Robert Kurinczuk and Nadir Matringe

A characterization of the relation between two $\ell$-modular correspondences


Published online: 15 June 2020

https://doi.org/10.5802/crmath.33

This article is licensed under the Creative Commons Attribution 4.0 International License.

http://creativecommons.org/licenses/by/4.0/
A characterization of the relation between two $\ell$-modular correspondences

Une caractérisation de la relation entre deux correspondances $\ell$-modulaires

Robert Kurinczuk$^a$ and Nadir Matringe$^b$

$^a$ Department of Mathematics, Imperial College London, SW7 2AZ, U.K.

$^b$ Université de Poitiers, Laboratoire de Mathématiques et Applications, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962, Futuroscope Chasseneuil Cedex. France.

E-mails: robkurinczuk@gmail.com, nadir.matringe@math.univ-poitiers.fr.

Abstract. Let $F$ be a non archimedean local field of residual characteristic $p$ and $\ell$ a prime number different from $p$. Let $V$ denote Vignéras’ $\ell$-modular local Langlands correspondence [7], between irreducible $\ell$-modular representations of $GL_n(F)$ and $n$-dimensional $\ell$-modular Deligne representations of the Weil group $W_F$. In [4], enlarging the space of Galois parameters to Deligne representations with non necessarily nilpotent operators allowed us to propose a modification of the correspondence of Vignéras into a correspondence $C$, compatible with the formation of local constants in the generic case. In this note, following a remark of Alberto Mínguez, we characterize the modification $C \circ V^{-1}$ by a short list of natural properties.

Résumé. Soit $F$ un corps local non archimédien de caractéristique résiduelle $p$ et $\ell$ un nombre premier différent de $p$. Soit $V$ la correspondance de Langlands $\ell$-modulaire définie par Vignéras en [7], entre représentations irréductibles $\ell$-modulaires de $GL_n(F)$ et représentations de Deligne $\ell$-modulaires de dimension $n$ du groupe de Weil $W_F$. Dans [4], l’élargissement de l’espace des paramètres galoisiens aux représentations de Deligne à opérateur non nécessairement nilpotent, nous a permis de proposer une modification de la correspondance de Vignéras en une correspondance notée $C$, compatible aux constantes locales des représentations génériques et de leur paramètre. Dans cette note rédigée à la suite d’une remarque d’Alberto Mínguez, nous caractérisons la modification $C \circ V^{-1}$ par une courte liste de propriétés naturelles.

Funding. The authors were supported by the Anglo-Franco-German Network in Representation Theory and its Applications: EPSRC Grant EP/R009279/1, the GDRI “Representation Theory” 2016-2020, and the LMS (Research in Pairs).
1. Introduction

Let $F$ be a non-archimedean local field with finite residue field of cardinality $q$, a power of a prime $p$, and $W_F$ the Weil group of $F$. Let $\ell$ be a prime number different from $p$. The $\ell$-modular local Langlands correspondence established by Vignéras in [7] is a bijection from isomorphism classes of smooth irreducible representations of $\text{GL}_n(F)$ and $n$-dimensional Deligne representations (Section 2.1) of the Weil group $W_F$ with nilpotent monodromy operator. It is uniquely characterized by a non-naive compatibility with the $\ell$-adic local Langlands correspondence ([1, 2, 5, 6]) under reduction modulo $\ell$, involving twists by Zelevinsky involutions. In [4], at the cost of having a less direct compatibility with reduction modulo $\ell$, we proposed a modification of the correspondence $V$ of Vignéras, by in particular enlarging its target to the space of Deligne representations with non necessarily nilpotent monodromy operator (it is a particularity of the $\ell$-modular setting that such operators can live outside the nilpotent world). The modified correspondence $C$ is built to be compatible with local constants on both sides of the correspondence ([3, 4]) and we proved that it is indeed the case for generic representations in [4]. Here, we show in Section 3 that if we expect a correspondence to have such a property, and some other natural properties, then it will be uniquely determined by $V$. Namely we characterize the map $C \circ V^{-1}$ by a list of five properties in Theorem 8. The map $C \circ V^{-1}$ endows the image of $C$ with a semiring structure because the image of $V$ is naturally equipped with semiring laws. We end this note by studying this structure from a different point of view in Section 4.

2. Preliminaries

Let $\nu : W_F \to F^{\times}_\ell$ be the unique character trivial on the inertia subgroup of $W_F$ and sending a geometric Frobenius element to $q^{-1}$, it corresponds to the normalized absolute value $\nu : F^{\times} \to F^{\times}_\ell$ via local class field theory.

We consider only smooth representations of locally compact groups, which unless otherwise stated will be considered on $F_\ell$-vector spaces. For $\mathcal{G}$ a locally compact topological group, we let $\text{Irr}(\mathcal{G})$ denote the set of isomorphism classes of irreducible representations of $\mathcal{G}$.

2.1. Deligne representations

We follow [4, Section 4], but slightly simplify some notation. A Deligne-representation of $W_F$ is a pair $(\Phi, U)$ where $\Phi$ is a finite dimensional semisimple representation of $W_F$, and $U \in \text{Hom}_{W_F}(\nu \Phi, \Phi)$; we call $(\Phi, U)$ nilpotent if $U$ is a nilpotent endomorphism over $F_F$.

The set of morphisms between Deligne representations $(\Phi, U), (\Phi', U')$ (of $W_F$) is given by $\text{Hom}_D(\Phi, \Phi') = \{ f \in \text{Hom}_{W_F}(\Phi, \Phi') : f \circ U = U' \circ f \}$. This leads to notions of irreducible and indecomposable Deligne representations. We refer to [4, Section 4], for the (standard) definitions of dual and direct sums of Deligne representations.

We let $\text{Rep}_{D,\text{ss}}(W_F)$ denote the set of isomorphism classes of Deligne-representations; and $\text{Indec}_{D,\text{ss}}(W_F)$ (resp. $\text{Irr}_{D,\text{ss}}(W_F)$, $\text{Nilp}_{D,\text{ss}}(W_F)$) denote the set of isomorphism classes of indecomposable (resp. irreducible, nilpotent) Deligne representations. Thus

$$\text{Irr}_{D,\text{ss}}(W_F) \subset \text{Indec}_{D,\text{ss}}(W_F) \subset \text{Rep}_{D,\text{ss}}(W_F), \quad \text{Nilp}_{D,\text{ss}}(W_F) \subset \text{Rep}_{D,\text{ss}}(W_F).$$

Let $\text{Rep}_{\text{ss}}(W_F)$ denote the set of isomorphism classes of semisimple representations of $W_F$, we have a canonical map $\text{Supp}_{W_F} : \text{Rep}_{D,\text{ss}}(W_F) \to \text{Rep}_{\text{ss}}(W_F), (\Phi, U) \mapsto \Phi$; we call $\Phi$ the $W_F$-support of $(\Phi, U)$.

For $\Psi \in \text{Irr}(W_F)$ we denote by $o(\Psi)$ the cardinality of the irreducible line $Z_\Psi = \{ \nu^k \Psi, k \in \mathbb{Z} \}$; it divides the order of $q$ in $F^{\times}_\ell$ hence is prime to $\ell$. We let $\text{I}(W_F) = \{ Z_\Psi : \Psi \in \text{Irr}(W_F) \}$. 

The fundamental examples of non-nilpotent Deligne representation are the cycle representations: let I be an isomorphism from $\psi^{\circ} \Psi$ to $\Psi$ and define $\mathcal{C}(\Psi, I) = (\Phi(\Psi), C_I) \in \text{Rep}_{ss}(D, \mathbb{F}_\ell)$ by

$$\Phi(\Psi) = \bigoplus_{k=0}^{\circ(\Psi)-1} v^k \Psi, \quad C_I(x_0, \ldots, x_{\circ(\Psi)-1}) = (I(x_0, \ldots, x_{\circ(\Psi)-1}), x_0, \ldots, x_{\circ(\Psi)-2}), \ x_k \in v^k \Psi.$$ 

Then $\mathcal{C}(\Psi, I) \in \text{Irr}_{ss}(D, \mathbb{F}_\ell)$ and its isomorphism class only depends on $(Z\Psi, I)$, by [4, Proposition 4.18].

To remove dependence on $I$, in [4, Definition 4.6 and Remark 4.9] we define an equivalence relation $\sim$ on $\text{Rep}_{D, ss}(W_F)$. We say that

$$(\Phi, U_1) \sim (\Phi, U_2)$$

for $(\Phi, U_1) \in \text{Rep}_{D, ss}(W_F)$ if they both admit a decomposition (it is unique up to re-ordering)

$$(\Phi, U_i) = \Phi^r \bigoplus_{k=1}^r (\Phi_{i,k}, U_{i,k})$$

as a direct sum of elements in $\text{Indec}_{D, ss}(W_F)$, such that $r_1 = r_2 =: r$ and for $k = 1, \ldots, r$ there exists $\lambda_k \in \mathbb{F}_\ell^\times$ such that

$$(\Phi_{2,k}, U_{2,k}) \simeq (\Phi_{1,k}, \lambda_k U_{1,k}).$$

The equivalence class of $\mathcal{C}(\Psi, I)$ is independent of $I$, and we set

$$\mathcal{C}(Z\Psi) := [\mathcal{C}(\Psi, I)] \in [\text{Irr}_{D, ss}(W_F)].$$

The sets $\text{Rep}_{D, ss}(W_F)$, $\text{Irr}_{D, ss}(W_F)$, $\text{Indec}_{D, ss}(W_F)$, and $\text{Nilp}_{D, ss}(W_F)$ are unions of $\sim$-classes, and if $X$ denotes any of them we set $[X] := X/\sim$. Similarly, for $(\Phi, U) \in \text{Rep}_{D, ss}(W_F)$ we write $[\Phi, U]$ for its equivalence class in $[\text{Rep}_{D, ss}(W_F)]$. On $\text{Nilp}_{D, ss}(W_F)$ the equivalence relation $\sim$ coincides with equality.

The operations $\oplus$ and $(\Phi, U) \mapsto (\Phi, U)^r$ on $\text{Rep}_{D, ss}(W_F)$ descend to $[\text{Rep}_{D, ss}(W_F)]$. Tensor products are more subtle; for example, tensor products of semisimple representations of $W_F$ are not necessarily semisimple. We define a semisimple tensor product operation $\otimes_{ss}$ on $[\text{Rep}_{D, ss}(W_F)]$ in [4, Section 4.4], turning $([\text{Rep}_{D, ss}(W_F)], \oplus, \otimes_{ss})$ into an abelian semiring.

The basic non-irreducible examples of elements of $\text{Nilp}_{D, ss}(W_F)$ are called segments: For $r \geq 1$, set $[0, r-1] := (\Phi(r), N(r))$, where

$$\Phi(r) = \bigoplus_{k=0}^{r-1} v^k, \quad N(r)(x_0, \ldots, x_{r-1}) = (0, x_0, \ldots, x_{r-2}), \ x_k \in v^k.$$

We now recall the classification of equivalence classes of Deligne representations of $W_F$ of [4].

**Theorem 1** ([4, Section 4.4]).

1. Let $\Phi \in \text{Irr}_{D, ss}(W_F)$, then there is either a unique $\Psi \in \text{Irr}(W_F)$ such that $\Phi = \Psi$, or a unique irreducible line $Z\Psi$ such that $[\Phi] = \mathcal{C}(Z\Psi)$.
2. Let $[\Phi, U] \in [\text{Indec}_{D, ss}(W_F)]$, then there exist a unique $r \geq 1$ and a unique $\Theta \in [\text{Irr}_{D, ss}(W_F)]$ such that $[\Phi, U] = [0, r-1] \otimes_{ss} \Theta$.
3. Let $[\Phi, U] \in [\text{Rep}_{D, ss}(W_F)]$, there exist $[\Phi_i, U_i] \in [\text{Indec}_{D, ss}(W_F)]$ for $1 \leq i \leq r$ such that $[\Phi, U] = \bigoplus_{i=1}^r [\Phi_i, U_i]$.

We recall the following classical result about tensor products of segments.

**Lemma 2.** For $n \geq m \geq 1$, one has

$$[0, n-1] \otimes_{ss} [0, m-1] = [0, n+m-2] \oplus [1, n+m-3] \oplus \cdots \oplus [m-1, n-1].$$
Lemma 4. Let $\Phi$ be a nilpotent operator in $\mathbb{Q}_\ell$ and by $[0, i-1]_{\mathbb{Q}_\ell}$ the $\ell$-adic Deligne representation with $W_F$-support $\Phi^{\perp_{k=0}}_{k=0} \mathbf{Q}_\ell$ and nilpotent operator $N(i)_{\mathbb{Q}_\ell}$ sending $(x_0, \ldots, x_{i-2})$ to $(0, x_0, \ldots, x_{i-2})$. The relation
\[
[0, n-1]_{\mathbb{Q}_\ell} \otimes [0, m-1]_{\mathbb{Q}_\ell} = [0, n + m - 2]_{\mathbb{Q}_\ell} \oplus [1, n + m - 3]_{\mathbb{Q}_\ell} \oplus \cdots \oplus [m - 1, n - 1]_{\mathbb{Q}_\ell} \tag{1}
\]
can be translated into a statement on tensor product of irreducible representations of $\text{SL}_2(\mathbb{C})$, which is well-known and easily checked by the highest weight theory. Because all powers of $v_{\mathbb{Q}_\ell}$ take values in $\mathbb{Z}_\ell^\times$, the canonical $\mathbb{Z}_\ell$-lattice in $\Phi^{\perp_{k=0}}_{k=0} \mathbf{Q}_\ell$ is stable under both the actions of $W_F$ and $N(i)_{\mathbb{Q}_\ell}$, and this defines a $\mathbb{Z}_\ell$-Deligne representation $[0, i-1]_{\mathbb{Z}_\ell}$. Taking the canonical lattices on both sides of Equation (1) we get
\[
[0, n-1]_{\mathbb{Z}_\ell} \otimes [0, m-1]_{\mathbb{Z}_\ell} = [0, n + m - 2]_{\mathbb{Z}_\ell} \oplus [1, n + m - 3]_{\mathbb{Z}_\ell} \oplus \cdots \oplus [m - 1, n - 1]_{\mathbb{Z}_\ell}. \tag{2}
\]
Now tensoring Equation (2) by $\mathbf{F}_\ell$ we obtain the relation
\[
[0, n-1] \otimes [0, m-1] = [0, n + m - 2] \oplus [1, n + m - 3] \oplus \cdots \oplus [m - 1, n - 1].
\]
Finally by definition (see [4, Definition 4.37]) one has $[0, n-1] \otimes_{\text{ss}} [0, m-1] = [0, n-1] \otimes [0, m-1]$, hence the sought equality. \qed

2.2. L-factors

We set $\text{Irr}_{\text{cusp}}(\text{GL}(F)) := \bigcap_{n \geq 0} \text{Irr}_{\text{cusp}}(\text{GL}_n(F))$ where $\text{Irr}_{\text{cusp}}(\text{GL}_n(F))$ is the set of isomorphism classes of irreducible cuspidal representations of $\text{GL}_n(F)$.

Let $\pi$ and $\pi'$ be a pair of cuspidal representations of $\text{GL}_n(F)$ and $\text{GL}_{m}(F)$ respectively. We denote by $L(X, \pi, \pi')$ the Euler factor attached to this pair in [3] via the Rankin–Selberg method, it is a rational function of the form $\frac{1}{Q(\mathbf{F}_\ell[X])}$ where $Q \in \mathbf{F}_\ell[X]$ satisfies $Q(0) = 1$. We recall that a cuspidal representation of $\text{GL}_n(F)$ is called banal if $\nu \otimes \pi \neq \pi$. The following is a part of [3, Theorem 4.9].

Proposition 3. Let $\pi, \pi' \in \text{Irr}_{\text{cusp}}(\text{GL}(F))$. If $\pi$ or $\pi'$ is non-banal, then $L(X, \pi, \pi') = 1$.

Let $[\Phi, U] \in [\text{Rep}_{D, ss}(W_F)]$, for brevity from now on we often denote such a class just by $\Phi$, we denote by $L(X, \Phi)$ the L-factor attached to it in [4, Section 5], their most basic property is that
\[
L(X, \Phi \oplus \Phi') = L(X, \Phi)L(X, \Phi')
\]
for $\Phi$ and $\Phi'$ in $[\text{Rep}_{D, ss}(W_F)]$. We need the following property of such factors.

Lemma 4. Let $\Psi \in \text{Irr}(W_F)$ and $a \leq b$ be integers, put $\Phi = [a, b] \otimes_{\text{ss}} \Psi$ and $\Phi' = [-b, -a] \otimes_{\text{ss}} \Psi^\vee$, then $L(X, \Phi \otimes_{\text{ss}} \Phi')$ has a pole at $X = 0$.

Proof. According to [4, Lemma 5.7], it is sufficient to prove that $L(X, \Psi \otimes_{\text{ss}} \Psi^\vee)$ has a pole at $X = 0$ for $\Psi \in \text{Irr}(W_F)$, but this property follows from the definition of the L-factor in question, and the fact that $\Psi \otimes_{\text{ss}} \Psi^\vee$ contains a nonzero vector fixed by $W_F$. \qed

2.3. The map CV

For $\Psi \in \text{Irr}(W_F)$ we set $S_0(\mathbb{Z}_\Psi) = \bigoplus_{k=0}^{0(\Psi)-1} \mathbf{Q}_\ell^k \Psi$. By Theorem 1, an element $\Phi \in \text{Nilp}_{D, ss}(W_F)$ has a unique decomposition
\[
\Phi = \Phi_{\text{acyc}} \oplus \bigoplus_{k \geq 1, Z \in l(W_F)} [0, k-1] \otimes_{\text{ss}} n_{Z \Psi, k} S_0(\mathbb{Z}_\Psi),
\]
where for all $k \geq 1$ and $Z \Psi \in l(W_F)$, $\Phi_{\text{acyc}}$ has no summand isomorphic to $[0, k-1] \otimes_{\text{ss}} S_0(\mathbb{Z}_\Psi)$; i.e. we have separated $\Phi$ into an acyclic and a cyclic part. Then following [4, Section 6.3], we set:
\[
\text{CV}(\Phi) = \Phi_{\text{acyc}} \oplus \bigoplus_{k \geq 1, Z \Psi \in l(W_F)} [0, k-1] \otimes_{\text{ss}} n_{Z \Psi, k} \mathcal{C}(\mathbb{Z}_\Psi).
\]
We denote by $C_{D,ss}(W_F)$ the image of $CV : \text{Nilp}_{D,ss}(W_F) \to [\text{Rep}_{D,ss}(W_F)]$, and call $C_{D,ss}(W_F)$ the set of $C$-parameters.

2.4. $\ell$-modular local Langlands

We let $\text{Irr}(\text{GL}(F)) = \bigsqcup_{n \geq 0} \text{Irr}(\text{GL}_n(F))$ where $\text{Irr}(\text{GL}_n(F))$ denotes the set of isomorphism classes of irreducible representations of $\text{GL}_n(F)$.

In [7], Vigneras introduces the $\ell$-modular local Langlands correspondence: a bijection

$$V : \text{Irr}(\text{GL}(F)) \to \text{Nilp}_{D,ss},$$

characterized in a non-naive way by reduction modulo $\ell$. For this note, we recall $\text{Supp}_{W_F} \circ V$, the semisimple $\ell$-modular local Langlands correspondence of Vigneras, induces a bijection between supercuspidal supports elements of $\text{Irr}(\text{GL}_n(F))$ and $\text{Rep}_{ss}(W_F)$ compatible with reduction modulo $\ell$.

In [4], we introduced the bijection

$$C = CV \circ V : \text{Irr}(\text{GL}(F)) \to C_{D,ss}(W_F);$$

which satisfies $\text{Supp}_{W_F} \circ V = \text{Supp}_{W_F} \circ C$. Moreover, the correspondence $C$ is compatible with the formation of $L$-factors for generic representations, a property $V$ does not share; in the cuspidal case:

**Proposition 5** ([4, Proposition 6.13]). For $\pi$ and $\pi'$ in $\text{Irr}_{\text{cusp}}(\text{GL}(F))$ one has $L(X, \pi, \pi') = L(X, C(\pi), C(\pi'))$.

We note another characterization of non-banal cuspidal representations:

**Proposition 6** ([4, Sections 3.2 and 6.2]). A representation $\pi \in \text{Irr}_{\text{cusp}}(\text{GL}(F))$ is non-banal if and only if $V(\pi) = \ell^k S_0(\mathbb{Z}_\psi)$, or equivalently $C(\pi) = \ell^k C(\mathbb{Z}_\psi)$, for some $k \geq 0$ and $\Psi \in \text{Irr}(W_F)$.

Amongst non-banal cuspidal representations, those for which $k = 0$ in the above statement, shall play a special role in our characterization. We denote by $\text{Irr}^*_{\text{cusp}}(\text{GL}(F))$ the subset of $\text{Irr}_{\text{cusp}}(\text{GL}(F))$ consisting of those $\pi \in \text{Irr}_{\text{cusp}}(\text{GL}(F))$ such that $C(\pi) = C(\psi)$, for some $\Psi \in \text{Irr}(W_F)$.

3. The characterization

In this section, we provide a list of natural properties which characterize $CV : \text{Nilp}_{D,ss}(W_F) \to [\text{Rep}_{D,ss}(W_F)]$.

**Proposition 7.** Let $CV : \text{Nilp}_{D,ss}(W_F) \to [\text{Rep}_{D,ss}(W_F)]$ be any map, and $C' := CV \circ V$. Suppose

(i) $\text{Supp}_{W_F} \circ C'$ is the semisimple $\ell$-modular local Langlands correspondence of Vigneras; in other words, $C'$ preserves the $W_F$-support;

(ii) $C'$ (or equivalently $CV'$) commutes with taking duals;

(iii) $L(X, \pi, \pi') = L(X, C'(\pi), C'(\pi'))$ for all non-banal representations $\pi \in \text{Irr}^*_{\text{cusp}}(\text{GL}(F))$.

Then for all $\Psi \in \text{Irr}(W_F)$, one has $CV'(S_0(\mathbb{Z}_\psi)) = C'(\mathbb{Z}_\psi)$.

**Proof.** Thanks to (i), $CV'(S_0(\mathbb{Z}_\psi))$ has $W_F$-support $\bigoplus_{k=0}^{q(\psi)} \mathbf{1}^k \mathbb{Z}_\psi$. Hence, by Theorem 1, its image under $CV'$ is either $C'(\mathbb{Z}_\psi)$ or a sum of Deligne representations of the form $[a, b] \otimes_{ss} \Psi$ for $0 \leq a < b < o(\psi) - 1$. If $\mathbf{1}$ is the only situation, writing $CV'(S_0(\mathbb{Z}_\psi)) = ([a, b] \otimes_{ss} \Psi) \oplus W$, we have $CV'(S_0(\mathbb{Z}_\psi)) = ([1, b, a] \otimes_{ss} \Psi)^L \oplus W^L$, thanks to (ii). However, writing $r$ for the non-banal cuspidal representation $V^{-1}(S_0(\mathbb{Z}_\psi))$, we have $L(X, \tau, \psi^L) = 1$ according to Theorem 3 and Proposition 6, whereas

$$L(X, C(\tau), C(\psi^L)) = L(X, ([a, b] \otimes_{ss} \Psi) \oplus W) \otimes_{ss} ([-b, -a] \otimes_{ss} \Psi^L) \oplus W^L)$$

$$= L(X, ([a, b] \otimes_{ss} \Psi) \otimes_{ss} ([-b, -a] \otimes_{ss} \Psi^L) W)L'(X)$$
for $L'(X)$ an Euler factor. Now, observe that $L(X, ((a, b) \otimes_{ss} \Psi) \otimes_{ss} (-b, -a) \otimes_{ss} \Psi^v)$ has a pole at $X = 0$ according to Lemma 4, hence cannot be equal to 1. The conclusion of this discussion, according to (iii) is $CV'( \text{St}_0 (Z_Ψ)) = \mathcal{C}(Z_Ψ).$ \hfill \Box

It follows that (i)–(iii) characterize $C|_{\text{Irr}_{\text{cusp}}(GL(F))}$ without reference to Vignéras' correspondence $V$.

On the other hand any map $CV'$ satisfying (i)–(iii) must send each $\nu^k \Psi$ to itself if $o(\Psi) > 1$ by (i). So there is no chance that $CV'$ will preserve direct sums because $\bigoplus_{k=0}^{\infty} CV'(\nu^k \Psi) \neq \mathcal{C}(Z_Ψ)$. In particular any compatibility property of $CV'$ with direct sums will have to be non-naive. Here is our characterization of the map $CV$:

**Theorem 8.** Suppose $CV' : \text{Nilp}_{D,ss}(W_F) \rightarrow [\text{Rep}_{D,ss}(W_F)]$ satisfies (i)–(iii) of Proposition 7, and suppose moreover

(a) If $\Phi' \in \text{Im}(CV')$ and $\Phi' = \Phi'_1 \oplus \Phi'_2$ in $[\text{Rep}_{D,ss}(W_F)]$ then $\Phi'_1, \Phi'_2 \in \text{Im}(CV')$. Moreover, if $\Phi' = CV'(\Phi)$, $\Phi'_1 = CV'(\Phi_i)$ for $\Phi_i \in \text{Nilp}_{D,ss}(W_F)$, and $\Phi' = \Phi'_1 \oplus \Phi'_2$, then $\Phi = \Phi_1 \oplus \Phi_2$.

(b) $CV'(\Psi) \in [\text{Rep}_{D,ss}(W_F)]$ for $j \in \mathbb{N}_{>1}$ and $\Phi \in \text{Nilp}_{D,ss}(W_F)$.

Then $CV' = CV$.

**Proof.** For $\Psi \in \text{Irr}(W_F)$, it follows at once from Proposition 7 and (b) that $CV'(\Psi) = \text{St}_0 (Z_Ψ) \otimes_{ss} \mathcal{C}(Z_Ψ)$. Hence $CV(\Psi)$ can be decomposed as a direct sum of elements in $\text{Im}(CV') \cap [\text{Indec}_{D,ss}]$, and (a) reduces the proof of the inclusion $\text{Im}(CV') \subset C_{D,ss}(W_F)$ to showing that $[0, j - 1] \otimes_{ss} \text{St}_0 (Z_Ψ) \otimes_{ss} \mathcal{C}(Z_Ψ)$ for $\Psi \in \text{Irr}(W_F)$, $j \geq 1$.

We first assume that $o(\Psi) = 1$, so $\text{St}_0 (Z_Ψ) = \Psi$. The only possible pre-image of $\Psi$ by $CV'$ is $\Psi$ by (i), however $CV'(\Psi) = \mathcal{C}(Z_Ψ)$ by Proposition 7 so $\text{St}_0 (Z_Ψ) \otimes_{ss} \mathcal{C}(Z_Ψ)$ for $\Psi \in \text{Irr}(W_F)$, $j \geq 1$. Also $[0, j - 1] \otimes_{ss} \text{St}_0 (Z_Ψ) \otimes_{ss} \mathcal{C}(Z_Ψ)$ also belongs to $\text{Im}(CV')$ thanks to Lemma 2. However as $o(\Psi) = 1$, the Deligne representation $[j - 1, j - 1] \otimes_{ss} \Psi$ is nothing else than $\Psi$, which does not belong to $\text{Im}(CV')$, contradicting (a).

If $o(\Psi) > 1$, then $CV'(\nu^k \Psi) = \nu^k \Psi$. If $\text{St}_0 (\Psi)$ belonged to $\text{Im}(CV')$ then (a) would imply that $\text{St}_0 (\Psi) = CV'(\bigoplus_{k=0}^{\infty} \nu^k \Psi)$, which is not the case thanks to Proposition 7. To see that $[0, j - 1] \otimes_{ss} \text{St}_0 (\Psi) \otimes_{ss} \mathcal{C}(Z_Ψ)$ for all $j \geq 2$ we use the same trick as in the $o(\Psi) = 1$ case.

Now take $\Phi \in \text{Nilp}_{D,ss}$, as we just noticed $CV' (\Phi)$ is a $C$-parameter and we write it

$$CV' (\Phi) = CV'(\Phi)_{\text{acyc}} \bigoplus_{k \geq 1, Z_Ψ \in \{ l(W_F) \}} [0, k - 1] \otimes_{ss} \text{St}_0 (Z_Ψ),$$

as in Section 2.3, where for each irreducible line $Z_Ψ$ we have fixed an irreducible $\Psi \in Z_Ψ$. Then (a) and the beginning of the proof imply that

$$\Phi = CV'(\Phi)_{\text{acyc}} \bigoplus_{k \geq 1, Z_Ψ \in \{ l(W_F) \}} [0, k - 1] \otimes_{ss} \text{St}_0 (Z_Ψ),$$

hence that $CV'(\Phi) = CV(\Phi)$. \hfill \Box

### 4. The semiring structure on the space of $C$-parameters

As $(\text{Nilp}_{D,ss}(W_F), \oplus, \otimes_{ss})$ is a semiring, the map $CV$ endows $C_{D,ss}(W_F)$ with a semiring structure by transport of structure. We show that this semiring structure on $C_{D,ss}(W_F)$ can be obtained without referring to $CV$ directly, thus shedding a slightly different light on the map $CV$.  

---

_C. R. Mathématique, 2020, 358, n° 2, 201-209_
We denote by $\mathcal{G}(\text{Rep}_{D,ss}(W_F))$ the Grothendieck group of the monoid $([\text{Rep}_{D,ss}(W_F)], \oplus)$. We set
\[
\mathcal{G}_0(\text{Rep}_{D,ss}(W_F)) = \langle [0, k - 1] \otimes_{ss} \text{St}_0(\Psi) - [0, k - 1] \otimes_{ss} \mathcal{E}(\Psi) \rangle_{\Psi \in \mathcal{I}(W_F)}, \quad k \in \mathbb{N}_{\geq 1},
\]
the additive subgroup of $\mathcal{G}(\text{Rep}_{D,ss}(W_F))$ generated by the differences $[0, k - 1] \otimes_{ss} \text{St}_0(\Psi) - [0, k - 1] \otimes_{ss} \mathcal{E}(\Psi)$ for $\Psi \in \mathcal{I}(W_F)$ and $k \in \mathbb{N}_{\geq 1}$.

**Proposition 9.** The canonical map $h_C : C_{D,ss}(W_F) \to \mathcal{G}(\text{Rep}_{D,ss}(W_F))/\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$, obtained by composing the canonical projection $h : \mathcal{G}(\text{Rep}_{D,ss}(W_F)) \to \mathcal{G}(\text{Rep}_{D,ss}(W_F))/\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$ with the natural injection of $C_{D,ss}(W_F) \hookrightarrow \mathcal{G}(\text{Rep}_{D,ss}(W_F))$, is injective. Moreover, its image is stable under the operation $\oplus$. In particular, this endows the set $C_{D,ss}(W_F)$ with a natural monoid structure.

**Proof.** Note that $h_C$ is the restriction of the canonical surjection $h$ to $C_{D,ss}(W_F)$. Let $\Phi, \Phi'$ be $C$-parameters, as in Section 2.3 and the last proof, we write
\[
\Phi = \bigoplus_{k \geq 1, \Psi \in \mathcal{I}(W_F)} [0, k - 1] \otimes_{ss} \text{St}_0(\Psi) \left( \bigoplus_{i=0}^{\sigma(\Psi) - 1} m_{\Psi,k,i} \Psi^i \right) \oplus n_{\Psi,k} \mathcal{E}(\Psi)
\]
\[
\Phi' = \bigoplus_{k \geq 1, \Psi \in \mathcal{I}(W_F)} [0, k - 1] \otimes_{ss} \text{St}_0(\Psi) \left( \bigoplus_{i=0}^{\sigma(\Psi) - 1} m'_{\Psi,k,i} \Psi^i \right) \oplus n'_{\Psi,k} \mathcal{E}(\Psi)
\]
where for each $(\Psi, k)$, there are $i, i'$ such that $m_{\Psi,k,i} = 0$ and $m'_{\Psi,k,i'} = 0$. Suppose that both $\Phi$ and $\Phi'$ have same the image under $h_C$, then $\Phi' - \Phi \in \text{Ker}(h) = \mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$. We thus get an equality of the form
\[
\Phi - \Phi' = \bigoplus_{k \geq 1, \Psi \in \mathcal{I}(W_F)} a_{\Psi,k} \left( [0, k - 1] \otimes_{ss} \text{St}_0(\Psi) - [0, k - 1] \otimes_{ss} \mathcal{E}(\Psi) \right),
\]
where all sums are finite. Set $f$ to be the set of pairs $(\Psi, k)$ such that $a_{\Psi,k} \geq 0$ and $f^-$ to be the set of pairs $(\Psi, k)$ such that $b_{\Psi,k} := -a_{\Psi,k} > 0$. We obtain
\[
\Phi \oplus \bigoplus_{(\Psi, k) \in f^-} b_{\Psi,k} [0, k - 1] \otimes_{ss} \text{St}_0(\Psi) \oplus \bigoplus_{(\Psi, k) \in f^+} a_{\Psi,k} [0, k - 1] \otimes_{ss} \mathcal{E}(\Psi) = \Phi' \oplus \bigoplus_{(\Psi, k) \in f^-} b_{\Psi,k} [0, k - 1] \otimes_{ss} \mathcal{E}(\Psi) \oplus \bigoplus_{(\Psi, k) \in f^+} a_{\Psi,k} [0, k - 1] \otimes_{ss} \text{St}_0(\Psi)
\]
in $[\text{Rep}_{D,ss}(W_F)]$. Now take $(\Psi, k) \in f^+$, there is $i$ such that $m_{\Psi,k,i} = 0$. Comparing the occurence of $[0, k - 1] \otimes_{ss} \Psi^i$ on the left and right hand sides of the equality we obtain
\[
0 = m'_{\Psi,k,i} + a_{\Psi,k} \Rightarrow a_{\Psi,k} = 0.
\]
Hence we just proved that $a_{\Psi,k} = 0$ for all $(\Psi, k) \in f^+$. The symmetric argument shows that for $(\Psi, k) \in f^-$, there is $i'$ such that
\[
m'_{\Psi,k,i'} + b_{\Psi,k} = 0 \Rightarrow b_{\Psi,k} = 0,
\]
which is impossible by assumption. Hence $f = f^+$ and $a_{\Psi,k} = 0$ for all $\Psi \in J$, which implies $\Phi = \Phi'$, so $h_C$ is indeed injective.

For the next assertion, suppose that $h_C([\Phi]_{\text{Inde}c_{D,ss}(W_F)}/n_{\Phi,0}) \in \text{Im}(h_C)$. Take $\Phi_0 \in \text{Inde}c_{D,ss}(W_F)$ and consider $h_C([\Phi]_{\text{Inde}c_{D,ss}(W_F)}/n_{\Phi,0}) \oplus h_C(\Phi_0)$. If $\Phi_0$ "completes a cycle" of $[\Phi]_{\text{Inde}c_{D,ss}(W_F)}/n_{\Phi,0}$, i.e. if $\Phi_0 = [0, k] \otimes_{ss} \Psi$ with $\Psi$ an irreducible representation $\Psi$ of $W_F$, and if all other elements of $[0, k] \otimes_{ss} \Psi$ appear in $[\Phi]_{\text{Inde}c_{D,ss}(W_F)}/n_{\Phi,0}$ as representations $[0, k] \otimes_{ss} \Psi^j$, $j = 1, \ldots, o(\Psi) - 1$, one gets
\[
h_C([\Phi]_{\text{Inde}c_{D,ss}(W_F)}/n_{\Phi,0}) \oplus h_C(\Phi_0) = h_C([\Phi]_{I}/n_0 \Phi \oplus [\Phi]_{I}(n_0 - 1) \otimes_{ss} \mathcal{E}(\Psi)).
\]
If $\Phi_0$ does not complete a cycle, one has
\[ h_C(\oplus_{\Phi \in \text{Indec}_{D,ss}(W_F)} n_{\Phi}) \oplus h_C(\Phi_0) = h_C(\oplus_{\Phi \in \text{Indec}_{D,ss}(W_F)} n_{\Phi} \Phi \oplus \Phi_0). \]

The assertion follows by induction.

In fact the tensor product operation descends on $\text{Im}(h_C)$.

**Proposition 10.** The additive subgroup $\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$ of the ring $\mathcal{G}(\text{Rep}_{D,ss}(W_F))$ is in fact an ideal. Moreover $\text{Im}(h_C)$ is stable under $\otimes_{ss}$. In particular this endows $C_{D,ss}(W_F)$ with a natural semiring structure, and $h_C$ becomes a semiring isomorphism from $C_{D,ss}(W_F)$ to $\text{Im}(h_C)$.

**Proof.** For the first part, taking $\Psi_0 \in \text{Irr}(W_F)$, it is enough to prove that for any $\Phi_1 \in \text{Irr}_{D,ss}(W_F)$ and $k$, $l \geq 0$, the tensor product $[0,k] \otimes_{ss} \text{St}_0(Z_{\Psi_0}) \otimes_{ss} [0,l] \otimes_{ss} \Phi_1$ belongs to $\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$. By associativity and commutativity of tensor product, and because $[0,l] \otimes_{ss} [0,j]$ is always a sum of segments by Lemma 2, it is enough to check that $(\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Phi_1 \otimes_{ss} [0,l]) \otimes_{ss} \Phi_1$ belongs to $\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$. Suppose first that $\Phi_1$ is nilpotent, i.e. $\Phi_1 = \Psi_1 \in \text{Irr}(W_F)$. Because $\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Psi_1$ is fixed by $\nu$ under twisting and because its Deligne operator is zero, we get that
\[ \text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Psi_1 = \bigoplus_{Z_{\Psi} \in \text{Irr}(W_F)} a_{Z_{\Psi}} \text{St}_0(Z_{\Psi}). \]

On the other hand because $\mathcal{C}(Z_{\Psi_0}) \otimes_{ss} \Psi_1$ is fixed by $\nu$ and because its Deligne operator is bijective we obtain
\[ \mathcal{C}(Z_{\Psi_0}) \otimes_{ss} \Psi_1 = \bigoplus_{Z_{\Psi} \in \text{Irr}(W_F)} b_{Z_{\Psi}} \mathcal{C}(Z_{\Psi}). \]

Now observing that both $\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Psi_1$ and $\mathcal{C}(Z_{\Psi_0}) \otimes_{ss} \Psi_1$ have the same $W_F$-support, it implies that $a_{Z_{\Psi}} = b_{Z_{\Psi}}$ for all lines $Z_{\Psi}$, form which we deduce that $(\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Phi_1 \otimes_{ss} \Phi_1$ belongs to $\mathcal{G}_0(\text{Rep}_{D,ss}(W_F))$. With the same arguments we obtain that $(\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Phi_1 \otimes_{ss} \Phi_1$ when $\Phi_1$ is of the form $\mathcal{C}(Z_{\Psi_1})$ (because in this case both $\text{St}_0(Z_{\Psi_0}) \otimes_{ss} \Phi_1$ and $\mathcal{C}(Z_{\Psi_0}) \otimes_{ss} \Phi_1$ have bijective Deligne operators).

The following proposition is proved in a similar, but simpler manner than the propositions above.

**Proposition 11.** Let $h_{\text{Nilp}}$ be the restriction of
\[ h : \mathcal{G}(\text{Rep}_{D,ss}(W_F)) \to \mathcal{G}(\text{Rep}_{D,ss}(W_F))/\mathcal{G}_0(\text{Rep}_{D,ss}(W_F)) \]
to $\text{Nilp}_{D,ss}(W_F)$, then $h_{\text{Nilp}}$ is a semiring isomorphism and $\text{Im}(h_{\text{Nilp}}) = \text{Im}(h_C)$.

The above propositions have the following immediate corollary.

**Corollary 12.** One has $CV = h_C^{-1} \circ h_{\text{Nilp}}$, in particular it is a semiring isomorphism from $\text{Nilp}_{D,ss}(W_F)$ to $C_{D,ss}(W_F)$.

**Acknowledgements**

We thank the referee for useful comments and corrections.

**References**


