Gamma} (-1)^{n+1-d(\sigma)} (1-t)^{k(\sigma)} p_{A_\sigma}(t) \\
 &= \sum_{\tau \leq \sigma \in \Gamma} (-1)^{n+1-d(\sigma)} (1-t)^{k(\sigma)-d(\sigma)} q_\sigma(t).
 \end{aligned} \tag{6}$$

3. Irreducible plane curve singularities

In this section, we elaborate on the case $n = 1$ where f defines an irreducible plane curve singularity $C = f^{-1}(0)$. We first consider the case of a single Puiseux pair and prove Proposition 5, then move on to the general case and prove Theorem 6.

Suppose first that C has a single Puiseux pair (p, q) . Then f is Newton non-degenerate with Newton diagram Γ consisting of a single line segment $[(p, 0), (q, 0)]$ and defines an irreducible plane curve singularity $C = f^{-1}(0)$. The function f is semiquasihomogeneous of weighted degree 1 with respect to weights

$$w_0 = \frac{1}{p}, \quad w_1 = \frac{1}{q}, \quad d := pq, \tag{6}$$

on variables z_0, z_1 and can be written explicitly as

$$f(z_0, z_1) = z_0^p + z_1^q + \sum_{iq+jp>d} a_{i,j} z_0^i z_1^j.$$

The normalization $\tilde{C} \rightarrow C$ is given by

$$\begin{aligned} \mathcal{O}_C = \mathcal{O}/\langle f \rangle &\hookrightarrow \mathcal{O}_{\tilde{C}} = \tilde{\mathcal{O}}_C \cong \mathbb{C}\{t\}, \\ \bar{z}_0 &\longmapsto t^q + \dots, \\ \bar{z}_1 &\longmapsto t^p + \dots. \end{aligned} \tag{7}$$

The valuation $v: \tilde{\mathcal{O}}_C \rightarrow \mathbb{N}$, $v(t) = 1$, defines the *value semigroup*

$$S := v(\mathcal{O}_C \setminus \{0\}) = \langle p, q \rangle \subset \mathbb{N}.$$

Due to the finiteness of the normalization S has a finite set of *gaps* $\mathbb{N} \setminus S$, which yields $k + \mathbb{N} \subset S$ for $k \gg 0$. The minimal such k is the *conductor* of S and equals the Milnor number (see [5, Prop. 1.2.1.1]))

$$\mu = (p - 1)(q - 1). \tag{8}$$

The Gorenstein property of C is reflected by the symmetry between elements and gaps (see [13])

$$\begin{aligned} S &\xrightarrow{1:1} \mathbb{Z} \setminus S, \\ a &\longmapsto \mu - 1 - a. \end{aligned} \tag{9}$$

The normalized valuation v/d induces the filtration \mathcal{O}_C defined by weights $w = (w_0, w_1)$ on z_0, z_1 . By assumption, this is the Newton filtration \mathcal{N} . Factoring (7) as

$$\begin{array}{ccc} \mathcal{O}_C & \hookrightarrow & \mathbb{C}\{t\} \\ \downarrow & & \downarrow \\ \mathcal{O}/\left\langle z_0 \frac{\partial f}{\partial z_0}, z_1 \frac{\partial f}{\partial z_1} \right\rangle & \longrightarrow & \mathbb{C}\{t\}/\langle t^d \rangle \end{array}$$

and using (4) yields a Newton filtered inclusion

$$\mathcal{O}/\mathcal{N}_{1+w_0+w_1}\mathcal{O} \xrightarrow[\cong]{dz_0 \wedge dz_1} \Omega_f/\mathcal{N}_1\Omega_f \hookrightarrow \mathbb{C}\{t\}/\langle t^d \rangle.$$

This identifies the corresponding ranges of spectral numbers and of values in the semigroup by means of

$$\begin{aligned} \{\alpha \in \{\alpha_1, \dots, \alpha_\mu\} \mid \alpha < 1 + w_0 + w_1\} &\xleftarrow{1:1} S/\langle d \rangle, \\ \alpha &\longmapsto d\alpha - p - q, \\ \frac{k}{d} + w_0 + w_1 &\longleftarrow k. \end{aligned} \tag{10}$$

The smallest spectral number $w_0 + w_1$ corresponds to $0 \in S$, and the gap $\mu - 1$ of S defining the Gorenstein symmetry (9) corresponds to the non-spectral number 1. It follows that (3) can be written explicitly as

$$p_g = \left| \left\{ (i, j) \in \mathbb{N}^2 \mid \frac{i+1}{p} + \frac{j+1}{q} < 1 \right\} \right| = |\mathbb{N} \setminus S| = \frac{\mu}{2}. \tag{11}$$

Under (9) the gap $1 \in \mathbb{N} \setminus S$ is the mirror of $\mu - 2 \in S$ and corresponds to the largest spectral number $1 - w_0 w_1 < 1$ by (10).

After these preparations we are ready to give the

Proof of Proposition 5.

(a). Using (6) and (8) we compute

$$\begin{aligned} \mu\Phi_f(w_0 + w_1) &= \frac{\mu}{2}(w_0 + w_1)^2 - 1 = \frac{(p-1)(q-1)}{2} \left(\frac{1}{p} + \frac{1}{q}\right)^2 - 1 \\ &= \frac{(p-1)(q-1)(p+q)^2 - 2p^2q^2}{2p^2q^2} \\ &= \frac{(pq - p - q + 1)(p^2 + 2pq + q^2) - 2p^2q^2}{2p^2q^2} \\ &= \frac{2pq + p^3q - 3p^2q - p^3 + p^2 + pq^3 - 3pq^2 - q^3 + q^2}{2p^2q^2} \\ &= \frac{1}{pq} + \frac{pq - 3q - p + 1}{2q^2} + \frac{pq - 3p - q + 1}{2p^2}, \end{aligned} \tag{12}$$

which tends to 0 for $p \rightarrow \infty$. If $p \geq 4$ and $q \geq 5$, then (12) is positive since

$$\begin{aligned} pq - 3q - p + 1 &= p(q-1) - 3q + 1 \geq 4q - 4 - 3q + 1 \geq q - 3 > 0, \\ pq - 3p - q + 1 &= p(q-3) - q + 1 \geq 4q - 12 - q + 1 \geq 3q - 11 > 0. \end{aligned}$$

If $p = 3$, then (12) becomes

$$\frac{1}{3q} - \frac{2}{2q^2} + \frac{2q-8}{18} = \frac{2q^3 - 8q^2 + 6q - 18}{18q^2},$$

which is positive if $q \geq 4$. Finally, if $p = 2$, then (12) becomes

$$\frac{1}{2q} - \frac{q+1}{2q^2} + \frac{q-5}{8} = \frac{q^3 - 5q^2 - 4}{8q^2},$$

which is positive if $q \geq 6$, but negative if $q \in \{3, 5\}$.

(b). Using (11) and (6) we compute

$$\Phi_f(1 - w_0w_1) = \frac{1}{2}(1 - w_0w_1)^2 - \frac{1}{2} = -\frac{1}{2d^2} = -\frac{1}{2p^2q^2} < 0,$$

which tends to 0 for $p \rightarrow \infty$. □

Consider now an irreducible plane curve singularity $C = f^{-1}(0)$ with arbitrary number g of Puiseux pairs. To prepare the proof of Theorem 6, we review some standard integer invariants (see [21, Ch. II, §1-3]): Let

$$\bar{\beta}_0 < \bar{\beta}_1 < \dots < \bar{\beta}_g$$

denote the minimal generators of the value semigroup of C and set

$$e_i := \gcd(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_i), \quad n_i := \frac{e_{i-1}}{e_i}, \quad q_i := \frac{\bar{\beta}_i}{e_i} \tag{13}$$

for $i = 0, \dots, g$. These greatest common divisors form a strictly decreasing sequence

$$\bar{\beta}_0 = e_0 > e_1 > \dots > e_g = 1. \tag{14}$$

Moreover, the minimal generators of the value semigroup satisfy inequalities

$$n_{i-1}\bar{\beta}_{i-1} < \bar{\beta}_i \tag{15}$$

for $i = 1, \dots, g$. The *characteristic Puiseux exponents* of C are defined recursively by

$$\beta_0 := \bar{\beta}_0, \quad \beta_1 := \bar{\beta}_1, \quad \beta_i := \bar{\beta}_i - n_{i-1}\bar{\beta}_{i-1} + \beta_{i-1}, \tag{16}$$

for $i = 2, \dots, g$. By (15) they form a strictly increasing sequence

$$1 \leq \beta_0 < \beta_1 < \dots < \beta_g. \tag{17}$$

The Milnor number of f can be written as (see [21, Ch. II, §3, (3.14)])

$$\mu = \sum_{i=1}^g \beta_i (e_{i-1} - e_i) - \beta_0 + 1. \quad (18)$$

On the other hand, A'Campo showed that (see [1, Thm. 3.(ii)])

$$\mu = \sum_{i=1}^g e_i \mu_i, \quad \mu_i := (n_i - 1)(q_i - 1). \quad (19)$$

Proof of Theorem 6. The case where $g = 1$ is covered by Proposition 5 (a). Using (14) and (17), we find a lower bound

$$\begin{aligned} \mu &= -\beta_g e_g + \sum_{i=1}^{g-1} (\beta_{i+1} - \beta_i) e_i + \beta_1 e_0 - \beta_0 + 1 \\ &\geq -\beta_g e_g + \beta_1 e_0 - \beta_0 + 1 \\ &= -\beta_g + \beta_0(\beta_1 - 1) + 1 \\ &> -\beta_g + \beta_0 \beta_1 - \beta_0. \end{aligned} \quad (20)$$

Suppose first that $g \geq 3$. Using (18) and (20), we compute

$$\begin{aligned} (\beta_0 + \beta_1)^2 \mu - 2\beta_0^2 \beta_1^2 &= (\beta_0^2 + 2\beta_0 \beta_1 + \beta_1^2) \mu - 2\beta_0^2 \beta_1^2 \\ &> \sum_{i=1}^g \beta_0^2 \beta_i (e_{i-1} - e_i) - \beta_0^3 + \beta_0^2 \\ &\quad + \sum_{i=1}^g \beta_1^2 \beta_i (e_{i-1} - e_i) - \beta_0 \beta_1^2 + \beta_1^2 \\ &\quad - 2\beta_0 \beta_1 \beta_g - 2\beta_0^2 \beta_1 \\ &> \sum_{i=1}^{g-1} \beta_0^2 \beta_i + \sum_{i=1}^{g-1} \beta_1^2 \beta_i - \beta_0^3 - \beta_0 \beta_1^2 \\ &\quad + (\beta_0 - \beta_1)^2 \beta_g - 2\beta_0^2 \beta_1 \\ &> 2\beta_0^2 \beta_1 + 2\beta_1^3 - \beta_0^3 - \beta_0 \beta_1^2 - 2\beta_0^2 \beta_1 > 0. \end{aligned}$$

It follows that

$$\Phi_f \left(\frac{1}{\beta_0} + \frac{1}{\beta_1} \right) = \frac{1}{2} \left(\frac{1}{\beta_0} + \frac{1}{\beta_1} \right)^2 - \frac{1}{\mu} = \frac{(\beta_0 + \beta_1)^2 \mu - 2\beta_0^2 \beta_1^2}{2\beta_0^2 \beta_1^2 \mu} > 0.$$

Suppose now that $g = 2$. By (19), (13), (14) and (15),

$$e_1 \mu_1 = (n_1 - 1)(\bar{\beta}_1 - e_1) = n_1 \bar{\beta}_1 - \bar{\beta}_1 - e_0 + e_1 < n_1 \bar{\beta}_1 \leq \bar{\beta}_2 - 1 = q_2 - 1$$

and hence

$$\begin{aligned} \mu - e_1^2 \mu_1 &= e_1 \mu_1 + e_2 \mu_2 - e_1^2 \mu_1 \\ &> e_1(1 + e_2(n_2 - 1) - e_1) \mu_1 \\ &= e_1(1 + e_1 - e_2 - e_1) \mu_1 \\ &= e_1(1 - e_2) \mu_1 = 0. \end{aligned} \quad (21)$$

If $(n_1, q_1) \notin \{(2, 3), (2, 5)\}$, then by (13), Proposition 5 and (21)

$$\Phi_f \left(\frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1} \right) = \frac{1}{e_1^2} \frac{1}{2} \left(\frac{1}{n_1} + \frac{1}{q_1} \right)^2 - \frac{1}{\mu} > \frac{1}{e_1^2} \frac{1}{\mu_1} - \frac{1}{\mu} = \frac{\mu - e_1^2 \mu_1}{e_1^2 \mu_1 \mu} > 0.$$

Otherwise, we have $n_1 = 2 \leq e_1 = n_2$ and (15) yields $q_2 > 2e_1q_1$. Using (19) it follows that

$$\begin{aligned} \mu &= (e_1 - 1)(q_2 - 1) + e_1(q_1 - 1) \\ &> (e_1 - 1)(2e_1q_1 - 1) + e_1(q_1 - 1) \\ &= 2e_1q_1(e_1 - 1) + e_1(q_1 - 2) + 1 \\ &= 2q_1e_1^2 - (q_1 + 2)e_1 + 1 \end{aligned}$$

and hence

$$(2 + q_1)^2\mu - 8e_1^2q_1^2 = \begin{cases} 78e_1^2 - 125e_1 + 25 & \text{if } q = 3, \\ 290e_1^2 - 343e_1 + 49 & \text{if } q = 5. \end{cases}$$

In both cases $e_1 \geq 2$ implies

$$\Phi_f\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) = \frac{(n_1 + q_1)^2\mu - 2e_1^2n_1^2q_1^2}{2e_1^2n_1^2q_1^2\mu} = \frac{(2 + q_1)^2\mu - 8e_1^2q_1^2}{8e_1^2q_1^2\mu} > 0$$

which tend to 0 for $n_g \rightarrow \infty$ since this entails $e_1 \rightarrow \infty$ by (14) and $\mu \rightarrow \infty$ by (19) □

4. Limit spectral distribution

In this section we return to the general setup of Section 2 and prove our main result Theorem 1. Our approach is to subdivide the Newton diagram and mimic an argument of K. Saito (see [16, (2.2), (3.7)]). We begin with his motivating

Example 9 (Brieskorn–Pham type singularities). Suppose first that $n = 0$ and $f = f(z) = z^d$ is quasihomogeneous of degree 1 with respect to the weight $w = 1/d$ on z with Milnor number $\mu = d - 1$. By (5), $H = \langle z \rangle \subset \mathbb{C}\{z\}/\langle z^d \rangle$ and hence

$$p_{H_f}(t) = \frac{t(t^{w-1} - 1)}{1 - t^w} = \frac{t - t^w}{t^w - 1}.$$

By Theorem 8, using L'Hôpital's rule in the second step,

$$\begin{aligned} \lim_{w \rightarrow 0} \chi_f(t) &= \lim_{w \rightarrow 0} \frac{p_H(T)}{\mu} & (22) \\ &= \lim_{w \rightarrow 0} \frac{w}{1 - w} \frac{\exp(2\pi it) - \exp(2\pi itw)}{\exp(2\pi itw) - 1} \\ &= \lim_{w \rightarrow 0} \frac{\exp(2\pi it) - \exp(2\pi itw) - 2\pi itw \exp(2\pi itw)}{1 - \exp(2\pi itw) + 2\pi it(1 - w) \exp(2\pi itw)} \\ &= \frac{\exp(2\pi it) - 1}{2\pi it} = \frac{\exp(\pi it)}{\pi t} \frac{\exp(\pi it) - \exp(-\pi it)}{2i} \\ &= \frac{\exp(\pi it)}{\pi t} \sin(\pi t) = \mathcal{F}(\varphi)(t). \end{aligned}$$

Consider now $f = f_0 + \dots + f_n$, where $f_j = f_j(z_j) = z_j^{d_j}$, which is quasihomogeneous of degree 1 with respect to weights $w_0 = 1/d_0, \dots, w_n = 1/d_n$ on the variables z_0, \dots, z_n with Milnor number $\mu = \mu_f = \prod_{j=0}^n \mu_{f_j}$. Then $H = H_f = H_{f_0} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} H_{f_n}$ and hence, by the first part and (1),

$$\lim_{w_0, \dots, w_n \rightarrow 0} \chi_f(t) = \prod_{i=0}^n \lim_{w_i \rightarrow 0} \chi_{f_i}(t) = \mathcal{F}(\varphi)^{n+1}(t) = \mathcal{F}(N_{n+1})(t).$$

In this sense the normalized spectrum converges in distribution to the continuous probability distribution N_{n+1} .

For our purpose we adapt the calculation (22) as follows.

Lemma 10. $\lim_{w \rightarrow 0} w \frac{1-T}{1-T^w} = \mathcal{F}(\varphi)(t).$

Proof. Using L'Hôpital's rule in the second step and (22), we compute

$$\begin{aligned} \lim_{w \rightarrow 0} w \frac{1 - T}{1 - T^w} &= \lim_{w \rightarrow 0} \frac{w \cdot (1 - \exp(2\pi i t))}{1 - \exp(2\pi i t w)} \\ &= \lim_{w \rightarrow 0} \frac{1 - \exp(2\pi i t)}{-2\pi i t \exp(2\pi i t w)} \\ &= \frac{1 - \exp(2\pi i t)}{-2\pi i t} = \mathcal{F}(\varphi)(t). \end{aligned} \quad \square$$

For the subdivision of the Newton diagram we rely on the following general result. The basis of a rational pointed cone σ are the irreducible integral vectors $\alpha_0, \dots, \alpha_k$ on its rays. If it extends to a lattice basis, then σ is called *regular*. In this case σ is a simplicial cone and the convex hull of $\{0, \alpha_0, \dots, \alpha_k\}$ has k -dimensional volume 1 (see [10, §1.1]). A rational fan is called *regular* if all its cones are regular. Varchenko [10, Thm. 1, Remark] pointed out the following

Theorem 11. *Any finite rational fan has a regular subdivision.* □

Finally, we make use of Kouchnirenko's formula [11, Thm. I] for the Milnor number of Newton non-degenerate singularities in terms of volumes.

Theorem 12 (Kouchnirenko). *The Milnor number of any Newton non-degenerate f in $n + 1$ variables can be written as*

$$\mu = (n + 1)!V_{n+1} - n!V_n + \dots + 1!(-1)^n V_1 + (-1)^{n+1},$$

where V_k is the sum of k -dimensional volumes of the intersection of the convex hull of $\Gamma \cup \{0\}$ with the k -dimensional coordinate planes. □

We are now ready for the

Proof of Theorem 1. By Theorem 11, Γ has a subdivision $\tilde{\Gamma}$ corresponding to a regular subdivision of its fan of cones. For any $\tau \in \tilde{\Gamma}$ let $w_0^\tau, \dots, w_k^\tau$ be the weights of the basis of $\mathbb{Q}_{\geq 0} \tau \cap \mathbb{Z}$. Then

$$p_{A_\sigma}(t) = \sum_{\tilde{\Gamma} \ni \tau \leq \sigma} (-1)^{d(\sigma) - d(\tau)} p_{A_\tau}(t), \quad p_{A_\tau}(t) = \prod_{j=0}^{\dim \tau} \frac{1}{(1 - t^{w_j^\tau})}.$$

Substituting into Steenbrink's formula from Theorem 8 yields

$$p_{H_f}(t) = \sum_{\tilde{\Gamma} \ni \tau \leq \sigma \in \Gamma} (-1)^{n+1 - d(\tau)} \frac{(1 - t)^{k(\sigma)}}{\prod_{j=0}^{\dim \tau} (1 - t^{w_j^\tau})}.$$

Passing to $\omega\Gamma$, w_j^τ is replaced by εw_j^τ where $\varepsilon\omega = 1$ and hence

$$p_{H_{f_\omega}}(t) = \sum_{\tilde{\Gamma} \ni \tau \leq \sigma \in \Gamma} (-1)^{n+1 - d(\tau)} \frac{(1 - t)^{k(\sigma)}}{\prod_{j=0}^{\dim \tau} (1 - t^{\varepsilon w_j^\tau})}.$$

By Theorems 7 and 12,

$$\lim_{\omega \rightarrow \infty} \chi_{f_\omega}(t) = \lim_{\omega \rightarrow \infty} \frac{p_{H_{f_\omega}}(T)}{\mu_{f_\omega}} \tag{23}$$

$$\begin{aligned} &= \sum_{\tilde{\Gamma} \ni \tau \leq \sigma \in \Gamma} (-1)^{n+1 - d(\tau)} \lim_{\omega \rightarrow \infty} \frac{1}{\mu_{f_\omega}} \frac{(1 - t)^{k(\sigma)}}{\prod_{j=0}^{\dim \tau} (1 - t^{\varepsilon w_j^\tau})}, \\ \mu_{f_\omega} &= \sum_{j=0}^{n+1} (-1)^{n+1-j} j! \omega^j V_j. \end{aligned} \tag{24}$$

Fix $\tilde{\Gamma} \ni \tau \leq \sigma \in \Gamma$. Let $V(\tau)$ be the $d(\tau)$ -dimensional volume of the convex hull of $\tau \cup \{0\}$. Note that

$$\sum_{\substack{\tau \in \tilde{\Gamma} \\ d(\tau)=n+1}} V(\tau) = V_{n+1}, \tag{25}$$

$$1/V(\tau) = d(\tau)! \prod_{j=0}^{d(\tau)} w_j^\tau. \tag{26}$$

The summand in (23) indexed by τ is then computed using (26), Lemma 10 and (24):

$$\begin{aligned} & \lim_{\omega \rightarrow \infty} \frac{1}{\mu_{f_\omega}} \frac{(1-T)^{k(\sigma)}}{\prod_{j=0}^{\dim \tau} (1-T^\varepsilon w_j^\tau)} \\ &= \lim_{\omega \rightarrow \infty} \frac{\prod_{j=0}^{\dim \tau} \frac{\omega}{w_j^\tau}}{\mu_{f_\omega}} (1-T)^{k(\sigma)-d(\tau)} \prod_{j=0}^{\dim \tau} \lim_{\varepsilon \rightarrow 0} \varepsilon w_j^\tau \frac{1-T}{1-T^\varepsilon w_j^\tau} \\ &= \lim_{\omega \rightarrow \infty} \frac{d(\tau)! V(\tau) \omega^{d(\tau)}}{\mu_{f_\omega}} (1-T)^{k(\sigma)-d(\tau)} \left(\lim_{w \rightarrow 0} w \frac{1-T}{1-T^w} \right)^{d(\tau)} \\ &= \lim_{\omega \rightarrow \infty} \frac{d(\tau)! V(\tau) \omega^{d(\tau)}}{\mu_{f_\omega}} (1-T)^{k(\sigma)-d(\tau)} \mathcal{F}(N_{d(\tau)})(t) \\ &= \begin{cases} \frac{V(\tau)}{V_{n+1}} \mathcal{F}(N_{n+1})(t) & \text{if } d(\tau) = n+1, \\ 0 & \text{if } d(\tau) < n+1. \end{cases} \end{aligned}$$

The claim now follows by substituting into (23) and applying (25). □

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